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Fano Polytopes and Gorenstein Polytopes

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Introduction

Given a normal projective variety X over a field k with $H^0(X,\mathcal{O}_X)=k$ and an ample $\mathbb{Q}$-*divisor* D, namely rational coefficient Weil divisor, we consider a normal graded ring R(X,D) defined by $R(X,D)=\oplus_{n\geq 0}H^0(X,\mathcal{O}_X(nD))T^n$ (T an indeterminate), where by $\mathcal{O}_X(nD)$ we denote the sheaves $\Gamma(U,\mathcal{O}_X(nD))=\{f\in K(X);\ \mathrm{div}_U(f)+nD|_U\geq 0\}$ for each open set U of X. These rings have been introduced by Demazure [De], to show that a normal graded domain over a field k is obtained in this way. For a Cohen-Macaulay graded ring R(X,D), Watanabe [Wa] has given a necessary and sufficient condition for the Gorenstein property, in terms of an ample $\mathbb{Q}$-divisor D and a canonical divisor K_X on X. Therefore, it is natural to ask what kind of normal projective variety X has an ample $\mathbb{Q}$-divisor D such that R(X,D) is Gorenstein.

This problem for ample Cartier divisors D has been treated by Goto and Watanabe [GW] and it has been shown that R(X,D) is Gorenstein if and only if there exists an ample Cartier divisor D such that $H^i(X,\mathcal{O}_X(nD))=0$ for $0<i<\dim X$ and for all $n\in\mathbb{Z}$ and $\mathcal{O}_X(aD)$ is isomorphic to the canonical sheaf ω_X on X for some integer a. However, as far as I know, there is not much known as yet about the answer to the problem for ample $\mathbb{Q}$-divisors, beyond the criterion of Watanabe [Wa]. Our purpose here is to give an answer to this problem in the case that X are normal projective torus embeddings, by constructing R(X,D) explicitly for *T-stable* ample $\mathbb{Q}$-divisors D on X, namely ample $\mathbb{Q}$-divisors which are stable under the torus action.

The first main result is a technical one, whose precise statement is given in (1.5). Roughly speaking, for a projective torus embedding X and a T-stable ample $\mathbb{Q}$-divisor D, we construct R(X,D) as a numerical semigroup ring from the data of the fan and the support function associated to X and D. To this end, we firstly relate the pairs of r-dimensional projective torus embeddings and T-stable ample $\mathbb{Q}$-divisors on them with the r-dimensional rational convex polytopes P in $\mathbb{R}^r$, according to Oda [Od2, chapter2]. Consequently, when we define a graded ring R(P) over a field k for a rational convex polytope P of dimension r in $\mathbb{R}^r$ by $R(P)=\bigoplus_{n\geq 0}(\sum k e(m))T^n$ ($m\in nP\cap\mathbb{Z}^r$, T an indeterminate), it turns out that there is a natural isomorphism from R(P) to a graded ring R(X(P),D(P)) for the T-stable ample $\mathbb{Q}$-divisor D(P) on the normal projective torus embedding X(P) associated to P.

This result provides us with some consequences, as well as the second main result, namely a vanishing theorem for T-stable ample $\mathbb{Q}$-divisors on normal projective torus embeddings (1.6), and an enumeration problem of integral points in rational convex polytopes (1.7).

The second main result is theorem (2.2), which is a criterion for a normal graded numerical semigroup ring R(P) to be Gorenstein, in terms of a rational polytope P or the projective torus embedding X(P) with the T-stable ample $\mathbb{Q}$-divisor D(P). As immediate consequences, we have two results which provide us with an answer to our problems:

Corollary 2.5. *Let* X *be a normal projective torus embedding. Then* *there exists an ample Cartier divisor* D *such that* R(X,D) *is Gorenstein if and only if the canonical sheaf* ω_X *on* X *is isomorphic to an invertible sheaf* $\mathcal{O}_X(-aD)$ *for some* $a \in \mathbb{N}$.

Theorem 2.6. *Every normal projective torus embedding* X *over a field* k *has a* T*-stable ample* $\mathbb{Q}$*-divisor* D *such that* R(X,D) *is Gorenstein.*

From two results above, for example, it turns out that minimal rational surfaces whose anticanonical divisors are not ample have ample $\mathbb{Q}$-divisors D such that R(X,D) are Gorenstein but do not have such ample (integral) divisors, because every minimal rational surface is a normal projective torus embedding (c.f.[Od1, Theorem 8.2]). But, in general, the situation for our problem would be still obscure.

In another direction, by theorem (2.2) together with a theorem of Stanley [St1, (4.4)], we recover results of Hibi [Hi1,2]. Our proof here makes clear why the condition (c2) in (2.2) is needed, in terms of Demazure's construction.

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§ 0. Preliminaries.

(0.1). $[a]$ denotes the greatest integer not greater than $a \in \mathbb{R}$. $\lceil a \rceil$ denotes $-[-a]$ for $a \in \mathbb{R}$.

(0.2). For notion of torus embeddings, we refer the reader to [Od2]. Let T be an r-dimensional algebraic torus over a field k. Let M,N be the group of characters and one-parameter subgroups, respectively. Set $M_{\mathbb{R}}=M\otimes_{\mathbb{Z}}\mathbb{R}$ and $N_{\mathbb{R}}=N\otimes_{\mathbb{Z}}\mathbb{R}$. Let $<\ ,\ >:M_{\mathbb{R}}\times N_{\mathbb{R}}\longrightarrow\mathbb{R}$ represent the natural non-degenerate pairing. For a complete fan Δ of N, $\Delta(i)$ denotes the i-dimensional cones in Δ. A one-dimensional cone $\rho\in\Delta(1)$ is generated by a unique integral primitive vector $n(\rho)$. We denote by $SF(N,\Delta)$ the additive group consisting of Δ-linear support functions (see [Od2,p66] for the definition). Set $SF(N,\Delta,\mathbb{Q})=SF(N,\Delta)\otimes_{\mathbb{Z}}\mathbb{Q}$. Its elements are also called Δ-linear support functions. Then we have two injections $M\longrightarrow SF(N,\Delta)$ sending m to $<m,\ >$, and $SF(N,\Delta)\longrightarrow\mathbb{Z}^{\Delta(1)}$ sending h to $(h(n(\rho)))$. Let X be a normal complete torus embedding $T_N emb(\Delta)$. By TDiv(X), TCDiv(X) and PDiv(X) we denote the groups of T-stable Weil divisors, T-stable Cartier divisors and principal divisors on X. The one-dimensional cones ρ in $\Delta(1)$ are in a one-to-one correspondence with the irreducible T-stable closed subvarieties $V(\rho)$ of codimension one in X. Therefore the map $\mathbb{Z}^{\Delta(1)}\longrightarrow TDiv(X)$ sending g to $D_g=-\sum_\rho g_\rho\cdot V(\rho)$ $(\rho\in\Delta(1))$ is a bijection, and induces two isomorphisms of groups, $SF(N,\Delta)\longrightarrow TDiv(X)$ and $M\longrightarrow PDiv(X)\cap TCDiv(X)$. As a result, we have two commutative diagrams:

$$\begin{array}{ccccc} M & \longrightarrow & SF(N,\Delta) & \longrightarrow & \mathbb{Z}^{\Delta(1)} \\ \downarrow & & \downarrow & & \downarrow \\ PDiv(X)\cap TCDiv(X) & \longrightarrow & TCDiv(X) & \longrightarrow & TDiv(X) \end{array} \qquad \begin{array}{ccc} SF(N,\Delta,\mathbb{Q}) & \longrightarrow & \mathbb{Q}^{\Delta(1)} \\ \downarrow & & \downarrow \\ TCDiv(X,\mathbb{Q}) & \longrightarrow & TDiv(X,\mathbb{Q}). \end{array}$$

§ 1. Rational Polytopes and Projective Torus Embeddings.

In the present section, we shall describe the relation between

$\mathbb{Q}$-divisors on a normal complete torus embedding and support functions. And we shall give a relationship between rational polytopes and normal projective torus embeddings with T-stable ample $\mathbb{Q}$-divisors.

Lemma 1.1. *Let* $X=T_N\mathrm{emb}(\Delta)$ *be an* r*-dimensional complete torus embedding over a field* k. *For* $g\in\mathbb{Q}^{\Delta(1)}$, *the set* $\square_g=\{m\in M_{\mathbb{R}};\langle m,n(\rho)\rangle\geq g_\rho$ for all $\rho\in\Delta(1)\}$ *is a (possibly empty) convex polytope in* $M_{\mathbb{R}}$. *The set* $H^0(X,\mathcal{O}_X(D_g))$ *of global sections of the divisorial* $\mathcal{O}_X$*-module* $\mathcal{O}_X(D_g)$ *is a finite dimensional* k*-vector space with* $\{e(m);m\in M\cap\square_g\}$ *as a basis. Moreover* $m\in\mathrm{int}(\square_g)$ *if and only if each coefficient* a_ρ *of a T-stable Weil divisor* $V(\rho)$ *in the* $\mathbb{Q}$*-divisor* $\mathrm{div}(e(m))+D_g=\sum_\rho a_\rho V(\rho)$ $(\rho\in\Delta(1))$ *is a positive rational number.*

Proof. The first part is the same as in the case of $g\in\mathbb{Z}^{\Delta(1)}$. Since $n(\rho)$ is a primitive vector and the pairing $\langle\ ,\ \rangle$ is non-degenerate, we have $\square_g\cap M=\square_{\lceil g\rceil}\cap M$, where $\lceil g\rceil$ denotes the integral vector $(\lceil g_1\rceil,\ldots,\lceil g_{\#(\Delta(1))}\rceil)$. On the other hand, we have $\mathcal{O}_X(D_g)=\mathcal{O}_X(D_{\lceil g\rceil})$ by definition. Hence we may assume that $g\in\mathbb{Z}^{\Delta(1)}$. In this case, the assertion follows from [TE,p41,theorem] (c.f.[Od2,lemma 2.3]). The rest is obvious. Q.E.D.

Recall that a Δ-linear support function $h\in SF(N,\Delta,\mathbb{Q})$ is said to be *strictly upper convex* with respect to Δ if h is upper convex, namely $h(n)+h(n')\leq h(n+n')$ for all $n,n'\in N_{\mathbb{R}}$, and Δ is the coarsest among the fans Δ' in N for which h is Δ'-linear.

Lemma 1.2. *Let* $X=T_N\mathrm{emb}(\Delta)$ *be an* r*-dimensional complete torus embedding over a field* k *and* $h\in SF(N,\Delta,\mathbb{Q})$. *Then* D_h is ample, *that is,*

bD_h *is an ample Cartier divisor for some positive integer* b, *if and only if* h *is strictly upper convex with respect to* Δ.

Proof. See [Od2,corollary 2.14]. Q.E.D.

Proposition 1.3. *Let* P *be a rational* r*-polytope in* $M_{\mathbb{R}}=\mathbb{R}^r$, *namely,* r*-dimensional convex polytope in* $M_{\mathbb{R}}$ *whose vertices have rational coordinates. Then there exists a unique finite complete fan* Δ_P *in* N *such that support function* $h_P:N_{\mathbb{R}}\longrightarrow\mathbb{R}$ *for* P *defined by* $h_P(n)=\inf\{<m,n>;m\in P\}$ $(n\in N_{\mathbb{R}})$ *is a strictly upper convex* Δ_P*-linear support function with respect to* Δ_P. *We denote the corresponding* r*-dimensional projective torus embedding* $T_N\mathrm{emb}(\Delta_P)$ *and the ample* T*-stable* $\mathbb{Q}$*-divisor* D_{h_P} *by* X(P) *and* D_{h_P}. *Conversely, every pair of a normal projective torus embedding and a* T*-stable ample* $\mathbb{Q}$*-divisor on it is obtained from a rational* r*-polytope in* $M_{\mathbb{R}}$ *in this way.*

Proof. The first part follows form [Od2, A.18 & A.19]. Then, by (1.2), D(P) is a T-stable ample $\mathbb{Q}$-divisor on X(P). Conversely, given a nomal projective torus embedding X with a T-stable ample $\mathbb{Q}$-divisor D, there exist a complete fan Δ and a Δ-linear support function h which is strictly upper convex with respect to Δ. Set $\square_h=\{u\in M_{\mathbb{R}};<u,n(\rho)>\geq h(n(\rho))$ for all $\rho\in\Delta(1)\}$. By the construction and [Od2, A.18 & A.19], we have $X=X(\square_h)$ and $D=D(\square_h)$. Q.E.D.

Remark 1.4. In (1.3), D(P) is a Cartier divisor if and only if P is integral. D(P) is a Weil divisor if and only if P is *facet-reticular*, that is, each supporting hyperplane carried by a facet (face of the maximal dimension) of P contains an element of M.

Proposition 1.5. *Let* P *be a rational* r-*polytope in* $M_{\mathbb{R}}$. *Then the graded semigroup ring* $R(P):=\bigoplus_{n\geq 0}\{\sum_m k e(m)\}T^n$ $(m\in nP\cap M)$ *over a field* k *is isomorphic to the graded ring* R(X(P),D(P)) *associated to the projective torus embedding* X(P) *over* k *and the ample* $\mathbb{Q}$-*divisor* D(P), *as graded k-algebras. Consequently,* Proj(R(P)) *is isomorphic to* X(P) *and the sheaf* $\mathcal{O}(n)=R(P)(n)^{\sim}$ *on* Proj(R(P)) *corresponds via this isomorphism to* $\mathcal{O}_{X(P)}(nD(P))$ *for all* $n\in\mathbb{Z}$.

Proof. Since $\square_{nh_P}=nP$ and $D(nP)=D_{nh_P}$ for all $n\in\mathbb{N}$, we have $H^0(X(P),\mathcal{O}_{X(P)}(nD(P)))=\sum_m k e(m)$ $(m\in nP\cap M)$ by (1.1). This implies $R(P)\simeq R(X(P),D(P))$. The rest follows from a standard argument in the theory of Demazure's construction (c.f.[Wa,(2.1)]). Q.E.D.

Corollary 1.6. *For an r-dimensional normal projective torus embedding* $X=T_N\mathrm{emb}(\Delta)$ *over a field* k *and a strict upper convex* Δ-*linear support function* $h\in SF(N,\Delta,\mathbb{Q})$ *with respect to* Δ, *we have:*

(a) $\dim_k H^0(X,\mathcal{O}_X(nD_h))=\begin{cases}\#(n\square_h\cap M) & \text{if } n\geq 0\\ 0 & \text{if } n<0;\end{cases}$

(b) $\dim_k H^i(X,\mathcal{O}_X(nD_h))=0$ *for* $0<i<r$ *and all* $n\in\mathbb{Z}$;

(c) $\dim_k H^r(X,\mathcal{O}_X(nD_h))=\begin{cases}0 & \text{if } n\geq 0\\ \#(\mathrm{int}(n\square_h)\cap M) & \text{if } n<0,\end{cases}$

where $\#(n\square_h\cap M)$ *is the number of the set* $n\square_h\cap M$ *of lattice points in the rational convex polytope* $n\square_h$, *and* $\mathrm{int}(n\square_h)$ *denotes the interior of the convex polytope* $n\square_h$.

Proof.(a):This follows from (1.1). (b):Since $R(X,D_h)$ is a normal numerical semigroup ring by (1.3) and (1.5), $R(X,D_h)$ is normal and Cohen-Macaulay by a theorem of Hochster [Ho]. Therefore, this follows from corollary (2.2) in [Wa]. (c):By Serre duality (c.f.[Wa,(2.7)]), we have $\mathrm{Hom}_k(H^r(X,\mathcal{O}_X(nD_h)),k)\simeq H^0(X,\mathcal{O}_X(-[nD_h]+K_X))$, where K_X denotes a canonical divisor on X. Since $K_X=-\sum_\rho V(\rho)$ $(\rho\in\Delta(1))$ and n_ρ is a primitive vector for each $\rho\in\Delta(1)$, this follows from (1.1). Q.E.D.

Remark 1.7. Let P be a rational r-polytope in $\mathbb{R}^r$ and $m=\min\{i\in\mathbb{N}; i>0$ and iP is integral$\}$. By (1.3), (1.5) and (1.6), we have $\#(nP\cap\mathbb{Z}^r)=\chi(X(P),\mathcal{O}_{X(P)}(nD(P)))$ for $n\geq 0$ and $\#(\mathrm{int}((-n)P)\cap\mathbb{Z}^r)=(-1)^r\chi\left(X(P),\mathcal{O}_{X(P)}(nD(P))\right)$ for $n<0$, where $\chi\left(X(P),\mathcal{O}_{X(P)}(nD(P))\right)$ denotes $\sum_{j=0}^{r}(-1)^j\dim_k H^j(X(P),\mathcal{O}(nD(P)))$. By a result due to Snapper and Kleiman, for every $n\in\mathbb{Z}$, there exists a polynomial $P_n(\lambda)$ with coefficients in $\mathbb{Q}$ such that $\chi\left(X(P),\mathcal{O}_{X(P)}((n+m\lambda)D(P))\right)=P_n(\lambda)$. Thus we recover the reciprocity theorem and see that Ehrhart quasi-polynomial is indeed a quasi-polynomial.

§ 2. Criteria for Gorenstein Property.

First, we prove the following lemma:

Lemma 2.1. *Let* Δ *be a complete fan in* N *and* h *be a strictly upper convex* Δ*-linear support function in* $SF(N,\Delta,\mathbb{Q})$ *with respect to* Δ. *Set* $\square_h=\{u\in M_{\mathbb{R}}; \langle u,n(\rho)\rangle\geq h(n(\rho))$ for each $\rho\in\Delta(1)\}$. *Suppose that* h *has negative values except at the origin or, equivalently,* $\square_h$ *contains the origin in its interior. Then the set of vertices of the polar* *convex polyhedral set* $(\square_h)^{o}:=\{v\in N_{\mathbb{R}};\langle u,v\rangle\geq -1$ for all $u\in\square_h\}$ *for* $\square_h$ *is* $\{-(1/h(n(\rho))n(\rho);\rho\in\Delta(1)\}$.

Proof. By [Od2,A.19], there exists a bijection from $\Delta(1)$ to the set $\mathcal{F}^{r-1}(\square_h)$ of $(r-1)$-dimensional faces of $\square_h$ sending $\rho\in\Delta(1)$ to $Q_\rho=\{u\in\square_h;\langle u,n(\rho)\rangle=h(n(\rho))\}$. Also, by [Od2,A.17], there exists a bijection from $\mathcal{F}^{r-1}(\square_h)$ to the set of vertices of $(\square_h)^{o}$ sending an $(r-1)$-dimensional face Q to $Q^*=\{v\in(\square_h)^{o};\langle u,v\rangle=-1$ for all $v\in Q\}$. Then we observe that $(Q_\rho)^*$ is $-(1/h(n(\rho)))n(\rho)$. Q.E.D.

Theorem 2.2. *For a rational* r*-polytope* P *in* $M_{\mathbb{R}}=\mathbb{R}^r$ *with* $M=\mathbb{Z}^r$ *and a positive integer* δ, *the following are equivalent:*

(a) *The semigroup ring* $R(P)=\bigoplus_{n\geq 0}\{\sum_m ke(m)\}T^n$ $(m\in nP\cap M)$ *over a field* k *is a Gorenstein ring with* $a(R(P))=-\delta$, where $a(R(P))$ is defined by $-\min\{m\in\mathbb{Z};(K_{R(P)})_m\neq 0$ for the canonical module $K_{R(P)}$ of $R(P)\}$. (For details concerning $a(\cdot)$, see [GW]).

(b) *The normal projective torus embedding* $X(P)=T_N\mathrm{emb}(\Delta_P)$ *over a field* k, *and the ample* $\mathbb{Q}$*-divisor* $D(P)=\sum_\rho(p_\rho/q_\rho)V(\rho)$ $(\rho\in\Delta_P(1)$, $q_\rho>0$, p_ρ *and* q_ρ *are coprime) satisfy the following:*

(b1) *There exist a positive integer* r_ρ *for each* $\rho\in\Delta_P(1)$ *and a character* $m\in M$ *such that* $\delta D(P)+\mathrm{div}(e(m))=\sum_\rho(1/r_\rho)V(\rho)$ $(\rho\in\Delta_P(1))$;

(b2) δ *and* q_ρ *are coprime for each* $\rho\in\Delta_P(1)$.

(c) P *satisfies the following:*

(c1) *There exists a character* $m\in M$ *such that the polar polyhedral set* $(\delta P-m)^{o}:=\{v\in N_{\mathbb{R}};\langle u,v\rangle\geq -1$ *for all* $u\in\delta P-m\}$ *for* $\delta P-m=\{\delta p-m\in M_{\mathbb{R}};p\in P\}$ *is an integral* r*-polytope, namely, an* r*-polytope whose vertices have integral coordinates;*

(c2) *The convex hull* $\tilde{P}$ *of the set* $P\times\{0\}\cup\{(0,\ldots,0,1/\delta)\}$ *in*

$M_{\mathbb{R}}\times\mathbb{R}$ *is facet-reticular* (*c.f.* (1.4)).

Proof. (a)<=>(b): By (1.5), R(P) is isomorphic to R(X(P),D(P)) and, therefore, R(X(P),D(P)) is Cohen-Macaulay by a theorem of Hochster[Ho]. Since a canonical divisor $K_{X(P)}$ on X(P) is $-\sum_{\rho}V(\rho)$ $(\rho\in\Delta_P(1))$ (see for example [TE,theorem 9,III.d]), it follows from a criterion of Watanabe [Wa,(2.9)] that R(P) is a Gorenstein ring with $a(R(P))=-\delta$ if and only if there exists a character $m\in M$ such that $\delta D(P)+\mathrm{div}(\mathbb{e}(m))=\sum_{\rho}(1/q_{\rho})\cdot V(\rho)$ $(\rho\in\Delta_P(1))$. Note that a semi-inveriant rational function $f\in K(X(P))^{*}$ corresponds to some character $m\in M$. We assume that (a) holds. By the preceding remark, we have the relation above and, therefore, (b1) holds. Rewriting the relation, we have $\mathrm{div}(\mathbb{e}(m))=\sum_{\rho}\{(1-\delta p_{\rho})/q_{\rho}\}V(\rho)$ $(\rho\in\Delta_P(1))$. Hence $(1-\delta p_{\rho})/q_{\rho}$ is an integer and, therefore, δ and q_{ρ} are coprime for each $\rho\in\Delta_P(1)$. Conversely, we assume that (b) holds. By the preceding remark, we claim that $r_{\rho}=q_{\rho}$ for each $\rho\in\Delta_P(1)$. Since r_{ρ} is a factor of q_{ρ}, $b_{\rho}:=(q_{\rho}/r_{\rho})$ is a positive integer. Then, by (b1), $(b_{\rho}-\delta p_{\rho})/(r_{\rho}b_P)$ is an integer and, therefore, b_{ρ} is a factor of δ or p_{ρ}. Hence we have $b_{\rho}=1$ for each $\rho\in\Delta_P(1)$, as required, because neither δ nor p_{ρ} has any common factor with q_{ρ}.

(b1)=>(c1): Set $g=\delta h_P-m\in SF(N,\Delta_P,\mathbb{Q})$. Since $D_g=\delta D(P)+\mathrm{div}(\mathbb{e}(m))$ and D_g is ample, g is strictly upper convex and $g(n(\rho))=-(1/r_{\rho})$ for each $\rho\in\Delta_P(1)$. Therefore, by (2.1), the set of vertices of the polar convex polyhedral set $(\square_g)^{O}$ is $\{-(1/g(n(\rho)))n(p);\rho\in\Delta_P(1)\}=\{r_{\rho}n(\rho);\rho\in\Delta_P(1)\}$. On the other hand, $\square_g=\delta P-m$ by definition. Therefore $(\delta P-m)^{O}$ is an integral convex polytope.

(c1)=>(b1): Set $g=\delta h_P-m\in SF(N,\Delta_P,\mathbb{Q})$. Since g is strictly upper convex with respect to Δ_P and $O\in\mathrm{int}(\delta P-m)$, it follows from (2.1) that the vertices set of $(\delta P-m)^{o}$ is $\{-(1/g(n(\rho)))n(p);\rho\in\Delta_P(1)\}$. Hence, by assumption, $-(1/g(n(\rho)))n(p)$ is an integral vector. Since $n(\rho)$ is a primitive vector and $g\in SF(N,\Delta_P,\mathbb{Q})$ is negative-valued, $r_\rho:=-1/(g(n(\rho)))$ is a positive integer for each $\rho\in\Delta_P(1)$ and $\delta D(P)+\mathrm{div}(e(m))=D_g=\sum_\rho(1/r_\rho)V(\rho)$.

(b2)<=>(c2): Since a supporting hyperplane carried by a facet of P corresponding to $\rho\in\Delta_P(1)$ is $H_\rho=\{u\in M_{\mathbb{R}};\langle u,n(\rho)\rangle=h_P(n(\rho))\}$, a supporting hyperplane carried by a facet of $\tilde{P}$ is of the form $\tilde{H}_\rho:=\{(u,x)\in M_{\mathbb{R}}\times\mathbb{R};\delta x+(1/h_P(n(\rho)))\langle u,n(\rho)\rangle=1\}$ or $\{(u,0)\in M_{\mathbb{R}}\times\mathbb{R}\}$. Since $h_P(n(\rho))=-(p_\rho/q_\rho)$ and $n(\rho)$ is a primitive vector, δ and q_ρ are coprime if and only if $\tilde{H}_\rho\cap M\times\mathbb{Z}$ is non-empty. Q.E.D.

Remark 2.3.1. Under the condition (b) in the theorem, suppose that there exist an integer $\delta'\leq\delta$, a character $m'\in M$ and a positive integer a_ρ for each $\rho\in\Delta_P(1)$ such that $\delta'D+\mathrm{div}(e(m'))=\sum_\rho(a_\rho/q_\rho)\cdot V(\rho)$ $(\rho\in\Delta_P(1))$. Then we have $\delta'=\delta$, $m'=m$ and $a_\rho=1$ for each $\rho\in\Delta_P(1)$. In other words, we have $\#(\mathbb{Z}^r\cap\mathrm{int}(nP))=0$ for each $0\leq n<\delta$ and $\#(\mathbb{Z}^r\cap\mathrm{int}(\delta P))=1$. In fact, we observe that $(\delta'-\delta)D+\mathrm{div}(e(m'-m))=\sum_\rho(a_\rho-1)/q_\rho\cdot V(\rho)$ $(\rho\in\Delta_P(1))$. But D is an ample $\mathbb{Q}$-Cartier divisor. So we have $\delta'=\delta$, $a_\rho=1$ for each $\rho\in\Delta_P(1)$, and $m'=m$.

Remark 2.3.2. Combining the equivalence between (a) and (c) in (2.2) and a theorem of Stanley [St1, theorem 4.4], we recover theorem of Hibi [Hi1,2]. Our proof makes clear why the condition (c2) in (2.2) is needed, in terms of Demazure's construction. Indeed, let R(X,D) be a Cohen-Macaulay graded ring obtained from a normal projective variety X and an ample $\mathbb{Q}$-divisor $D=\sum_V(p_V/q_V)V$ (V runs through irreducible subvarieties of codimension 1, $q_V>0$ and p_V,q_V are coprime for each V). Then it follows form [Wa,(2.9)] that R(X,D) is Gorenstein if the Veronese subring $R(X,D)^{(d)}$ of order d is Gorenstein for a integer d such that $a\equiv 0 \pmod d$ and that d and q_V are coprime for each V.

Corollary 2.4. *For a rational* r*-polytope* P *in* $M_{\mathbb{R}}=\mathbb{R}^r$ *with* $M=\mathbb{Z}^r$ *and an integer* δ, *the following are equivalent:*

(a) P *is integral and there exists a character* $m\in M$ *such that the polar polyhedral set* $(\delta P-m)^{\circ}$ *for* $\delta P-m$ *is an integral* r*-polytope;*

(b) *The* $\mathbb{Q}$*-divisor* D(P) *on the normal projective torus embedding* X(P) *over a field* k *is an ample Cartier divisor. And the invertible sheaf* $\mathcal{O}_X(-\delta D(P))$ *is isomorphic to the canonical sheaf* $\omega_{X(P)}$.

Proof. It follows from (1.4) and (2.2) that (a) holds if and only if D(P) is a Cartier divisor and there exists a character $m\in M$ such that $\delta D(P)+\mathrm{div}(e(m))=\sum_\rho V(\rho)$ $(\rho\in\Delta_P(1))$. Since a canonical divisor $K_{X(P)}$ on X(P) is $-\sum_\rho V(\rho)$ $(\rho\in\Delta_P(1))$, (a) is equivalent to (b). Q.E.D.

Since every Cartier divisor on a normal complete torus embedding is linearly equivalent to a T-stable Cartier divisor (c.f.[Od1, (6.1)]), we have:

Corollary 2.5. *Let* X *be a normal projective torus embedding. Then there exists an ample Cartier divisor* D *such that* R(X,D) *is Gorenstein if and only if the canonical sheaf* ω_X *on* X *is isomorphic to an invertible sheaf* $\mathcal{O}_X(-aD)$ *for some* $a\in\mathbb{N}$.

Theorem 2.6. *Every normal projective torus embedding* X *over a field* k *has a* T-*stable ample* $\mathbb{Q}$-*divisor* D *such that* R(X,D) *is Gorenstein.*

Proof. By assumption, we may assume that $X=T_N\text{emb}(\Delta)$ has a T-stable ample Cartier divisor D of the form $D=\sum_\rho a_\rho V(\rho)$, $a_\rho>0$ ($\rho\in\Delta(1)$). Set $c=\text{L.C.M.}\{a_\rho;\rho\in\Delta(1)\}$. By (1.3), we may assume that (X,(1/c)D) corresponds to a rational polytope P in $M_{\mathbb{R}}$. Then, by (1.5) and (2.2), R(X,(1/c)D) is a Gorenstein ring with a(R(X,(1/c)D)=-1, as required.

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