数理解析研究所講究録
第 737 巻 1990 年 50-64

Kazhdan-Lusztig Conjecture for Kac-Moody Lie Algebras.

TOSHIYUKI TANISAKI
(谷崎俊之)

College of General Education, Osaka University, Toyonaka 560, Japan.

The purpose of this note is to give a survey of the papers [**K1**], [**K2**], [**KT**] which prove the Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras (cf. also [**C1**]).

0. Introduction

0.1. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and W the Weyl group. For an element $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module with highest weight $\lambda - \rho$ and $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$, where ρ is the half of the sum of the positive roots.

Algebraic theory of these infinite dimensional highest weight modules was initiated by Verma [**V**], and several important works concerning them appeared in 70's. Bernstein-Gel'fand-Gel'fand [**BGG**] determined the condition for $L(\mu)$ to be a composition factor of $M(\lambda)$, and Jantzen [**J**] investigated properties of the multiplicities of $L(\mu)$ in $M(\lambda)$ (see the exposition [**T1**]).

Although Jantzen's method was very powerful and determined the mulitiplicities in many cases, a general multiplicity formula was not known until the remarkable paper of Kazhdan-Lusztig [**KL1**] appeared in 1979. Kazhdan-Lusztig defined certain polynomials $P_{y,w}(q)$ $(y, w \in W)$, called the Kazhdan-Lusztig polynomials, through a combinatorics in the Hecke algebra of the Weyl group, and proposed a conjectural multiplicity formula using these polynomials. They also give a geometric interpretation of these polynomials in terms of the intersection cohomologies of the Schubert varieties ([**KL2**]). This conjecture was proved in 1981 by Beilinson-Bernstein [**BeB**] and Brylinski-Kashiwara [**BK**] independently by relating the highest weight modules with the intersection cohomologies of the Schubert varieties via D-modules on the flag varieties (see the expositions [**Se**], [**T2**], [**T3**]). The precise statement is the following:

Theorem A ([BeB], [BK]). *If λ is a regular dominant integral weight, then*

(0.1)
$$\operatorname{ch} M(w\lambda) = \sum_{z \geqq w} P_{w,z}(1) \operatorname{ch} L(z\lambda).$$

Equivalently, if μ is a regular antidominant integral weight, then

(0.2)
$$\operatorname{ch} L(w\mu) = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch} M(y\mu).$$

Here w, y, z are elements of the Weyl group W, $\geqq$ is the Bruhat order, $\ell \colon W \to \mathbf{Z}_{\geqq 0}$ is the length function, and ch *denotes the characters.*

The equivalence of (0.1) and (0.2) follows from the inversion formula of the Kazhdan-Lusztig polynomials.

0.2. It is natural to ask whether the similar results hold for a Kac-Moody Lie algebra, which is an infinite dimensional analogue of the finite dimensional semisimple Lie algebra. We also have highest weight modules, Weyl groups, Hecke algebras and Kazhdan-Lusztig polynomials in this setting.

Algebraic theory of highest weight modules for Kac-Moody Lie algebras was developped by Kac-Kazhdan [**KK**] and Deodhar-Gabber-Kac [**DGK**]. Especially it was conjectured in [**DGK**] that the formula (0.1) holds for symmetrizable Kac-Moody Lie algebras. Note that in this setting the formulas (0.1) and (0.2) give different statements since dominant weights and antidominant weights are not conjugate under the action of the Weyl group. This conjecture, given in 1982, was proved by Kashiwara [**K2**], Kashiwara-Tanisaki [**KT**] and independently by Casian [**C1**] in 1989, i.e.

Theorem B ([K2], [KT], and [C1]). *The formula* (0.1) *holds for symmetrizable Kac-Moody Lie algebras.*

0.3. The proof goes along the line of the original finite-dimensional case. The algebraic part [**K2**] relates the highest weight modules with D-modules on the flag variety. This reduces the problem, via the Riemann-Hilbert correspondence, to the calculation of intersection cohomologies of the Schubert varieties, and it is done in the topological part [**KT**]. A geometric foundation of [**K2**] and [**KT**] is the scheme theoretic construction of the flag variety given by Kashiwara [**K1**]. A technical difficulty in these works comes from the fact that the flag variety is infinite dimensional, and hence some parts of [**K2**] and [**KT**] are devoted to developping sufficient theories of D-modules and Hodge modules on infinite dimensional varieties. Besides this the proof is more or less the same as the finite dimensional case except for some points in [**K2**]. The most important point we should mention here is the lack of the Beilinson-Bernstein correspondence in the Kac-Moody setting. Hence the proof in the algebraic part [**K2**] is similar to (but more comlicated than) the one in [**BK**] rather than the one in [**BeB**].

The proof of Casian [**C1**] is somewhat different. He uses an analogue of the *composite* of the localization functor and the de Rham functor. The localization functor relates highest weight modules with D-modules, and the de Rham functor relates D-modules with perverse sheaves. Hence D-modules are hidden and only the perverse sheaves appear in his paper.

0.4. Very recently we have received Casian's preprint [C2] whose main theorem is the formula (0.2) for affine Lie algebras.

1. Infinite dimensional schemes

1.1. We give some examples of infinite dimensional schemes.

(1) Infinite dimensional affine space $\mathbf{A}^\infty = \operatorname{Spec} \mathbf{C}[x_i | i \in \mathbf{N}] = \varprojlim \mathbf{A}^n$. The set of the closed points is given by $\mathbf{A}^\infty(\mathbf{C}) = \{(x_i)_{i\in\mathbf{N}} \,|\, x_i \in \mathbf{C}\}$.

(2) Infinite dimensional projective space $\mathbf{P}^\infty$. The set of the closed points is given by

$$\begin{aligned}\mathbf{P}^\infty(\mathbf{C}) =& \{\text{line in the vector space } \mathbf{C}^\infty = \varprojlim \mathbf{C}^n\} \\ =& \{(x_i)_{i\in\mathbf{N}} \,|\, x_i \neq 0 \text{ for some } i \in \mathbf{N}\}/\mathbf{C}^*,\end{aligned}$$

and the scheme structure is naturally defined so that $\mathbf{P}^\infty$ is covered by open subsets U_i for $i \in \mathbf{N}$ with $U_i(\mathbf{C}) = \{(x_i)_{i\in\mathbf{N}} \,|\, x_i \neq 0\}/\mathbf{C}^*$ and $U_i \simeq \mathbf{A}^\infty$. Note that $\mathbf{P}^\infty$ is not quasi-compact since the open covering $\mathbf{P}^\infty = \cup_{i\in\mathbf{N}} U_i$ does not have a finite subcovering.

(3) Infinite dimensional Grassmann variety. Let V be a $\mathbf{C}$-vector space and $\{V_i\}_{i\in\mathbf{Z}}$ a strictly decreasing sequence of subspaces of V satisfying the following conditions:

(a) $V = \cup_{i\in\mathbf{Z}} V_i$,

(b) $\cap_{i\in\mathbf{Z}} V_i = \{0\}$,

(c) $\dim V_i/V_{i+1} < \infty$ for any $i \in \mathbf{Z}$,

(d) $V \simeq \varprojlim V/V_i$.

We denote by $\mathcal{F}(V)$ the set of subspaces U of V such that $V_{-i} \supset U \supset V_i$ for a sufficiently large i. The set of the closed point of the Grassmann variety $\mathrm{Grass}(V)$ is given by:

$$(\mathrm{Grass}(V))(\mathbf{C}) = \{\text{subspace } W \subset V \mid V = W \oplus U \text{ for some } U \in \mathcal{F}(V)\},$$

and the scheme structure is naturally defined so that $\mathrm{Grass}(V)$ is covered by open subsets $\mathrm{Grass}_U(V)$ for $U \in \mathcal{F}(V)$ with $(\mathrm{Grass}_U(V))(\mathbf{C}) = \{W \in (\mathrm{Grass}(V))(\mathbf{C}) \mid V = W \oplus U\}$ and $\mathrm{Grass}_U(V) \simeq \mathbf{A}^\infty$. For $k \in \mathbf{Z}$ set $\mathcal{F}_k(V) = \{U \in \mathcal{F}(V) \mid \dim U/(U \cap V_0) - \dim V_0/(U \cap V_0) = k\}$. Then $\mathrm{Grass}^d(V) := \cup_{U \in \mathcal{F}_{-d}(V)} \mathrm{Grass}_U(V)$ is a connected component of $\mathrm{Grass}(V)$ for each $d \in \mathbf{Z}$.

1.2. We define certain classes of infinite dimensional schemes.

Definition. Let X be a scheme over $\mathbf{C}$.

(a) X is said to be *pro-smooth* if X is covered by open subsets U such that $U \simeq \varprojlim S_n$ for some projective system $\{S_n\}_{n \in \mathrm{N}}$ satisfying the following conditions:

(a1) S_n is quasi-compact and smooth over $\mathbf{C}$ for any n.

(a2) The morphism $S_n \to S_{n-1}$ is smooth and affine for any n.

(b) X is said to be *essentially finite dimensional* if X is covered by open subsets isomorphic to $Z \times \mathbf{A}^\infty$ for some finite dimensional varieties Z.

(c) X is said to be *essentially smooth* if X is pro-smooth and essentially finite dimensional.

It is shown that a $\mathbf{C}$-scheme is essentially smooth if and only if it is covered by open subsets isomorphic to $Z \times \mathbf{A}^\infty$ for some $\mathbf{C}$-smooth schemes Z. The infinite dimensional schemes given in Section 1.1 are all essentially smooth.

1.3. Some notions concerning finite dimensional varieties are extended to the above classes of infinite dimensional schemes. For example, holonomic D-modules are defined for pro-smooth schemes, corresponding analytic spaces and perverse sheaves are defined for essentially finite dimensional schemes, and Hodge modules are defined for essentially smooth schemes (see [**K2**], [**KT**]).

2. The flag manifolds

2.0. We give the scheme theoretic definition of the flag variety of a Kac-Moody Lie algebra (see [**K1**]).

2.1. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra, W the Weyl group, $h_i \in \mathfrak{h}$ $(i \in I)$ the simple coroots, $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ the simple roots, Δ the set of roots, and Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots. We denote the root space corresponding to $\alpha \in \Delta$ by $\mathfrak{g}_\alpha$. Set

$$\begin{aligned}
\mathfrak{n}^\pm &= \oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha,\\
\mathfrak{b}^\pm &= \mathfrak{n}^\pm \oplus \mathfrak{h},\\
\mathfrak{l}_i &= \mathfrak{h} + \mathfrak{g}_{\alpha_i} + \mathfrak{g}_{-\alpha_i} \quad (i \in I)\\
\mathfrak{n}_i^\pm &= \oplus_{\alpha\in\Delta^\pm-\{\pm\alpha_i\}}\mathfrak{g}_\alpha \quad (i \in I),\\
\mathfrak{p}_i^\pm &= \mathfrak{n}_i^\pm \oplus \mathfrak{l}_i \quad (i \in I).
\end{aligned}$$

2.2. We give some group schemes corresponding to subalgebras of $\mathfrak{g}$. Fix a $\mathbf{Z}$-lattice P in $\mathfrak{h}^*$ such that $\alpha_i \in P$ and $\lambda(h_i) \in \mathbf{Z}$ for any $\lambda \in P$. Let

$$\begin{aligned}
T &= \mathrm{Spec}(\mathbf{C}[P])\\
U^\pm &= \varprojlim_k \exp(\mathfrak{n}^\pm/(\mathrm{ad}\,\mathfrak{n}^\pm)^k\mathfrak{n}^\pm)\\
B^\pm &= (\text{the semidirect product of } T \text{ and } U^\pm),\\
L_i &= (\text{the group corresponding to } \mathfrak{l}_i \text{ with } \mathrm{Lie}(\mathfrak{h}) = T) \quad (i \in I),\\
U_i^\pm &= \varprojlim_k \exp(\mathfrak{n}_i{}^\pm/(\mathrm{ad}\,\mathfrak{n}^\pm)^k\mathfrak{n}_i{}^\pm) \quad (i \in I)\\
P_i^\pm &= (\text{the semidirect product of } L_i \text{ and } U_i^\pm) \quad (i \in I).
\end{aligned}$$

2.3. For $k \in \mathbf{Z}$ let

$$\mathfrak{g}_k = \begin{cases} \oplus_{\alpha\in\Delta^+,\mathrm{ht}(\alpha)\geqq k}\mathfrak{g}_\alpha & (k > 0)\\ (\oplus_{\alpha\in\Delta^-,\mathrm{ht}(\alpha)\leqq -k}\mathfrak{g}_\alpha) \oplus \mathfrak{b}^+ & (k \leqq 0), \end{cases}$$

where $\mathrm{ht}(\alpha) = \sum_{i\in I} m_i$ for $\alpha = \sum_{i\in I} m_i\alpha_i \in \Delta$. Set $\hat{\mathfrak{g}} = \varprojlim_j(\mathfrak{g}/\mathfrak{g}_j)$ and $\hat{\mathfrak{g}_k} = \varprojlim_j(\mathfrak{g}_k/\mathfrak{g}_j)$. Then $(\hat{\mathfrak{g}}, \{\hat{\mathfrak{g}_k}\}_{k\in\mathbf{Z}})$ satisfies the condition (a)–(d) in Section 1.1 (3), and we can consider the Grassmann variety $\mathrm{Grass}(\hat{\mathfrak{g}})$.

Note that the group schemes B^+ and P_i^+ naturally act on $\mathrm{Grass}(\hat{\mathfrak{g}})$. Let $x_0 \in \mathrm{Grass}(\hat{\mathfrak{g}})$ be the point corresponding to $\mathfrak{b}^- \subset \hat{\mathfrak{g}}$. Then the flag variety is defined by

$$X = \cup_{w\in W} wB^+x_0,$$

where the action of $w \in W$ is given by taking a reduced expression of w and choosing representatives of the simple reflections $s_i \in W$ in $L_i \subset P_i$.

Remark. There is another construction of the flag variety as the quotient scheme $X = G/B^-$. Here G is a scheme (not a group scheme) with locally free B^--action defined as the spectrum of a certain subring of the dual space of the enveloping algebra $U(\mathfrak{g})$ (note that the comultiplication $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defines a ring structure on $U(\mathfrak{g})^*$). See **[K1]** for the details.

2.4. Set $X_w = B^+wx_0$ for $w \in W$. As in the finite dimensional case we have the following properties of the Scubert cells X_w.

Proposition 1 ([K1]). (i) $X_w \simeq \mathbf{A}^\infty$.

(ii) X_w is an affine scheme with codimension $\ell(w)$ in X.

(iii) $X = \sqcup_{w\in W} X_w$.

(iv) $\overline{X}_w = \sqcup_{z\geqq w} X_z$.

Here $\ell(w)$ is the length of a reduced expression of w and $\geqq$ is the Bruhat order on W.

Especially, X is essentially smooth.

2.5. As an example we give a description of the flag variety of $\mathfrak{g} = A_{n-1}^{(1)}$.

Let $\mathbf{C}[[t]]$ be the formal power series ring and $\mathbf{C}((t))$ its quotient field. Let V be the $\mathbf{C}((t))$-vector space with basis $\{e_0, \ldots, e_{n-1}\}$ and set $V^+ = \oplus_{k=0}^{n-1}\mathbf{C}[[t]]e_k$. Let Y be the set of $\mathbf{C}[t^{-1}]$-submodules U of V so that the kernel and the cokernel of $U \to V/V^+$ are finite dimensional over $\mathbf{C}$. Set

$$Y_k = \{U \in Y \mid \dim \mathrm{Cok}(U \to V/V^+) - \dim \mathrm{Ker}(U \to V/V^+) = k\}$$

for $k \in \mathbf{Z}$. The flag varety X is given by

$$\begin{aligned}
&X(\mathbf{C}) \\
=&\{(U_k)_{k\in\mathbf{Z}} \mid U_k \in Y_k, U_k \supset U_{k+1}, U_{k+n} = t^{-1}U_k\} \\
\simeq&\{(U_0 \supset U_1 \supset \cdots \supset U_{n-1} \supset t^{-1}U_0 \mid U_0 \in Y_0 \text{ and} \\
&\qquad U_1 \ldots U_{n-1} \text{ are } \mathbf{C}\text{-subspaces of } U_0 \text{ satisfying } \dim(U_0/U_k) = k\}.
\end{aligned}$$

Identify W with the group consisting of the permutations σ of $\mathbf{Z}$ satisfying $\sigma(k+n) = \sigma(k) + n \quad (k \in \mathbf{Z})$ and $\sum_{k=0}^{n-1}(\sigma(k) - k) = 0$. Set $V_i = (\oplus_{k<i}\mathbf{C}[[t]]t^{-1}e_k) \oplus (\oplus_{k\geqq i}\mathbf{C}[[t]]e_k)$ for $i = 0, \ldots, n-1$. Then the Schubert cells are given by

$$\begin{aligned}
&X_\sigma(\mathbf{C}) \\
=&\{(U_k)_{k\in\mathbf{Z}} \in X(\mathbf{C}) \mid \dim(U_k \cap V_i) = \sharp((-\infty, i] \cap \sigma[k, \infty)) \quad (0 \leqq i \leqq n-1)\}.
\end{aligned}$$

3. Kazhdan-Lusztig conjecture

3.1. We recall basic facts concerning Verma modules. Let ρ be an element of $\mathfrak{h}^*$ such that $\rho(h_i) = 1$ for any $i \in I$. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module with highest weight $\lambda - \rho$, i.e.,

$$M(\lambda) = U(\mathfrak{g})/(U(\mathfrak{g})\mathfrak{n}^+ + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - (\lambda - \rho)(h))).$$

The $U(\mathfrak{g})$-module $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$, and other composition factors of $M(\lambda)$ are of the form $L(\mu)$ with $\lambda - \mu \in (\sum_{i \in I} \mathbf{Z}_{\geqq 0}\alpha_i) - \{0\}$. A main problem of the algebraic theory of highest weight modules is to detremine the multiplicities of $L(\mu)$ in $M(\lambda)$, and this is equivalent to giving a character formula of the form $\operatorname{ch} M(\lambda) = \sum_\mu a_{\lambda,\mu} \operatorname{ch} L(\mu)$ or $\operatorname{ch} L(\lambda) = \sum_\mu b_{\lambda,\mu} \operatorname{ch} M(\mu)$. The answer to this problem will be given below when $\mathfrak{g}$ is symmetrizable and λ is regular dominant integral.

3.2. In order to write down the character formula, we need to recall basic facts concerning the Hecke algebra.

The Hecke algebra $H(W)$ is a $\mathbf{Z}[q, q^{-1}]$-algebra with basis $\{T_w\}_{w \in W}$ satisfying the following relations:

$$(T_s + 1)(T_s - q) = 0 \qquad (s \in S),$$

$$T_{w_1}T_{w_2} = T_{w_1w_2} \qquad (\ell(w_1) + \ell(w_2) = \ell(W_1w_2)),$$

(see Iwahori [**I**], and [**Bou**]).

In [**KL1**] Kazhdan-Lusztig introduced a new basis $\{C_w^*\}_{w \in W}$ of $H(W)$. It is characterized by the following properties:

(a) $C_w^* = \sum_{y \leqq w} P_{y,w}(q)T_y = q^{\ell(w)} \sum_{y \leqq w} P_{y,w}(q^{-1})T_{y^{-1}}^{-1}$,

(b) $P_{y,w}(q) \in \mathbf{Z}[q]$ with $P_{w,w}(q) = 1$ and $\deg P_{y,w}(q) \leqq (\ell(w) - \ell(y) - 1)/2$ for $y < w$.

Define $Q_{y,w}(q) \in \mathbf{Z}[q]$ for $y \leqq w$ by

$$\sum_{y \leqq w \leqq z} (-1)^{\ell(w)-\ell(z)} Q_{y,w} P_{w,z} = \delta_{y,z} \qquad (y \leqq z).$$

The polynomials $P_{y,w}(q)$ (resp. $Q_{y,w}(q)$) are called the Kazhdan-Lusztig polynomials (resp. the inverse Kazhdan-Lusztig polynomials).

3.3. The main result of [**K2**] and [**KT**] is the following.

Thoerem. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. Then for any regular dominant integral weight $\lambda \in \mathfrak{h}^*$ we have:*

$$\operatorname{ch} M(w\lambda) = \sum_{z \geqq w} P_{w,z}(1) \operatorname{ch} L(z\lambda),$$

or equivalently

$$\operatorname{ch} L(w\lambda) = \sum_{z \geqq w} (-1)^{\ell(z)-\ell(w)} Q_{w,z}(1) \operatorname{ch} L(z\lambda).$$

We need the symmetrizability condition since the following fact is used in [**K2**].

Proposition 2 (Deodhar-Gabber-Kac [DGK]). *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, and $\lambda \in \mathfrak{h}^*$ a regular dominant integral weight. Then any composition factor of $M(w\lambda)$ is isomorphic to $L(z\lambda)$ for some $z \in W$.*

Proposition 2 is proved using the Casimir operator. I do not know whether the symmetrizability condition is really necessary in Theorem (or in Proposition 2).

3.4. We describe an outline of the proof of Theorem. For the sake of simplicity we assume that $\lambda = \rho$.

Let X be the flag variety as in Section 2, and let D_X be the sheaf of linear differential operators on X. We denote by $M(D_X, B^+)$ the abelian category of admissible holonomic D_X-modules with B^+-actions. An admissible holonomic D-module on an essentially smooth $\mathbf{C}$-scheme Y is a D_Y-module $\mathcal{M}$ satisfying the following condition:

(*) For any $y \in Y$ there exist an open neighborhood U of y, a smooth $\mathbf{C}$-scheme Z, and a holonomic D_Z-module $\mathcal{N}$ such that $U \simeq Z \times \mathbf{A}^\infty$ and $\mathcal{M} \simeq \mathcal{N} \boxtimes \mathcal{O}_{\mathbf{A}^\infty}$.

For $w \in W$ we define objects $\mathcal{B}_w$, $\mathcal{M}_w$, $\mathcal{L}_w$ of $M(D_X, B^+)$ as follows. $\mathcal{B}_w$ is the local cohomology sheaf $\mathcal{H}^{\ell(w)}_{X_w}(\mathcal{O}_X)$, $\mathcal{M}_w$ is the dual holonomic D_X-module $(\mathcal{H}^{\ell(w)}_{X_w}(\mathcal{O}_X))^*$, and $\mathcal{L}_w$ is the image of the natural homomorphism $\mathcal{M}_w \to \mathcal{B}_w$.

Let $M(\mathfrak{g}, B^+)$ be the abelian category of $U(\mathfrak{g})$-modules with B^+-actions. For $\mu \in \mathfrak{h}^*$ let $M^*(\mu)$ be the $\mathfrak{h}$-finite part of the dual of the Verma module with lowest weight $-\mu$. The character of $M^*(\mu)$ cincides with that of $M(\mu)$. For $\mu \in P$ the $U(\mathfrak{g})$-modules $M(\mu)$, $L(\mu)$, $M^*(\mu)$ are objects of $M(\mathfrak{g}, B^+)$.

Define an additive functor $\tilde{\Gamma}$: $M(D_X, B^+) \to M(\mathfrak{g}, B^+)$ by

$$\tilde{\Gamma}(X, \mathcal{M}) = \oplus_{\mu \in P}(\varprojlim_{\Omega}(\text{the weight space of } \Gamma(\Omega, \mathcal{M}) \text{ with weight } \mu)),$$

wehere Ω ranges over B^+-stable quasi-compact open subsets of X. Note that a B^+-stable open subset of X is quasi-compact if and only if it consists of finitely many X_w's.

Proposition 3 ([K2]). (i) $\tilde{\Gamma}$ *is an exact functor.*

(ii) $\tilde{\Gamma}(X, \mathcal{M}_w) = M(w\rho)$.

(iii) $\tilde{\Gamma}(X, \mathcal{B}_w) = M^*(w\rho)$.

(ii) $\tilde{\Gamma}(X, \mathcal{L}_w) = L(w\rho)$.

Therefore, in order to prove Theorem it is sufficient to show

$$[\mathcal{L}_w|\Omega] = \sum_{z \geqq w} (-1)^{\ell(z)-\ell(w)} Q_{w,z}(1)[\mathcal{B}_z|\Omega]$$

for any B^+-stable quasi-compact open subset Ω of X. Here $[\mathcal{L}_w|\Omega]$ and $[\mathcal{B}_z|\Omega]$ are elements of the Grothendieck group of the abelian category of B^+-equivariant holonomic D_Ω-modules. By the Riemann-Hilbert correspondence we have

$$\mathrm{Sol}(\mathcal{L}_w) = {}^\pi\mathbf{C}_{X_w}[-\ell(w)],$$
$$\mathrm{Sol}(\mathcal{B}_w) = \mathbf{C}_{X_w}[-\ell(w)],$$

and hence the proof is reduced to showing:

$$[{}^\pi\mathbf{C}_{X_w}[-\ell(w)]|\Omega] = \sum_{z \geqq w} (-1)^{\ell(z)-\ell(w)} Q_{w,z}(1)[\mathbf{C}_{X_z}[-\ell(z)]|\Omega]$$

in the Grothendieck group of the abelian category of B^+-equivarivant perverse sheaves on Ω. This follows from the following.

Proposition 4 ([KT]). *Let w, z be elements of W such that $z \geqq w$, and let $i\colon X_z \to \overline{X}_w$ be the inclusion. Set $Q_{w,z}(q) = \sum_j c_j q^j \quad (c_j \in \mathbf{Z})$.*

(i) $\mathcal{H}^{2j+1}(i^*({}^\pi\mathbf{C}_{X_w})) = 0$ *for any* $j \in \mathbf{Z}$.

(ii) *For any $j \in \mathbf{Z}$ we have $c_j \geqq 0$, and $\mathcal{H}^{2j}(i^*({}^\pi\mathbf{C}_{X_w})) \simeq \mathbf{C}_{X_z}^{\oplus c_j}$.*

The proof of Proposition 4 uses the theory of mixed Hodge modules (Saito [Sa]) as in [T4].

References

[**BeB**] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, Comptes Rendus **292** (1981), 15–18.

[**BGG**] J. Bernstein, I. M. Gel'fand, S. I. Gel'fand, *Structures of representations generated by vectors of highest weight*, Funct. Anal. App. **5** (1971), 1–8.

[**Bou**] N. Bourbaki, "Groupes et algébres de Lie," Ch. 4–6, Hermann, 1968.

[**BK**] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387-410.

[**C1**] L. Casian, *Proof of the Kazhdan-Lusztig conjecture for Kac-Moody algebras (Symmetrizable case)*, preprint.

[**C2**] L. Casian, *Kazhdan-Lusztig conjecture in the negative level case (Kac-Moody algebras of affine type)*, preprint.

[**DGK**] V. Deodhar, O. Gabber and V. Kac, *Structure of some categories of representations of infinite-dimensional Lie algebras*, Adv. Math. **45** (1982), 92–116.

[**I**] N. Iwahori, *On the structure of the Hecke ring of a Chevalley group over a finite field*, J. Fac. Sci. Univ. Tokyo **10** (1964), 215–236.

[**J**] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Springer LNM 750 (1979).

[**KK**] V. Kac and D. Kazhdan, *Structures of representations with highest weight of infinite dimensional Lie algebras*, Adv. Math. **45** (1982), 92–116.

[**K1**] M. Kashiwara, *The flag manifold of Kac-Moody Lie algebras*, in "Algebraic analysis, Geometry, and Number theory," Johns Hopkins Univ. Press, 1990.

[**K2**] M. Kashiwara, *Kazhdan-Lusztig conjecture for symmeytrizable Kac-Moody Lie algebra*, to be published in the volume in honor of A. Grothendieck's sixtieth birthday.

[**KT**] M. Kashiwara and T. Tanisaki, *Kazhdan-Lusztig conjecture for symmeytrizable Kac-Moody Lie algebra II*, to be published in the volume in honor of J.

Dixmier's sixty-fifth birthday.

[**KL1**] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[**KL2**] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, Proc. Symp. in Pure Math. **36** (1980), 185–203.

[**Sa**] M. Saito, *Mixed Hodge Modules*, Publ. RIMS, Kyoto Univ. **26** (1989).

[**Se**] J. Sekiguchi, "Applications of the theory of differential equations to the representation theory," (in Japanese) Sophia Univ. kokyuroku 27, (1988).

[**T1**] T. Tanisaki, *Highest weight modules of semisimple Lie algebras*, (in Japanese), in "Proc. Symp. on algebraic groups held at Kawatabi," 1980, pp. 45–77.

[**T2**] T. Tanisaki, *Representation theory of complex semisimple Lie algebras and D-modules*, (in Japanese), in "Proc. Symp. on algebra held at Kinosaki," 1982, pp. 67–163.

[**T3**] T. Tanisaki, *D-modules and group representations*, (in Japanese), in "RIMS kokyuroku 668," 1988, pp. 63–119.

[**T4**] T. Tanisaki, *Hodge modules, equivariant K-theory and Hecke algebras*, Publ. RIMS, Kyoto Univ. **23** (1987), 841–879.

[**V**] D. N. Verma, *Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc. **74** (1968), 160–166.