数理解析研究所講究録
第 734 巻 1990 年 123-142

The initial value problem for the equations of motion of general fluids with general slip boundary condition

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§1. Introduction and Main Theorem

In this communication we are concerned with the initial-boundary value problem for compressible viscous isotropic Newtonian fluids (say, general fluids) which happen to slip on the solid boundary.

The motion of general fluids filled in a bounded domain $\Omega \subset \boldsymbol{R}^3$ is governed by the so-called compressible Navier-Stokes equations:

$$
(1)\quad \begin{cases} \dfrac{D\rho}{Dt} = -\rho \ \nabla\cdot v, \\ \rho \ \dfrac{Dv}{Dt} = \nabla\cdot \boldsymbol{P} + \rho f, \qquad x\in\Omega, \quad t>0, \\ \rho\theta \dfrac{DS}{Dt} = \nabla\cdot(\kappa \nabla\theta) + \mu'(\nabla\cdot v)^2 + 2\mu \boldsymbol{D}(v):\boldsymbol{D}(v). \end{cases}
$$

Here $\rho=\rho(x,t)$ is the density, $v=v(x,t)=(v_1,v_2,v_3)$ is the velocity vector field, $\theta=\theta(x,t)$ is the absolute temperature, $f=f(x,t)$ is a vector field of external forces, $\boldsymbol{P}=(-p+ +\mu' \nabla\cdot v)\mathrm{I}_3+2\mu\boldsymbol{D}(v)$ is the stress tensor, $\boldsymbol{D}(v)$ is the velocity deformation tensor with the elements

$$
D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \boldsymbol{D}(v):\boldsymbol{D}(v) = D_{jk}\ D_{jk},
$$

$p=p(\rho,\theta)$ is a pressure, $S=S(\rho,\theta)$ is an entropy, μ, μ', κ are, respectively, coefficient of viscosity, second coefficient of viscosity, coefficient of heat conductivity, which are all assumed to be constants satisfying $\mu>0$, $2\mu+3\mu'\geq 0$, $\kappa>0$, $D/Dt=\partial/\partial t+v\cdot\nabla$ and I_3 is an identity matrix of degree 3.

Here and in what follows we use the well-known notation of vector analysis and the summation convention. And we should refer to [7,8] for the notation not stated here explicitely.

We have already studied the initial-boundary value problem for (1) in the perfect slip case $K=0$ and $\kappa_e=1$ or $\kappa_e<1$ in [10].

Here we consider the general slip boundary condition, which is formulated as follows:

$$v\cdot n=0, \quad v\cdot\tau=KPn\cdot\tau,$$

or equivalently,

$$v\cdot n=0, \quad v=K[Pn-(Pn\cdot n)n], \tag{2}$$

where n and τ are a unit inward normal and a unit tangential vector, respectively, such that $n\times\tau=1$ and K is assumed to be a positive function defined on $\Gamma_T=\Gamma\times[0,T]$ (Γ is a boundary of Ω, T is any, but fixed, positive number). Dividing both sides of (2) by $1+\mu K$ and using the same letter K in place of $1/(1+\mu K)$, we deduce from (2)

$$v\cdot n=0, \quad \frac{1}{\mu}(1-K)[Pn-(Pn\cdot n)n]-Kv=0, \quad 1>K\geq 0.$$

Similarly, the boundary condition for θ

$$-\kappa\,\nabla\theta\cdot n=\kappa_e(\theta_e-\theta)+g, \quad \kappa_e\geq 0$$

implies that, using the same letters κ_e and g in place of $\kappa_e/(\kappa+\kappa_e)$ and $g/(\kappa+\kappa_e)$, respectively,

$$(1-\kappa_e)\,\nabla\theta\cdot n-\kappa_e(\theta-\theta_e)=g,\quad 1>\kappa_e\geq 0.$$

The aim of this paper is to establish the unique solvability, local in time, of the initial-boundary value problem (1) with the initial condition

(3) $$(\rho,\,v,\,\theta)|_{t=0}=(\rho_0,\,v_0,\,\theta_0)(x),$$

and the boundary conditions

(4) $$\begin{cases} v\cdot n=0, \quad \dfrac{1}{\mu}(1-K)[\boldsymbol{P}n-(\boldsymbol{P}n\cdot n)n]-Kv=0, \\ (1-\kappa_e)\,\nabla\theta\cdot n-\kappa_e(\theta-\theta_e)=g, \end{cases}$$

where $(K,\kappa_e)=(K,\kappa_e)(x,t)$, $1\geq K,\kappa_e\geq 0$.

The following is our main theorem:

<u>Theorem</u>. Let T be an arbitrary positive number and Ω be a bounded domain in $\boldsymbol{R}^3$ with boundary Γ of class $C^{2+\alpha}$, $\alpha\in(0,1)$. Furthermore, we assume that

(i) $(\rho_0,\,v_0,\,\theta_0)\in C^{1+\alpha}(\bar{\Omega})\times C^{2+\alpha}(\bar{\Omega})\times C^{2+\alpha}(\bar{\Omega})$, $\rho_0'\leq\rho_0(x)\leq\rho_0''$, $\theta_0'\leq\theta_0(x)\leq\theta_0''$ (ρ_0', ρ_0'', θ_0' and θ_0'' are positive constants);

(ii) $(K,\,\kappa_e)=(K,\,\kappa_e)(x,\,t)\in C^{1+\alpha,\,(1+\alpha)/2}_{x,t}(\Gamma_T)$, $0\leq K,\,\kappa_e\leq 1$;

(iii) $(\theta_e,\,g)=(\theta_e,\,g)(x,\,t)\in C^{1+\alpha,\,(1+\alpha)/2}_{x,t}(\Gamma_T)$, moreover,

$(\theta_e,\,g)\in C^{2+\alpha,\,1+\alpha/2}_{x,t}(\Gamma_T')$ $\quad \Gamma_T'=\cup\{x\in\Gamma\,|\,\kappa_e(x,\,t)=1;0\leq t\leq T\}$;

(iv) $f=f(x,\,t)\in C^{\alpha,\,\alpha/2}_{x,t}(\bar{Q}_T\equiv\bar{\Omega}\times[0,\,T])$;

(v) μ, μ' and κ are constants satisfying the relations $2\mu+3\mu'\geq$ ≥ 0, $\mu>0$, $\kappa>0$ and $(p,\,S)=(p,\,S)(\rho,\,\theta)\in$ $\in C^{2+\alpha}([\beta\rho_0',\,\beta^{-1}\rho_0'']\times[\beta\theta_0',\,\beta^{-1}\theta_0''])$ for some positive constant $\beta<1$ such that $S_\theta(\equiv\partial S/\partial\theta)>0$;

(vi) the compatibility conditions between the system (1) and the initial and the boundary conditions (3), (4) are valid.

Then there exists a unique solution (ρ, v, θ) of (1), (3), (4), which belongs to $[B^{1+\alpha}(\bar{Q}_{T'})\cap\{\beta\rho_0'\leq\rho(x,t)\leq\beta^{-1}\rho_0''\}]\times$ $\times C^{2+\alpha,1+\alpha/2}_{x,t}(\bar{Q}_{T'})\times[C^{2+\alpha,1+\alpha/2}_{x,t}(\bar{Q}_{T'})\cap\{\beta\theta_0'\leq\theta(x,t)\leq\beta^{-1}\theta_0''\}]$ for some $T'\in(0,T]$.

§2. Outline of the Proof of Theorem

First of all, we introduce the characteristic transformation Π^x_ξ: $x\rightarrow\xi\equiv X(0;x,t)$, where $X(\tau;x,t)$ $(0\leq\tau\leq t, x\in\bar{\Omega})$ is the solution of the system of equations

$$\text{(5)} \qquad \frac{d}{d\tau}X(\tau;x,t)=v(X(\tau;x,t),\tau), \quad X(t;x,t)=x.$$

If v is suitably smooth, then (5) has a unique solution curve by virtue of the basic theorem of ordinary differential equations. It gives us the relation between x and ξ:

$$\text{(6)} \qquad x=X(t;\xi,0)=\xi+\int_0^t u(\xi,\tau)\,d\tau\equiv X_u(\xi,t),$$

where $u(\xi,t)=v(X(t;\xi,0),t)$.

According to the boundary condition $v\cdot n=0$ on Γ_T, it is clear that Π^x_ξ is an one-to-one mapping from $\bar{Q}_T$ onto $\bar{Q}_T$.

In a similar way to that in [10], we use this transformatioin only for the first equation in (1), whence the unique solution of $(1)^1$ is given by

$$(7)\qquad \rho(x,t)=\Pi_x^{\xi}\ \rho_0(\xi)\ \exp[-\int_0^t \nabla_u\cdot u(\xi,\tau)d\tau]$$

provided that $u\in C_{x,t}^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$ is given.

Here Π_x^{ξ} is the inverse mapping of Π_{ξ}^{x}, $\nabla_u=G\nabla_{\xi}$, $G=(g_{jk})=$

$={}^t(\partial X_u/\partial\xi)^{-1}$, $\nabla_{\xi}=(\frac{\partial}{\partial\xi_1},\ \frac{\partial}{\partial\xi_2},\ \frac{\partial}{\partial\xi_3})$.

Hence the problem (1),(3),(4) can be reduced to the following initial-boundary value problems with respect to $w=v-v_0$ and with respect to $\sigma=\theta-\theta_0$:

$$(8)\quad \begin{cases} \dfrac{\partial w}{\partial t}=A(x,t,w;\nabla)w+\Phi(x,t,w,\sigma) & \text{in } Q_T,\\ w|_{t=0}=0 & \text{on } \Omega,\\ B(x,t;\nabla)w=-B(x,t;\nabla)v_0 & \text{on } \Gamma_T,\end{cases}$$

$$(9)\quad \begin{cases} \dfrac{\partial \sigma}{\partial t}=A'(x,t,w,\sigma)\Delta\sigma+\Psi(x,t,w,\sigma) & \text{in } Q_T,\\ \sigma|_{t=0}=0 & \text{on } \Omega,\\ (1-\kappa_e)\nabla\sigma\cdot n-\kappa_e\sigma=g-\kappa_e\theta_e-(1-\kappa_e)\nabla\theta_0\cdot n+\kappa_e\theta_0 & \text{on } \Gamma_T,\end{cases}$$

where the principal parts A and A', the lower order terms Φ and Ψ, the boundary operator $B=(B_{jk})_{j,k=1,2,3}$ are given by the formulae:

$$A=\left(\frac{\mu+\mu'}{\rho}\nabla_j\nabla_k+\frac{\mu}{\rho}\nabla_l^{\,2}\right)_{j,k=1,2,3},\quad A'=\frac{\kappa}{\rho\theta S_\theta},$$

$$\Phi=-\frac{1}{\rho}\nabla p+f-(v\cdot\nabla)v-Av_0,$$

$$\Psi=\frac{1}{\rho\,\theta\,S_\theta}[\mu'(\nabla\cdot v)^2+2\mu D(v):D(v)+\rho^2\theta S_\rho\nabla\cdot v]-(v\cdot\nabla)\theta-$$

$-\frac{\kappa}{\rho\,\theta\,S_\theta}\Delta\theta_0$ with ρ and (v,θ) replaced by (7) and

$(w+v_0,\sigma+\theta_0)$, respectively,

$$B_{jk}=\begin{cases} n_k & (j=1;\ k=1,2,3),\\ (1-K)(n_k\delta_{j-1\ l}+n_l\delta_{j-1\ k}-2n_{j-1}n_k n_l)\nabla_l- & \\ -K\delta_{j-1\ k} & (j=2,3;\ k=1,2,3),\end{cases}$$

(δ_{jk} is Kronecker's delta).

2.1 Linearized problem of (8) and (9)

First of all, we consider the following linearized problem of (8):

$$(10)\quad\begin{cases}\dfrac{\partial w}{\partial t}=A(x,t,w';\nabla)w+\Phi(x,t,w',\sigma') & \text{in } Q_T,\\ w|_{t=0}=0 & \text{on } \Omega,\\ B(x,t;\nabla)w=-B(x,t;\nabla)v_0 & \text{on } \Gamma_T.\end{cases}$$

Here (w',σ') is a given function belonging to the class

$$\mathscr{S}_T\equiv\{(w,\sigma)\in C^{2+\alpha,\,1+\alpha/2}_{x,\,t}(\bar{Q}_T)\,|\,(w,\sigma)|_{t=0}=0,\ \|(w,\sigma)\|^{(2)}_{\bar{Q}_T}<M_1,$$

$$\sum_{|s|=2}|D^s_x(w,\sigma)|^{(\alpha)}_{x,\bar{Q}_T}<M_2\},$$

where M_1 is an arbitrary positive number, M_2 is a positive number determined later, $\|u\|^{(m)}_{\bar{Q}_T}\equiv\sum^{m}_{2r+|s|=0}\|D^r_t D^s_x u\|^{(0)}_{\bar{Q}_T}$, $\|u\|^{(0)}_{\bar{Q}_T}\equiv$

$\equiv \sup\{|u(x,t)|;(x,t)\in\bar{Q}_T\}$ and $|u|^{(\alpha)}_{x,\bar{Q}_T} \equiv \sup\{|u(x,t)-$

$-u(x',t)||x-x'|^{-\alpha};(x,t),(x',t)\in\bar{Q}_T,\ x\neq x'\}.$

Then the following fact holds.

Lemma 1. The system of differential equations (10) is uniformly parabolic in the sense of Petrowsky with modulo of parabolicity δ if we take T in such a way that

$$M_1T<\theta_0',\ \ 0<M_3\equiv(M_1+\|v_0\|^{(1)}_{\Omega})T/[1-(M_1+\|v_0\|^{(1)}_{\Omega})T]<M_0,\ \ M_3T<1,$$

where M_0 is a positive root of the equation $1-3x-6x^2-6x^3=0$.

Proof. Since

$$\det[A(x,t,w';i\xi)-\lambda I_3]=(\lambda+\frac{|\xi|^2}{a_1})^2(\lambda+\frac{|\xi|^2}{a_3})\quad(a_1=a_2=\rho/\mu,$$

$a_3=\rho/(2\mu+\mu'))$, and the estimates

$$(11)\quad\begin{cases}\|u'\|^{(0)}_{\bar{Q}_T}\leq M_1,\quad \sum_{|s|=1}\|D^s_x u'\|^{(0)}_{\bar{Q}_T}\leq M_3,\\ \sum_{|s|=2}\|D^s_x u'\|^{(0)}_{\bar{Q}_T}\leq M_3(1+M_3)^2,\\ |g_{jk}-\delta_{jk}|\leq\dfrac{M_3+4M_3^2+6M_3^3}{1-3M_3-6M_3^2-6M_3^3}\equiv C_1(M_1,T)\quad(j,k=1,2,3)\end{cases}$$

follow from (6) for $u'=\Pi^x_\xi w'$, $(w',\sigma')\in\mathscr{S}_T$, it is sufficient to take δ in such a way that

$$\delta=\mu\rho_0'^{-1}\exp[-3(1-3C_1)TM_3]$$

by virtue of (7). □

The following complementing condition in the case of $\Omega=\boldsymbol{R}^3_+$, $\Gamma=\{x\in\boldsymbol{R}^3 \mid x_3=0\}$ is essential throughout our investigation.

Lemma 2. There exists a positive constant δ' smaller than δ such that for any $\xi'=(\xi_1,\xi_2)\in\boldsymbol{R}^2$ and $\nu\in\boldsymbol{C}^1$ satisfying

(12) $$\mathrm{Re}\ \nu\geqq-\delta'\xi'^2,\quad \xi'^4+|\nu|^2>0,$$

the row vectors of the matrix $B(x,t;i\xi)\hat{A}(x,t,w';i\xi,\nu)$ $((x,t)\in\Gamma_T$, fixed) are linearly independent modulo $M=\prod_{r=1}^{3}(\xi_3-\xi_3^{+(r)}(\xi',\nu))$, where $\hat{A}(x,t,w';i\xi,\nu)$ is an adjugate matrix of $A(x,t,w';i\xi)-\nu I_3$ and $\xi_3^{+(r)}$'s are the roots in ξ_3 of $\det[A(x,t,w';i\xi)-\nu I_3]=0$ with positive imaginary parts.

Proof. Let $\sum_{s=1}^{3}\alpha^{(s)}\xi_3^{s-1}$ be the remainder term when we divide $B(x,t;i\xi)\hat{A}(x,t,w';i\xi,\nu)$ by M. Then after some lengthy calculations we have

(13) $$\det\alpha^{(3)}=-a_1^{-3}a_3^{-3}J_{11}^2J_{31}(\xi_3^{+(1)}+\xi_3^{+(3)})(a\xi_3^{+(1)}+\xi_3^{+(3)})\times$$

$$\times[(1-K)i\xi_3^{+(1)}-K][(1-K)i\xi_3^{+(3)}(\xi_3^{+(1)}+\xi_3^{+(3)})-K(a\xi_3^{+(1)}+\xi_3^{+(3)})],$$

where $a=a_3/a_1=\mu/(2\mu+\mu')$, $J_{pq}=\xi_3^{+(p)}-\xi_3^{-(q)}$.

Since $\mathrm{Im}\ \xi_3^{+(r)}>0$ $(r=1,2,3)$ follows from the assumption (12), it is obvious that $|\det\alpha^{(3)}|>0$. □

Moreover, we can extend the domain of definition (12) of $\det\alpha^{(3)}$ to $\{(\zeta'=\xi'+i\eta',q)\in\boldsymbol{C}^2\times\boldsymbol{C}^1 \mid \xi'^4+|q|^2>0,\ \mathrm{Re}\ q\geqq-\beta'|\mathrm{Im}\ q|,$ $|\eta'|\leqq\beta''(\xi'^4+|q|^2)^{1/2}\}$ for some positive constants β' and β'', so that there det $\alpha^{(3)}$ is estimated from below

$$(14)\quad |\det \alpha^{(3)}|(\zeta', q-\delta'\zeta'^2) \geqq$$

$$\geqq C_2(\xi'^4+|q|^2)^{\frac{3}{2}}[K+(1-K)(\xi'^4+|q|^2)^{\frac{1}{2}}]^2,$$

hence we have the estimates of the inverse matrix $(\alpha_3^{(j,k)})_{j,k=1,2,3}$ of $\alpha^{(3)}$:

$$(15)\quad |\alpha_3^{(j,k)}|(\zeta', q-\delta'\zeta'^2) \leqq$$

$$\leqq C_3(\xi'^4+|q|^2)^{-\frac{1}{2}} \begin{cases} 1 & (j=1,2,3;\ k=1), \\ [K+(1-K)(\xi'^4+|q|^2)^{\frac{1}{4}}]^{-1} & (j=1,2,3;\ k=2,3). \end{cases}$$

From these estimates, we can construct the Poisson kernel H_1 and the Green matrix H_0 in the half space $\boldsymbol{R}^3_+$:

$$\hat{H}_1(y,\nu) = (2\pi i)^{-1}\int_{\gamma_+} \hat{A}\,(x,t,w';iy',i\xi_3,\nu)\,\alpha_3(y',\xi_3,\nu)\times$$

$$\times e^{iy_3\xi_3}/M(y',\xi_3,\nu)\,d\xi_3,$$

$$H_1(y,\tau) = -i(2\pi)^{-3}\int_{\boldsymbol{R}^2} e^{i(y',\xi')}\,d\xi'\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\tau\nu}\hat{H}_1(\xi',y_3,\nu)\,d\nu$$

$$(\varepsilon > -\delta'\xi'^2),$$

$$H_0(y,\tau;\xi,\tau_0) = Z_0(y-\xi,\tau-\tau_0;x,t;w',\sigma') -$$

$$-\int_{\tau_0}^{\tau} d\tau' \int_{\boldsymbol{R}^2} H_1(y-\eta',\tau-\tau')\,B(x,t;\nabla_\eta)\cdot$$

$$\cdot Z_0(\eta-\xi,\tau'-\tau_0;x,t;w',\sigma')|_{\eta_3=0}\,d\eta'.$$

where γ_+ is a contour enclosing all $\xi_3^{+(r)}$ $(r=1,2,3)$ and Z_0 is the fundamental solution of the system of equations

$$\frac{\partial W}{\partial t}=A(x,\ t,\ w';\ \nabla_y)W.$$

Then, tracing the proof of Lemma 3.14 in [7], we obtain

<u>Lemma 3</u>

$$|D_\tau^r D_y^s (H_1)_{jk}| \leq C_4 \tau^{-(2r+|s|+4)/2} \exp\left[-d\frac{|y|^2}{\tau}\right] \begin{cases} [K+(1-K)\tau^{-\frac{1}{2}}]^{-1} \\ \quad (j=1,2,3;\ k=1), \\ 1 \quad (j=1,2,3;\ k=2,3), \end{cases}$$

$$|D_\tau^r D_y^s H_0| \leq C_4(\tau-\tau_0)^{-(2r+|s|+3)/2} \exp\left[-d\frac{|y-\xi|^2}{\tau-\tau_0}\right].$$

In the present problem just unlike the previous one [10], it is necessary to introduce two systems of covering $\{\omega_k(t)\}$ and $\{\Omega_k(t)\}$ of $\bar{\Omega}$ depending on the time variable t.

Let λ be an arbitrary small positive number. We construct $\{\omega_k(t)\}$ and $\{\Omega_k(t)\}$ as follows (cf. [7]):

(i) $\omega_k(t) \subset \Omega_k(t) \subset \bar{\Omega}$, $\cup_k \omega_k(t) = \cup_k \Omega_k(t) = \bar{\Omega}$;

(ii) for any $x \in \bar{\Omega}$, there exists $\omega_k(t)$ such that $x \in \omega_k(t)$ and $\mathrm{dist}(x, \bar{\Omega}-\omega_k(t)) \geq \beta_1 \lambda$ for some $\beta_1 > 0$;

(iii) for any $\lambda > 0$, there exists a number N_0 independent of λ such that

$$\bigcap_{k=1}^{N_0+1} \Omega_k(t) = \phi;$$

(iv-1) if $\Omega_k(t) \cap \Gamma = \phi$ (in this case, we shall denote $k=k'$), then $\omega_{k'}(t)$ and $\Omega_{k'}(t)$ are the cubes with the same center and with the length of their edges, in a parallel direction with axes, equal to $\lambda/2$ and λ, respectively (indeed, $\Omega_{k'}(t)$ and

$\omega_{k'}(t)$ do not depend on t)

(iv-2) if $\omega_k(t)\cap\Gamma\neq\phi$, then we construct $\omega_k(t)$ and $\Omega_k(t)$ by means of the local rectangular coordinate system $\{y\}$ with the origin at some point $\xi_k\in\Gamma$, *i.e.*, we take the inner normal to Γ at $\xi_k\in\Gamma$ as the y_3-axis and place the y_1-, y_2-axis in the tangential plane at ξ_k. Let $\gamma(t)=\{x\in\Gamma \mid K(x,t)=1\}$.

For $\xi_k\in\Gamma-\gamma(t)$ (in this case, let us denote $k=k'$), we define by the local rectangular coordinate system $\{y\}$

$$(16)\begin{cases}\omega_{k''}(t)=\Pi_x^y\{|y_j|\leq\frac{1}{2}\beta_2\lambda\,(j=1,2),\ 0\leq y_3-F(y';\xi_{k''})\leq\beta_2\lambda\},\\ \\ \Omega_{k''}(t)=\Pi_x^y\{|y_j|\leq\beta_2\lambda\,(j=1,2),\ 0\leq y_3-F(y';\xi_{k''})\leq 2\beta_2\lambda\},\end{cases}$$

where the equation $y_3=F(y';\xi_{k''})$ $(y'=(y_1,y_2))$ represent the boundary Γ in the neighborhood of the point $\xi_{k''}$ and β_2 is a positive constant independent of λ. If $\gamma(t)$ is covered by $\cup_{k''}(\omega_{k''}(t)\cap\Gamma)$, then it is clear that $\overline{\Omega}$ is covered by $\{\omega_k(t)\}$ and $\{\Omega_k(t)\}$ constructed above.

Otherwise (in this case, we shall denote $k=k'''$), we define $\omega_{k'''}(t)$ and $\Omega_{k'''}(t)$ by the same way as (16) with anoter positive constant $\beta_3\,(\leq\beta_2)$ also independent of λ so that $\gamma(t)-\cup_{k''}(\omega_{k''}(t)\cap\Gamma)\subset\cup_{k'''}(\Omega_{k'''}(t)\cap\Gamma)\subset\gamma(t)$.

Now we introduce two families of smooth functions $\{\zeta_k(x)\}$ and $\{\eta_k(x)\}$ associated with the coverings $\{\omega_k(t)\}$, $\{\Omega_k(t)\}$:

$$\zeta_k(x)=\begin{cases}1 & \text{if } x\in\omega_k(t),\\ \\ 0 & \text{if } x\in\overline{\Omega}-\Omega_k(t),\end{cases}\qquad 0\leq\zeta_k(x)\leq 1,$$

$$|D_x^s\zeta_k(x)|\leq C_5\lambda^{-|s|},\qquad \eta_k(x)=\zeta_k(x)/\sum_k\zeta_k(x)^2.$$

Then similarly to [7,8], the regularizer R of the problem

$$(10)_{\tau,\tau+h}\quad \begin{cases} \dfrac{\partial w}{\partial t}=A(x,t,w';\nabla)w+\Phi & \text{in } Q_{\tau,\tau+h}\equiv\Omega\times(\tau,\tau+h), \\ w|_{t=\tau}=0 & \text{on } \Omega, \\ B(x,t;\nabla)w=\varphi & \text{on } \Gamma_{\tau,\tau+h}\equiv\Gamma\times(\tau,\tau+h) \end{cases}$$

($\forall\tau\geq 0$, $0<\forall h\leq T-\tau$) can be constructed and has the following properties.

Lemma 4. Assume that $\Gamma\in C^{2+\alpha}$ and $h=\chi\lambda^2$ ($\chi(>0)$ and λ are sufficiently small). Then $R_k\Phi\in\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x,t}(Q^{(k)}_{\tau,\tau+h}\equiv\Omega_k\times[\tau,\tau+h])$ provided $\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}(\bar{Q}_{\tau,\tau+h}\equiv\bar{\Omega}\times[\tau,\tau+h])$. Furthermore the following estimates hold:

$$|D_t^r D_x^s R_k\Phi|\leq C_6(t-\tau)^{(2-2r-|s|+\alpha)/2}\|\Phi\|^{(\alpha)}_{\bar{Q}_{\tau,\tau+h}}\qquad(2r+|s|\leq 2),$$

$$|\Delta_x^{x'} D_t^r D_x^s R_k\Phi|\leq C_6|x-x'|^{\alpha}\|\Phi\|^{(\alpha)}_{\bar{Q}_{\tau,\tau+h}}\qquad(2r+|s|=2),$$

$$|\Delta_t^{t'} D_t^r D_x^s R_k\Phi|\leq C_6|t-t'|^{(2-2r-|s|+\alpha)/2}\|\Phi\|^{(\alpha)}_{\bar{Q}_{\tau,\tau+h}}\qquad(0<2r+|s|\leq 2),$$

where

$$R_{k'}\Phi=\int_\tau^t d\tau'\int_{\Omega_{k'}} Z_0(x-\bar{x},t-\tau;\xi_{k'},\tau;w',\sigma')\times$$

$$\times\zeta_{k'}(\bar{x},\tau)\Phi(\bar{x},\tau')\,d\bar{x},$$

$$R_{k''}\Phi = \Pi^z_x \bar{R}_{k''}\Phi, \qquad R_{k'''}\Phi = \Pi^z_x \bar{R}_{k'''}\Phi,$$

$$\bar{R}_{k''}\Phi = \int_\tau^t d\tau' \int_{K_1} H_0^{(k'')}(y, t; z, \tau')\bar{\zeta}_{k''}(z)\bar{\Phi}(z, \tau')\,dz,$$

$$\bar{R}_{k'''}\Phi = \int_\tau^t d\tau' \int_{K_2} H_0^{(k''')}(y, t; z, \tau')\bar{\zeta}_{k'''}(z)\bar{\Phi}(z, \tau')\,dz,$$

$$(\bar{\zeta}_{k''}(z) = \Pi^x_z \zeta_{k''}(x), \quad \bar{\Phi}(z, \tau) = \Pi^x_z \Phi(x, \tau), \quad K_1 = \Pi^x_z \Omega_{k''},$$

$$K_2 = \Pi^x_z \Omega_{k'''}, \quad \Delta^{x', t'}_{x, t} g(x, t) = g(x, t) - g(x', t'), \quad \Delta^{x'}_x = \Delta^{x', t}_{x, t},$$

$$\Delta^{t'}_t = \Delta^{x, t'}_{x, t}, \quad z_j = y_j\ (j = 1, 2), \quad z_3 = y_3 - F(y'; \xi_{k''}))$$

$H_0^{(k'')}$ and $H_0^{(k''')}$ are the Green matrix for $A = \Pi^x_y A(\xi_{k''}, \tau, w'; \nabla_x)$ and $A = \Pi^x_y A(\xi_{k'''}, \tau, w'; \nabla_x)$ with ∇_y replaced by ∇_z in $(10)_{\tau, \tau+h}$, respectively.

Lemma 5. Under the same assumptions as those in Lemma 4, $R'_{k''}\varphi \in \overset{\circ}{C}{}^{2+\alpha, 1+\alpha/2}_{x, t}(Q^{(k'')}_{\tau, \tau+h})$ if $\varphi \in \overset{\circ}{C}{}^{1+\alpha, (1+\alpha)/2}_{x, t}(\Gamma^{(k'')}_{\tau, \tau+h} = (\Gamma \cap \Omega_{k''}) \times [\tau, \tau+h])$ and satisfies the estimates

$$|D^r_t D^s_x R'_{k''}\varphi| \le C_7 (t-\tau)^{(2-2r-|s|+\alpha)/2} \|\varphi\|^{(1+\alpha)}_{(k'')\tau, \tau+h} \quad (2r + |s| \le 2),$$

$$|\Delta^{x'}_x D^r_t D^s_x R'_{k''}\varphi| \le C_7 |x - x'|^\alpha \|\varphi\|^{(1+\alpha)}_{(k'')\tau, \tau+h} \quad (2r + |s| = 2),$$

$$|\Delta^{t'}_t D^r_t D^s_x R'_{k''}\varphi| \le C_7 |t - t'|^{(2-2r-|s|+\alpha)/2} \|\varphi\|^{(1+\alpha)}_{(k'')\tau, \tau+h} \quad (0 < 2r + |s| \le 2),$$

where

$$R'_{k''}\varphi = \Pi^z_x \bar{R}'_{k''}\varphi, \quad K'_1 = \Pi^z_x(\Omega_{k''}\cap\Gamma),$$

$$\bar{R}'_{k''}\varphi = \int_\tau^t d\tau' \int_{K'_1} H_1^{(k'')}(z-\bar{z}', t-\tau')\zeta_{k''}(\bar{z}')\bar{\varphi}(\bar{z}', \tau')d\bar{z}',$$

$\|\cdot\|^{(n+\alpha)}_{(k'')\tau,\tau+h}$ means the norm of the space $C^{n+\alpha,(n+\alpha)/2}_{x,t}(\Gamma^{(k'')}_{\tau,\tau+h})$

and $H_1^{(k'')}$ is the Poisson kernel for $A = \Pi^x_y A(\xi_{k''}, \tau, w'; \nabla_x)$ and $B = \Pi^x_y B(\xi_{k''}, \tau; \nabla_x)$ with ∇_y replaced by ∇_z in $(10)_{\tau,\tau+h}$.

The similar assertions to those in Lemma 5 are true in the case $k = k'''$.

Lemma 6. Under the same assumptions as those in Lemma 4, $R'_{k'''}\varphi \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x,t}(Q^{(k''')}_{\tau,\tau+h})$ if $\varphi \in \overset{\circ}{C}{}^{2+\alpha,(2+\alpha)/2}_{x,t}(\Gamma^{(k''')}_{\tau,\tau+h} \equiv (\Gamma\cap\Omega_{k'''})\times[\tau,\tau+h])$ **and satisfies the estiimates**

$$|D^r_t D^s_x R'_{k'''}\varphi| \leq C_7(t-\tau)^{(2-2r-|s|+\alpha)/2}\|\varphi\|^{(2+\alpha)}_{(k''')\tau,\tau+h} \quad (2r+|s|\leq 2),$$

$$|\Delta^{x'}_x D^r_t D^s_x R'_{k'''}\varphi| \leq C_7|x-x'|^\alpha\|\varphi\|^{(2+\alpha)}_{(k''')\tau,\tau+h} \quad (2r+|s|=2),$$

$$|\Delta^{t'}_t D^r_t D^s_x R'_{k'''}\varphi| \leq C_7|t-t'|^{(2-2r-|s|+\alpha)/2}\|\varphi\|^{(2+\alpha)}_{(k''')\tau,\tau+h} \quad (0<2r+|s|\leq 2),$$

where

$$R'_{k'''}\varphi = \Pi^z_x \bar{R}'_{k'''}\varphi, \quad K'_2 = \Pi^z_x(\Omega_{k'''}\cap\Gamma),$$

$$\bar{R}'_{k'''}\varphi = \int_\tau^t d\tau' \int_{K'_2} H_1^{(k''')}(z-\bar{z}', t-\tau')\zeta_{k'''}(\bar{z}')\bar{\varphi}(\bar{z}', \tau')d\bar{z}',$$

$\|\cdot\|^{(n+\alpha)}_{(k^{m})\tau,\tau+h}$ means the norm of the space $C^{n+\alpha,(n+\alpha)/2}_{x,t}(\Gamma^{(k^{m})}_{\tau,\tau+h})$ and $H^{(k^{m})}_{1}$ is the Poisson kernel for $A=\Pi^{x}_{y}A(\xi_{k^{m}},\tau,w';\nabla_{x})$ and $B=\Pi^{x}_{y}B(\xi_{k^{m}},\tau;\nabla_{x})$ with ∇_{y} replaced by ∇_{z} in $(10)_{\tau,\tau+h}$.

These lemmas and the same arguments as those in [7,8] yield the following theorem:

Theorem 7. Suppose that $\Gamma\in C^{2+\alpha}$, $\Phi\in C^{\alpha,\alpha/2}_{x,t}(\bar{Q}_{T})$, $\varphi=(\varphi_{1},\varphi_{2},\varphi_{3})$, $\varphi_{1}\in C^{2+\alpha,1+\alpha/2}_{x,t}(\Gamma_{T})$, $\varphi_{2},\varphi_{3}\in C^{1+\alpha,(1+\alpha)/2}_{x,t}(\Gamma_{T})$, $\varphi_{2},\varphi_{3}\in C^{2+\alpha,1+\alpha/2}_{x,t}(\gamma_{T})$, $\gamma_{T}=\cup_{0\leq t\leq T}\{x\in\Gamma\,|\,K(x,t)=1\}$.

Then there exists a unique solution $w\in C^{2+\alpha,1+\alpha/2}_{x,t}(\bar{Q}_{T})$ of $(10)_{0,T}$ which satisfies

$$|D^{r}_{t}D^{s}_{x}w|\leq(C_{9}+C_{10}M_{2})^{N_{1}}\,t^{(2-2r-|s|+\alpha)/2}\{\|\Phi\|^{(\alpha)}_{Q_{T}}+\|\varphi_{1}\|^{(2+\alpha)}_{\Gamma_{T}}+$$

$$+\|(\varphi_{2},\varphi_{3})\|^{(1+\alpha)}_{\Gamma_{T}}+\|(\varphi_{2},\varphi_{3})\|^{(2+\alpha)}_{\gamma_{T}}\}_{A}\quad(2r+|s|\leq2),$$

$$|\Delta^{x'}_{x}D^{r}_{t}D^{s}_{x}w|\leq(C_{9}+C_{10}M_{2})^{N_{1}}|x-x'|^{\alpha}\{\cdots\}_{A}\quad(2r+|s|=2),$$

$$|\Delta^{t'}_{t}D^{r}_{t}D^{s}_{x}w|\leq(C_{9}+C_{10}M_{2})^{N_{1}}|t-t'|^{(2-2r-|s|+\alpha)/2}\{\cdots\}_{A}$$
$$(0<2r+|s|\leq2),$$

where $C_{9}=C_{9}(T,M_{1})(\geq1)$ and $C_{10}=C_{10}(T,M_{1})$ increase monotonically in T and M_{1}, $C_{10}\to0$ as $T\to0$ and $N_{1}=N_{1}(T,M_{1},M_{2})$ increases monotonically in T, M_{1} and M_{2}.

Returning to the problem (10), it is clear that

$\varphi = -B(x, t; \nabla) v_0$ implies that $\varphi_1 = 0$,

$$\|(\varphi_2, \varphi_3)\|_{\Gamma_T}^{(1+\alpha)}, \quad \|(\varphi_2, \varphi_3)\|_{\gamma_T}^{(2+\alpha)} \leqq C_{11}.$$

From (6), (7) and (11) it follows that

$$\|\rho\|_{\bar{Q}_T}^{(1+\alpha)} \leqq C_{12}(T, M_1) + C_{13}(T, M_1) M_2,$$

hence

$$\|\Phi\|_{\bar{Q}_T}^{(\alpha)} \leqq C_{12}(T, M_1) + C_{13}(T, M_1) M_2,$$

where $C_{12}(\geqq 1)$ and C_{13} have the same properties as C_9 and C_{10} respectively.

Therefore we obtain

$$(17) \begin{cases} \|w\|_{\bar{Q}_T}^{(2)} \leqq [C_9(T, M_1) + C_{10}(T, M_1) M_2]^{N_1(T, M_1, M_2)} (T^{\alpha} + T^{1+\alpha/2}) \times \\ \qquad \times [C_{11} + C_{12}(T, M_1) + C_{13}(T, M_1) M_2], \\ \Sigma_{|s|=2} |D_x^s w|_{x, \bar{Q}_T}^{(\alpha)} \leqq [C_9(T, M_1) + C_{10}(T, M_1) M_2]^{N_1(T, M_1, M_2)} \times \\ \qquad \times [C_{11} + C_{12}(T, M_1) + C_{13}(T, M_1) M_2]. \end{cases}$$

Next let us consider the following linearized problem of (9):

$$(18) \begin{cases} \dfrac{\partial \sigma}{\partial t} = A'(x, t, w', \sigma') \Delta \sigma + \Psi(x, t, w', \sigma') & \text{in } Q_T, \\ \sigma|_{t=0} = 0 & \text{on } \Omega, \\ (1-\kappa_e) \nabla \sigma \cdot n - \kappa_e \sigma = \psi(x, t) & \text{on } \Gamma_T. \end{cases}$$

Here $(w', \sigma') \in \mathscr{S}_T$.

The similar, but easier, arguments to those for (10) yield

Theorem 8. Suppose that $\Gamma \in C^{2+\alpha}$, $\Psi \in C^{\alpha, \alpha/2}_{x,t}(\bar{Q}_T)$, $\psi \in$ $\in C^{1+\alpha, (1+\alpha)/2}_{x,t}(\Gamma_T)$ and moreover $\psi \in C^{2+\alpha, 1+\alpha/2}_{x,t}(\Gamma'_T)$ (For Γ'_T, see Theorem in §1).

Then there exists a unique solution $\sigma \in C^{2+\alpha, 1+\alpha/2}_{x,t}(\bar{Q}_T)$ of (18) which satisfies

$$|D_t^r D_x^s \sigma| \leq (C_{14}+C_{15}M_2)^{N_2} t^{(2-2r-|s|+\alpha)/2} \{\|\Psi\|^{(\alpha)}_{\bar{Q}_T} + \\ +\|\psi\|^{(1+\alpha)}_{\Gamma_T} + \|\psi\|^{(2+\alpha)}_{\Gamma'_T}\}_B \quad (2r+|s|\leq 2),$$

$$|\Delta_x^{x'} D_t^r D_x^s \sigma| \leq (C_{14}+C_{15}M_2)^{N_2} |x-x'|^{\alpha} \{\cdots\}_B \quad (2r+|s|=2),$$

$$|\Delta_t^{t'} D_t^r D_x^s \sigma| \leq (C_{14}+C_{15}M_2)^{N_2} |t-t'|^{(2-2r-|s|+\alpha)/2} \{\cdots\}_B \quad (0<2r+|s|\leq 2),$$

where $C_{14}=C_{14}(T, M_1)(\geq 1)$, $C_{15}=C_{15}(T, M_1)$ and $N_2=N_2(T, M_1, M_2)$ have the same properties of C_9, C_{10} and N_1, respectively.

Therefore we obtain

$$\left\{ \|\sigma\|^{(2)}_{\bar{Q}_T} \leq [C_{14}(T, M_1)+C_{15}(T, M_1)M_2]^{N_2(T, M_1, M_2)}(T^{\alpha}+T^{1+\alpha/2}) \times \\ \times [C_{16}+C_{17}(T, M_1)+C_{18}(T, M_1)M_2], \right.$$

$$(19)\left\{ \Sigma_{|s|=2} |D_x^2 \sigma|_{x,Q_T}^{(\alpha)} \le [C_{14}(T,M_1)+C_{15}(T,M_1)M_2]^{N_2(T,M_1,M_2)} \times \right.$$
$$\times [C_{16}+C_{17}(T,M_1)+C_{18}(T,M_1)M_2],$$

where C_{17} and C_{18} have the same properties as C_9 and C_{10} respectively.

From the estimates (17) and (19) we conclude that the solutions w and σ of (10) and of (18) belong to $\mathscr{S}_{T_0}$ for some $T_0 \in (0,T]$.

Indeed, it is sufficient to choose a constant M_2 so as to be larger than

$$[C_9(T,M_1)+M]^{N_1(T,M_1,M)}[C_{11}+C_{12}(T,M_1)+M]+$$
$$+[C_{14}(T,M_1)+M]^{N_2(T,M_1,M)}[C_{16}+C_{17}(T,M_1)+M]$$

for any positive number M, and then $T_0 \in (0,T]$ such that

$$\{[C_9(T_0,M_1)+M]^{N_1(T_0,M_1,M)}[C_{11}+C_{12}(T_0,M_1)+M]+$$
$$+[C_{14}(T_0,M_1)+M]^{N_2(T_0,M_1,M)}[C_{16}+C_{17}(T_0,M_1)+M]\}(T_0^{\alpha}+T_0^{1+\alpha/2})\le$$
$$\le M_1,$$

$$C_{10}(T_0,M_1)M_2,\quad C_{13}(T_0,M_1)M_2,\quad C_{15}(T_0,M_1)M_2,$$
$$C_{18}(T_0,M_1)M_2 \le M.$$

For simplicity, we take $T=T_0$ from the beginning.

2.2 Nonlinear problem (8) and (9)

We construct the sequence $\{(w_n, \sigma_n)(x, t)\}$ of the successive approximate solutions as follows

$$\begin{cases} (w_0, \sigma_0) \equiv 0 \in \mathscr{S}_T, \\ w_n \text{ and } \sigma_n \text{ are defined as the solutions } w \text{ and } \sigma \text{ of (10)} \\ \text{of (18) assuming } (w', \sigma')=(w_{n-1}, \sigma_{n-1})\in\mathscr{S}_T, \text{ respectively.} \end{cases}$$

Then the results in §2.1 imply that $(w_n, \sigma_n)(x, t)$ uniquely exists and belongs to $\mathscr{S}_T$ $(n=0,1,2,\cdots)$.

Applying the estimates in §2.1 to the equations concerning $w_n - w_{n-1}$ and $\sigma_n - \sigma_{n-1}$ we obtain

$$\|(w_n, \sigma_n)-(w_{n-1}, \sigma_{n-1})\|_{Q_T}^{(2+\alpha)} \leq$$

$$\leq C_{19}(T, M_1, M_2)\|(w_{n-1}, \sigma_{n-1})-(w_{n-2}, \sigma_{n-2})\|_{Q_T}^{(2+\alpha)},$$

where $C_{19} \rightarrow 0$ as $T \rightarrow 0$.

Therefore the sequence $\{(w_n, \sigma_n)\}$ converges to some function (w, σ) uniformly if we take $T' \in (0, T]$ so as to satisfy $C_{19}(T', M_1, M_2)<1$.

The uniqueness of the solution to the problem (8) and (9) is proved by the fact that the difference of two solutions supposed to exist satisfy the inequality analogous to (20).

The positivity and the boundedness of ρ and θ are obvious from our method for constructing the solution.

Thererefore our main theorem has been proved.

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