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Normal Forms of the Bifurcation Equations in the Problem of Capillary-Gravity Waves*

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Abstract. We analyze two dimensional capillary-gravity waves on surface of irrotational flow. Two sets of algebraic equations are proposed as normal forms. We show that these equations provide bifurcation diagrams, which are qualitatively the same as those obtained numerically. Thus we establish a rigorous basis for the numerical results in [**1,2,17,18**]. We also show that study of not only physically acceptable solutions but also self-intersecting unphysical solutions is necessary for complete understanding of the phenomena.

§1. Introduction.

The purpose of the present paper is to give a mathematical account of the bifurcation diagrams arising in the theory of progressive water waves of two dimensional irrotational flow. In this paper we consider nonlinear capillary-gravity waves, which was first analyzed by Wilton [**21**]. By capillary-gravity waves we mean that we take into account of both surface tension and gravity. There is a well-established (both physical and mathematical) theory on this problem ([**9,13-16,19-21**]). Recent researches, however, showed new phenomena in the capillary-gravity waves.

The problem is a bifurcation problem and it is of a primary importance to determine bifurcation diagrams, which are sometimes called solution diagrams. Among many references, Chen and Saffman [**1,2**], Schwartz and Vanden-Broeck [**17**] and the second author of the present paper [**18**] computed capillary-gravity waves numerically. In particular, [**18**] presented various bifurcation diagrams for various parameters. A mathematical analysis of the bifurcation was given by Pierson and Fife [**15**], Reeder and Shinbrot [**16**] and Toland and Jones [**9,20**]. These authors gave a rigorous mathematical background to the physical theory by Wilton [**21**] and others. Later [**13**] showed that some of the results in [**9,15,16,20**] can be derived more simply by making use of the results in Fujii et al. [**4,5**]. However, the numerical computation in [**18**] shows that there are considerably complicated structures in the bifurcation diagrams, which can not be found in the previous papers.

In this paper we give a normal form of the bifurcation equation, which turns out to be a set of polynomials in two independent variables with a bifurcation parameter. Our normal form is more degenerate than the bifurcation equations in [**9,20**]. By this introduction of degeneracy we can explain a semi-global structure of progressive waves in [**18**], which can not be found by the previous works. Actually, our task is to examine the agreement of the numerically computed diagrams in

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[18] and those given by abstract bifurcation equations in the present paper. Our normal forms of the bifurcation equations are derived by utilizing the theorems in Golubitsky et al. [6,7,8]

We emphasize a rather unexpected fact that we should take account of solutions in which the wave profiles have self-intersections. (We call such a solution a overlap solution or an unphysical solution.) In fact, in some bifurcation diagrams, some classes of non-overlap (thus physical) solutions are disconnected from the other classes in the phase space, if we take a look at only physical solutions. If, however, we consider both physical and unphysical solutions, all classes of the solutions are *connected* . The authors think this is an important step towards a new development in the theory of capillary-gravity waves. For example, the disconnected part of the solutions will not be traced if we neglect unphysical solutions.

This paper is composed of five sections. In §2 we introduce an equation originally due to Levi-Civita [10], which is a fundamental equation for the progressive water waves. At the end of this section, a rough description of our result is presented. In §3 we consider bifurcation equations and its simplification by means of its O(2)-equivariance. The bifurcation equations are classified by a pair of distinct positive integers, which we call a mode. We consider only the cases of mode (1,2) and (1,3) . We discuss the validity of our normal forms in §4 in the case of (1,2). The case of (1,3) is discussed in §5.

§2. The fundamental equation.

We consider a two dimensional irrotational flow of inviscid incompressible fluid with a free surface. What we consider is a problem of progressive waves, by which we mean a fluid motion with free surface whose configuration is constant if viewed in a coordinate system moving with the same speed as that of wave. Accordingly, in this moving frame, the fluid particles move in a opposite direction while the wave profile remains at rest. In this moving frame, we take (x, y) coordinate system with x horizontally to the right and y vertically upwards. We let c denote the propagation speed and let $y = h(x)$ represent the free surface. Our further assumption is that the wave profile is periodic in x with a period, say, L and that the wave profile is symmetric with respect to the y-axis. We assume that the flow is infinitely deep. By this and the periodicity assumption, we have only to consider the fluid in $-L/2 < x < L/2, -\infty < y < h(x)$. Then the problem is to find a wave profile function $y = h(x)$ and a complex potential $f \equiv U + iV$, where U is a velocity potential and V a stream function, such that $U + iV$ is a holomorphic function of $z \equiv x + iy$ and satisfies

$$
\begin{aligned}
&U\left(\pm\frac{L}{2}, y\right) = \pm\frac{cL}{2}, &&\text{on} \quad -\infty < y \leq h\left(\pm\frac{L}{2}\right), \text{ respectively,}\\
&\frac{\partial U}{\partial n} = 0 &&\text{on} \quad y = h(x) \quad \text{and} \quad y = 0,\\
&\frac{df}{dz} \longrightarrow c &&\text{as} \quad y \to -\infty
\end{aligned}
$$

$$(2.1) \qquad \frac{1}{2}\left|\frac{df}{dz}\right|^2 + gy + \frac{T}{d}K = \text{constant}, \qquad \text{on} \quad y = h(x),$$

where, $\partial/\partial n$ denotes the outward normal derivative, g is the gravity acceleration, T the surface tension coefficient, d the mass density, and K is the curvature of the boundary $y = h(x)$. K is represented as follows:

$$K = -\left(\frac{h'}{\sqrt{1+h'^2}}\right)',$$

where the prime means the differentiation with respect to x.

This is a free boundary problem, in which a boundary portion $y = h(x)$ should be sought. There is a mathematical device found by Stokes [**19**], which makes it considerably easy to handle the free boundary problem above. The idea is to regard f as an independent variable rather than the dependent one. The new formulation is described as follows:

Find a 2π-periodic function $\theta = \theta(\sigma)$ such that

$$(2.2) \qquad e^{2H\theta}\frac{dH\theta}{d\sigma} - pe^{-H\theta}\sin(\theta) + q\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0 \qquad (0 \le \sigma < 2\pi),$$

where H is a linear operator defined through the Fourier series as follows:

$$(2.3) \qquad H\left(\sum_{n=1}^{\infty}\left(a_n \sin n\sigma + b_n \cos n\sigma\right)\right) = \sum_{n=1}^{\infty}\left(- a_n \cos n\sigma + b_n \sin n\sigma\right).$$

Note that the equation (2.2) depends on two parameters p and q. These parameters are defined as

$$p = \frac{gL}{2\pi c^2}, \qquad q = \frac{2\pi T}{dc^2 L},$$

(see, [**13**]). The physical meaning restricts those parameters to be nonnegative:

$$(p, q) \in [0, \infty) \times [0, \infty).$$

This formulation (2.2,3), which is equivalent to the original one, is derived in [**13**] and used for proving the existenxe of non-zero solution θ. Since we define θ and τ in Appendix by

$$u - iv = ce^{-i\theta+\tau},$$

where (u, v) is the velocity vector, the function θ represents the angle made by the tangent vector to the free boundary and the x-axis.

Although the derivation of (2.2) is given in [**13**], we give an outline of the derivation in the Appendix for the sake of the reader's convenience.

In order to apply a mathematical theory of bifurcation such as Golubitsky and Schaeffer [**6,7,8**] or others, we use a mathematical setting with function spaces. Namely we define, for a nonnegative integer k, Banach spaces X_k by

$$X_k = \left\{ f = \sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma) \;\middle|\; \sum_{n=1}^{\infty} n^{2k}\left(|a_n|^2 + |b_n|^2\right) < \infty \right\}$$

or we can write $X_k = H^k(S^1)/\mathbb{R}$. Sobolev's imbedding theorem yields that $X_k \subset C^{k-1}(S^1)$ for $k = 1, 2, \cdots$. We then define a nonlinear operator $F : X_2 \to X_0$ by

$$F(p,q;\theta) = e^{2H\theta}\frac{dH\theta}{d\sigma} - pe^{-H\theta}\sin(\theta) + q\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right).$$

Thus, our task is the determination of the zeros of F. Clearly, $\theta \equiv 0$ is a solution to $F = 0$ for all the values of the parameters p and q. Our aim is, therefore, to find non-zero solutions. We remark that $\theta \equiv 0$ corresponds to $(u, v) = (c, 0)$ and the corresponding free boundary is flat. Using this F, [**13**] gave a simple proof of existence of non-zero solutions which bifurcate from $\theta \equiv 0$.

In the present paper our aim is to reproduce the numerical bifurcation diagrams in [**18**] in an abstract way. Therefore we reformulate our equation so that we can compare the results here and those in [**18**]. We divide F by p and define parameters κ and μ by $\mu = 1/p$ and $\kappa = q/p$. We then redefine F as follows.

$$F(\mu,\kappa\,;\theta) = \mu e^{2H\theta}\frac{dH\theta}{d\sigma} - e^{-H\theta}\sin(\theta) + \kappa\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right). \tag{2.4}$$

The following (A-C) are easy to prove:

(A). *Although F is physically meaningful only for $0 \le \kappa < \infty$, $0 < \mu < \infty$ the mapping F is a well-defined smooth mapping from $\mathbb{R}^2 \times X_2$ into X_0 and satisfies $F(\mu,\kappa\,;0) \equiv 0$.*

(B). *Its Fréchet derivative at $\theta = 0$ is given by*

$$F_\theta(\mu,\kappa\,;0)w = \mu\frac{dHw}{d\sigma} - w + \kappa\frac{d^2w}{d\sigma^2} \qquad (w \in X_2).$$

This formula yields

$$F_\theta(\mu,\kappa\,;0)(\sin n\sigma) = \left(n\mu - 1 - n^2\kappa\right)\sin n\sigma \qquad (n \in \mathbb{N}). \tag{2.5}$$

and a similar formula with sin replaced by cos.

(C). *Therefore, $F_\theta(\mu,\kappa\,;0)$ has a nontrivial null space if and only if (μ,κ) satisfies*

$$n\mu = 1 + n^2\kappa \tag{2.6}$$

for some positive integer n.

Since the proof of these facts are easy we do not give them here (see [**13**], where the same result is proved in the context of the original F using (p, q)). We, however, note that $H^1(S^1)$ and $H^2(S^1)$ are Banach algebras, which leads to (A). By arguing just in the same way as in [**13**], we see that $(\mu_0, \kappa_0; 0)$ which satisfies (2.6) is a bifurcation point. Namely there is a nontrivial solution to $F = 0$ in any neighborhood of $(\mu_0, \kappa_0, 0)$ in $\mathbb{R}^2 \times X_2$. It is, however, very important to notice that some of the bifurcation points differ from others by the dimension of the null space. In fact, some (μ, κ) satisfies (2.6) for two distinct n's. For a (μ, κ) satisfying (2.6) for n and m ($n \neq m$), the kernel of $F_\theta(\mu, \kappa; 0)$ is a 4-dimensional space spanned by $\sin m\sigma$, $\cos m\sigma$, $\sin n\sigma$ and $\cos n\sigma$. We call such a point a double bifurcation point and the other a simple bifurcation point. Thus a double bifurcation point is characterized by

$$m\mu = 1 + m^2\kappa, \qquad n\mu = 1 + n^2\kappa,$$

where m and n are distinct positive integers. We call the pair of positive integers m and n a mode. We define

$$\mu_0(m, n) = \frac{m+n}{nm}, \tag{2.7}$$

$$\kappa_0(m, n) = \frac{1}{mn}. \tag{2.8}$$

Consequently the bifurcation points of mode (m, n) is characterized as $(\mu, \kappa; \theta) = (\mu_0(m, n), \kappa_0(m, n); 0)$. In a neighborhood of the simple bifurcation point, we can only prove the existence of waves whose profile has n troughs and n crests with equal height and depth (Fig. 1). On the other hand, any neighborhood of the double bifurcation point has solutions in which the wave profiles are of mixed nature (Fig. 2). This fact is essentially known early in this century (Wilton [**21**]). However, the global structure of the solution set has not been well understood until the studies by [**1,2,17,18**].

Fig. 1

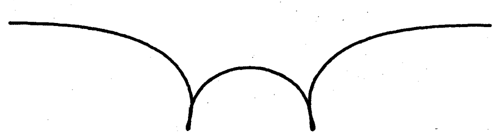


Fig. 2

The result in this paper is summarized as follows:

The paper [**18**] *computed zeros of F numerically and found a number of new bifurcation diagrams. In this paper, we consider the Lyapunov-Schmidt reduction for $F = 0$ and obtain a bifurcation equation. We then simplify the equation by change of variables to obtain a normal form which turns out to be polynomials (see (4.12, 13) and (5.13, 14)). We draw the figures of zeros of the normal form, changing two parameters called unfolding parameters. We then verify that these figures agree with the numerical diagrams in* [**18**].

§3. O(2)-equivariance.

In this section, we prove that F satisfies a certain property called O(2)-equivariance and that this property forces f to be of a special simple form ((3.5-8) below). We first define an action of the orthogonal group O(2) on X_0 as follows: let us recall that O(2) is generated by rotations of angle $\alpha \in [0, 2\pi)$ and a reflection. Accordingly,

$$\gamma_\alpha \theta(\sigma) = \theta(\sigma - \alpha) \qquad (0 \leq \alpha < 2\pi)$$
$$\gamma_- \theta(\sigma) = -\theta(-\sigma)$$

defines an action of O(2) on X_0, where γ_α represents the element of O(2) representing the rotation of angle α and γ_- the reflection. Then we have

PROPOSITION 3.1. *The mapping $F : X_2 \to X_0$ is O(2)-equivariant, by which we mean*

$$F(\mu, \kappa\, ; \gamma\theta) = \gamma F(\mu, \kappa\, ; \theta) \qquad (\quad \gamma \in O(2) \quad).$$

The proof is given in [**13**], though it is straightforward.

Proposition 3.1 enables us to simplify the bifurcation equation. We make a bifurcation equation as follows: Let $(\mu_0, \kappa_0; 0)$ be a double bifurcation point of mode (m, n) $\;(0 < m < n)$. Let P denote the L^2-projection from $L^2(S^1)$ onto the four dimensional subspace spanned by $\sin m\sigma, \cos m\sigma, \sin n\sigma$ and $\cos n\sigma$. Then, the equation

$$(3.1) \quad (I - P)F\big(\mu_0 + \mu_1, \kappa_0 + \kappa_1\, ; x \sin m\sigma + y \cos m\sigma + z \sin n\sigma + w \cos n\sigma + \phi(\mu_1, \kappa_1; x, y, z, w)\big) = 0.$$

uniquely defines a $(I - P)X_2$-valued mapping ϕ in some open set containing $(0, 0; 0, 0, 0, 0)$. We define G by

$$(3.2) \qquad G(\mu_1, \kappa_1; x, y, z, w) = PF(*),$$

where the arguments of F, $*$, is the same as in (3.1). This mapping G is a bifurcation equation.

Remark. From now on, we write as $f : \mathbb{R} \to \mathbb{R}$, even when the defining domain of f is some small open set of $\mathbb{R}$. For instance, we consider mapping germs at the origin, although we write as if it were defined in the whole space.

By Proposition 3.1 and the fact that the bifurcation equation inherits the group equivariance from the basic differential equation ([7]), we see that G too has an O(2)-equivariance. To represent this more conveniently, we identify $(x, y, z, w) \in \mathbb{R}^4$ with $(\xi, \zeta) \in \mathbb{C}^2$ in the way that $\xi = x + iy, \quad \zeta = z + iw$. Therefore, we can regard G as a mapping on (some open subset of) $\mathbb{R}^2 \times \mathbb{C}^2$. Similarly, we can regard that G takes its value in $\mathbb{C}^2$. Let (G_1, G_2) be the componentwise expression of G in $\mathbb{C}^2$. We now have

PROPOSITION 3.2. *The mapping G above is O(2)-equivariant in the sense that the following (3.3,4) hold true.*

$$G(\mu_1, \kappa_1; e^{im\alpha}\xi, e^{in\alpha}\zeta) = \left(e^{im\alpha}G_1(\mu_1, \kappa_1; \xi, \zeta), e^{in\alpha}G_2(\mu_1, \kappa_1; \xi, \zeta)\right), \quad (\alpha \in [0, 2\pi)) \tag{3.3}$$

$$G(\mu_1, \kappa_1; \overline{\xi}, \overline{\zeta}) = \left(\overline{G_1(\mu_1, \kappa_1; \xi, \zeta)}, \overline{G_2(\mu_1, \kappa_1; \xi, \zeta)}\right). \tag{3.4}$$

For the Proof, see [**13**]. Proposition 3.2 forces the mapping G to be of a special form. Let us prepare some symbols.

Definition. Let k be a positive integer. We call a function $f : \mathbb{R}^k \times \mathbb{C}^2 \to \mathbb{R}$ O(2)-invariant if

$$f(a; e^{im\alpha}\xi, e^{in\alpha}\zeta) \equiv f(a; \xi, \zeta) \qquad (\alpha \in [0, 2\pi))$$

and

$$f(a; \overline{\xi}, \overline{\zeta}) \equiv f(a; \xi, \zeta)$$

are satisfied. Here $a \in \mathbb{R}^k$, $\xi, \zeta \in \mathbb{C}$.

The set of all germs (at the origin) of O(2)-invariant C^∞-functions is a commutative ring with a unit. Let $\mathcal{E}$ denote this ring. The set of all the mapping $G : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ satisfying (3.3,4) constitutes an $\mathcal{E}$-module. Let E denote this $\mathcal{E}$-module. In order to give a simple expression to $\mathcal{E}$ and E, we need to introduce two positive integers n' and m'. We define them to be coprime positive integers satisfying $n/m = n'/m'$. We now have

PROPOSITION 3.3. *Any element $f \in \mathcal{E}$ is of the following form*

$$f(a; \xi, \zeta) = g(a; u, v, s)$$

where g is a C^∞ function of $k + 3$ variables and u, v, s are defined by

$$u = |\xi|^2, \qquad v = |\zeta|^2, \qquad s = Re[\overline{\xi}^{n'} \zeta^{m'}].$$

PROPOSITION 3.4. *The module E is generated over $\mathcal{E}$ by the following four elements:*

$$X_1 = (\xi, 0), \qquad X_2 = (0, \overline{\xi}^{n'-1}\zeta^{m'}), \qquad X_3 = (0, \zeta), \qquad X_4 = (0, \xi^{n'}\overline{\zeta}^{m'-1}).$$

The proofs of these Propositions 3.3,4 can be found in [**11**](see also [**4,8**]).

COROLLARY 3.1. *The mapping G at the bifurcation point of mode* $(1,2)$ *is of the following form*

$$G_1 = f_1\xi + f_2\bar{\xi}\zeta, \tag{3.5}$$

$$G_2 = f_3\zeta + f_4\xi^2, \tag{3.6}$$

where f_j are of the following form

$$f_j = f_j(\mu_1, \kappa_1; |\xi|^2, |\zeta|^2, Re[\bar{\xi}^2\zeta]). \qquad (j = 1,2,3,4)$$

COROLLARY 3.2. *The mapping G at the bifurcation point of mode* $(1,3)$ *is of the following form*

$$G_1 = g_1\xi + g_2\bar{\xi}^2\zeta, \tag{3.7}$$

$$G_2 = g_3\zeta + g_4\xi^3, \tag{3.8}$$

where g_j are of the following form

$$g_j = g_j(\mu_1, \kappa_1; |\xi|^2, |\zeta|^2, Re[\bar{\xi}^3\zeta]). \qquad (j = 1,2,3,4)$$

§4. Bifurcation equation of mode (1,2).

In this section we give a normal form of the bifurcation equation of mode (1,2). We show that a theorem in [**11**] is applicable to the problem now in question.

The bifurcation equation $G : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ is now written as (3.5,6). We use the theory in [**6,7,8**] in which mapping germs containing one parameter are considered. Our mapping, however, has two parameters. In [**18**], μ is used as a bifurcation parameter and κ a splitting parameter. We thereby freeze κ_1. We introduce λ and consider $G(\lambda, 0; \xi, \zeta)$. We then regard G as a mapping germ of (λ, ξ, ζ). We can now write as

$$G_1 = f_1(\lambda; u, v, s)\xi + f_2(\lambda; u, v, s)\bar{\xi}\zeta, \tag{4.1}$$

$$G_2 = f_3(\lambda; u, v, s)\zeta + f_4(\lambda; u, v, s)\xi^2. \tag{4.2}$$

If we have shown that this mapping is finitely determined and if we have computed universal unfoldings, then the equation for small but nonzero κ_1 can be realized by one of the unfolded mappings ([**6,7,8**]). Thus we are lead to the analysis of (4.1,2).

Since G is a bifurcation equation, all the derivatives of first order vanishes at the origin. Accordingly $f_j(0; 0,0,0) = 0 \quad (j = 1,3)$. In order to go further, we need to compute $f_2(0; 0,0,0)$ and $f_4(0; 0,0,0)$. This is done in [**13**]. The result is, in the present notation,

$$f_2(0; 0,0,0) = -3/4, \qquad f_4(0; 0,0,0) = -3.$$

See also Proposition 4.1 below.

In this situation we can use Theorem 3.2 and 3.3 of [**11**]. According to this theorem, the qualitative behaviour of the set of the solutions is described by

$$(4.3) \qquad (\epsilon\lambda + \alpha + bv)\xi + \bar{\xi}\zeta = 0, \qquad (\delta\lambda + \hat{b}v)\zeta + \xi^2 = 0,$$

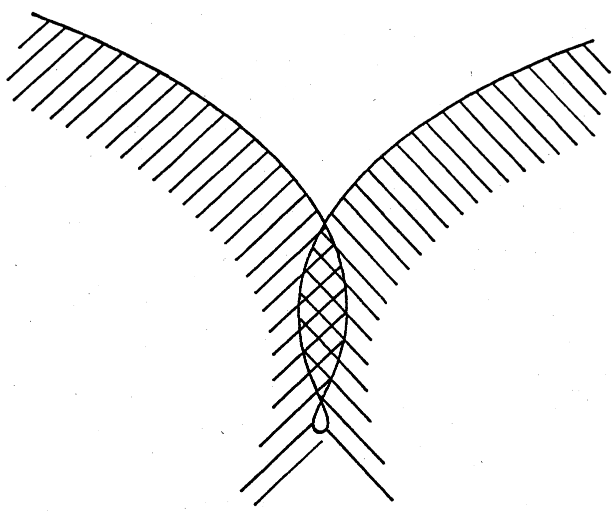
wherer $\epsilon, \alpha, \delta, b, \hat{b}$ are real constants.

We now turn to the original problem of the progressive water waves. As in [**18**], we assume that the wave profile is symmetric with respect to the y-axis. This assumption is equivalent to the oddness of θ in σ ([**13**]). The oddness of θ is written by $\gamma_{-}\theta = \theta$. This is interpreted as $(\bar{\xi}, \bar{\zeta}) = (\xi, \zeta)$. Consequently we consider only real solutions to $G(\lambda, \xi, \zeta) = 0$. We therefore put $\xi = x, \quad \zeta = z$ in (4.3). Thus we have to solve

$$(4.4) \qquad (\epsilon\lambda + \alpha + bz^2)x + xz = 0,$$

$$(4.5) \qquad (\delta\lambda + \hat{b}z^2)z + x^2 \qquad = 0.$$

We now see the numerical bifurcation diagrams. Fig. 3-7 are borrowed from [**18**]. Each of Fig. 3-7 has different κ and is drawn in the frame of (μ, A_1, A_2). In each figure, the picture below shows the diagram viewed in the direction of A_1-axis. κ decreases as the figure number increases. The variables x and z in this paper correspond to A_1 and A_2 in [**18**], respectively. In these diagrams, there are two kinds of curves: solid curves and broken curves. The both represent zeros of $F = 0$. The difference is that the solutions on the broken curves have self-intersections in the wave profile like:



On the other hand, those on the solid curves do not have any self-intersection. So, they look like Fig. 2. At the boundary of solid curve and broken curve, the wave has contact points, which resembles Crapper's pure capillary wave ([3]). Fig. 2 corresponds to the point a in Fig.4. References [1,2,17] reached this point. Before [18], however, both physicists and mathematicians neglected the solutions to $F = 0$ which have self-intersections simply because they represent unphysical states. However, [18] revealed that there are some limit points (turning points) in the physically meaningful regions of the diagrams (see Fig. 5) while they do not appear for relatively large κ (see Fig. 3,4). It indicates that total analysis of both physical and unphysical solutions is necessary for the explanation of the existence or nonexistence of limit points. The necessity becomes clearer when we consider waves of mode (1,3) in the next section.

By a simple computation, we find that (4.4,5) yields a bifurcation diagram listed in [4]. Fig. 8 is borrowed from [4]. In this figure, ξ and η correspond to x and z in our notation, respectively. The bifurcation parameter is taken vertically. We notice that these diagrams are only a subset of those in [18]. Therefore, *the normal form (4.4,5) can reproduce solutions only a small portion of the numerical solutions in [18].*

In order to capture wider class of solutions, we now use an idea in [4 - 7]: *Suppose we are given a problem to find zeros of $F(a_1, \cdots, a_k; u) = 0$ with k parameters. If there exists a set of solutions which is described by a normal form of codimension $\geq k$, then the problem requires more than k parameters.*

Following this principle, we assume as follows:

In addition to κ and μ, there is a parameter, say, ν, and a smooth mapping $\hat{F}(\kappa, \mu, \nu; u)$ such that $\hat{F}(\kappa, \mu, 0; u) \equiv F(\kappa, \mu; u)$. In other words, our problem is embedded in an extended system.

We do not know what is the additional parameter. Rather than identifying the parameter, we continue our abstract analysis under this assumption. We can then regard $f_2(0;0,0,0)$ and $f_4(0;0,0,0)$ depend on ν. We finally assume that: $f_2(0;0,0,0)$ vanishes at some $\nu = \nu_0$ and that $f_4(0;0,0,0)$ does not at $\nu = \nu_0$.

Under this assumption, we are allowed to divide G_2 by f_4 in some neighborhood of $(\kappa_1, \mu_1, \nu; u) = (0, 0, \nu_0; 0)$ ([11]). Then it holds that

$$G_1 = (\epsilon\lambda + au + bv + cs + f_5)\xi + (\eta\lambda + d_1u + d_2v + d_3s + f_6)\bar{\xi}\zeta, \tag{4.6}$$

$$G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}s + f_7)\zeta + \xi^2, \tag{4.7}$$

where $\epsilon, \delta, \eta, d_j \quad (j = 1, 2, 3), a, b, c, \hat{a}, \hat{b}, \hat{c}$ are real constants and f_j are terms of order ≥ 2. We are now in a position to quote some results in [11]:

THEOREM 4.2. *Under some nondegeneracy conditions on $\epsilon, \delta, a, b, \hat{a}, \hat{b}$ and d_2, the mapping G in (4.6,7) is O(2)-equivalent to*

$$G_1 = (\epsilon\lambda + a'u + b'v + c's)\xi + d'v\bar{\xi}\zeta, \tag{4.8}$$

$$G_2 = (\delta\lambda + \hat{a}'u + \hat{b}'v)\zeta + \xi^2, \tag{4.9}$$

where $a', b', c', d', \hat{a}'$ and $\hat{b}'$ are real constants. A universal O(2)-unfolding of the mapping (4.8,9) is given by

$$F_1 = [\epsilon\lambda + \alpha + (a' + \tau_1)u + (b' + \tau_2)v + (c' + \tau_3)s]\xi + [\beta + (d' + \tau_4)v]\bar{\xi}\zeta, \tag{4.10}$$

$$F_2 = [\delta\lambda + \hat{a}'u + \hat{b}'v]\zeta + \xi^2, \tag{4.11}$$

where $\alpha, \beta, \tau_1, \cdots, \tau_4$ are unfolding parameters.

This theorem enables us to neglect the higher order terms and some of the first order terms in (u, v, s) without losing any information on the bifurcation diagrams. As for the precise statement of the nondegeneracy assumption, see [**11**]. This assumption includes $\epsilon\delta \neq 0$, $\hat{b} \neq 0$, etc. We assume that these holds. Since the assumption is generic in the sense that it is expressed as "$\cdots$" $\neq 0$, our assumption is a reasonable one.

We now consider real solutions, whence we have

$$(\epsilon\lambda + \alpha + ax^2 + bz^2 + cx^2z)x + (\beta + dz^2)xz = 0, \tag{4.12}$$

$$(\delta\lambda + \hat{a}x^2 + \hat{b}z^2)z + x^2 = 0. \tag{4.13}$$

In order for (4.12,13) to be a normal form of the bifurcation equations, we must reproduce Fig. 3-7 by choosing parameters appropriately. We now show this is possible. Fig. 9-13 are the set of zeros of (4.12, 13) for suitable parameters. These are drawn by a personal computer and a lazer beam printer. We see that Fig 9-13 completely reproduce Fig. 3-7.

Remark. By the same idea, Fujii et al. [**5**] finds bifurcation diagrams similar to Fig. 9-13 in the problem of reaction diffusion equations. They do not, however, obtain the normal form (4.12,13).

We finally remark that these figures do not automatically prove that (4.12,13) has zeros like Fig. 3-7. In fact, we are dealing with a mapping germ defined by (4.12, 13). Therefore we must prove that figures like Fig. 3-7 can be drawn in any small neighborhood of the origin by choosing appropriately small α and β. This is, however, checked by the method described in [**12**].

Although we believe that there is a parameter ν which makes our assumption valid, we do not know the identity of ν. This is a very week point of our theory. We first thought that the aspect ratio serves as the parameter. The aspect ratio is a parameter which represent the ratio of the wave length and the depth of the flow. This means that we consider a generalized problem in which H in (2.3) is replaced by

$$H\left(\sum_{n=1}^{\infty}\left(a_n \sin n\sigma + b_n \cos n\sigma\right)\right) = \sum_{n=1}^{\infty}\frac{1+r^n}{1-r^n}\left(-a_n \cos n\sigma + b_n \sin n\sigma\right)$$

(see [13]). Here $r \in [0,1)$ is a new parameter. As $r \to 0$, the depth becomes deeper. When $r = 0$ this generalized problem reduces to the original problem of flows of infinite depth. Although only $0 \le r < 1$ is physically allowed, the formulation itself has a well-defined meaning for all $r \in (-1,1)$. Therefore we expected the existence of some r at which $f_2(0;0,0,0)$ vanish. The calculation, however, showed:

PROPOSITION 4.1. *It holds that*

$$f_2(0;0,0,0) = -\frac{3(1+r)}{4(1-r)}, \quad f_4(0;0,0,0) = -\frac{3(1+r)}{1-r}.$$

PROOF: We note that (2.6) is replaced by

$$n\frac{1+r^n}{1-r^n}\mu = 1 + n^2\kappa.$$

Similarly, (2.7,8) are replaced by

$$\mu_0 = \frac{3(1-r^2)}{2+8r+2r^2}, \qquad \kappa_0 = \frac{(1-r)^2}{2+8r+2r^2}, \tag{4.14}$$

since $m = 1$ and $n = 2$. We note that

$$f_2(0;0,0,0) = \frac{\partial^2 G_1}{\partial x \partial z}(0;0,0).$$

It holds that

$$\frac{\partial G}{\partial x}(\lambda;\xi,\zeta) = PF_\theta(\mu_0+\lambda,\kappa_0,r;\#)(\sin\sigma + \phi_x), \tag{4.15}$$

where F now depends upon r as well as μ and κ and $\#$ denotes $x\sin\sigma + y\cos\sigma + z\sin 2\sigma + w\cos 2\sigma + \phi$. Differentiating (4.15) in z, we obtain

$$\frac{\partial^2 G}{\partial x \partial z}(0;0,0) = PF_\theta(\mu_0,\kappa_0,r;0)(\phi^0_{xz}) + PF_{\theta\theta}(\mu_0,\kappa_0,r;0)(\sin\sigma,\sin 2\sigma).$$

Since ϕ is $(I-P)X_2$-valued and since $F_\theta(\kappa_0,\mu_0,r;0)$ commutes with P by (2.5), the first term of the right hand side vanishes. Hence $f_2(0;0,0)$ is the coefficient of $\sin\sigma$ of $F_{\theta\theta}(\mu_0,\kappa_0,r;0)(\sin\sigma,\sin 2\sigma)$. We now compute the second order Fréchet derivative of F. We have

$$F_{\theta\theta}(\mu_0,\kappa_0,r;0)(f,g) = 2\mu_0\frac{d}{d\sigma}(HfHg) + fHg + gHf + \kappa_0\frac{d}{d\sigma}\left(Hf\frac{dg}{d\sigma} + Hg\frac{df}{d\sigma}\right)$$

for all $f, g \in X_2$. This formula yields

$$\begin{aligned} F_{\theta\theta}(\mu_0,\kappa_0,r;0)(\sin\sigma,\sin 2\sigma) &= 2\mu_0\frac{1+r}{1-r}\frac{1+r^2}{1-r^2}\frac{d}{d\sigma}(\cos\sigma\cos 2\sigma) \\ &- \left(\frac{1+r^2}{1-r^2}\cos 2\sigma\sin\sigma + \frac{1+r}{1-r}\cos\sigma\sin 2\sigma)\right) \\ &+ \kappa_0\frac{d}{d\sigma}\left(-\frac{1+r}{1-r}\cos\sigma\cdot 2\cos 2\sigma - \frac{1+r^2}{1-r^2}\cos 2\sigma\cos\sigma\right). \end{aligned}$$

Consequently we obtain

$$PF_{\theta\theta}(\mu_0,\kappa_0,r;0)(\sin\sigma,\sin 2\sigma)=\mu_0\frac{1+r}{1-r}\frac{1+r^2}{1-r^2}(-\sin\sigma)$$
$$+\left(\frac{1}{2}\frac{1+r^2}{1-r^2}-\frac{1}{2}\frac{1+r}{1-r}\right)\sin\sigma+\kappa_0\left(\frac{1+r}{1-r}+\frac{1}{2}\frac{1+r^2}{1-r^2}\right)\sin\sigma.$$

By this equality and (4.14) we have

$$f_2(0;0,0,0)=-\frac{3(1+r)}{4(1-r)}.$$

In a similar way we obtain

$$f_4(0;0,0,0)=PF_{\theta\theta}(\mu_0,\kappa_0,r;0)(\sin\sigma,\sin\sigma)=-\frac{3(1+r)}{1-r}.$$

This completes the proof. ∎

We are puzzled by this computation: we must look for another parameter. Although there is a possibility that the parameter may be a one representing vorticity, a one representing the viscosity or a density ratio in the two phase problem, we do not have any definite idea for the moment.

§5. Bifurcation equation of mode (1,3).

In this section we consider (3.7,8) and introduce a certain degeneracy which makes (3.7,8) yield bifurcation diagrams in [**18**].

First of all, we assume that

(H1) $$g_2(0;0,0,0)\neq 0,\qquad g_4(0;0,0,0)\neq 0.$$

We divide (3.7) by g_2, and (3.8) by $-g_4$. This operation preserves O(2)-equivariance (see [**11**]) and give rise to the following O(2)-equivalent mapping:

(5.1) $$G_1=(\epsilon\lambda+au+bv+cs+g_5)\xi+\bar{\xi}^2\zeta,$$

(5.2) $$G_2=(\delta\lambda+\hat{a}u+\hat{b}v+\hat{c}s+g_6)\zeta-\xi^3,$$

where $\epsilon,\delta,a,\cdots,\hat{c}$ are real constants and g_5 and g_6 are of order ≥ 2. Before doing further simplification of the mapping we prepare some symbols for the notational convenience. Let $\mathcal{M}$ be the set of all $g\in\mathcal{E}$ satisfying $g(0;0,0,0)=0$. The symbol $\mathcal{M}^2$ implies the ideal of $\mathcal{E}$ generated by $\{fg; f\in\mathcal{M}, g\in\mathcal{M}\}$. Using this notation, we can say that $g_5,g_6\in\mathcal{M}^2$. Let $\mathcal{E}_\lambda$ be the set of all the C^∞-germs of function $\phi(\lambda)$ at $0\in\mathbf{R}$. We designate by $\mathcal{M}_\lambda$ the set of all $\phi\in\mathcal{E}_\lambda$ satisfying $\phi(0)=0$. Note that the elements of $\mathcal{E}_\lambda$ is a germ of a function of λ only. The ideal in $\mathcal{E}$ generated by $f_1,\cdots,f_r$ is denoted by $\langle f_1,\cdots,f_r\rangle$.

Let us write as $g_5 = \lambda^2\eta_1(\lambda) + g_7, \quad g_6 = \lambda^2\eta_2(\lambda) + g_8$, where $\eta_1, \eta_2 \in \mathcal{E}_\lambda$ and $g_7, g_8 \in \mathcal{M}^2 \cap \langle u, v, s\rangle$. We now perform the following change of variables:

$$(\xi, \zeta) \to \Big((1 + \phi(\lambda))\xi,\ \zeta\Big),$$

where $\phi \in \mathcal{M}_\lambda$ is to be determined later. This change preserves the O(2)-equivariance (see [**11**]). Applying this change and dividing (5.1) by $(1+\phi)^2$, (5.2) by $(1+\phi)^3$, we obtain

$$G_1' = \frac{1}{1+\phi}(\epsilon\lambda + \lambda^2\eta_1 + au + bv + cs + g_9)\xi + \overline{\xi}^2\zeta,$$
$$G_2' = \frac{1}{(1+\phi)^3}(\delta\lambda + \lambda^2\eta_2 + \hat{a}u + \hat{b}v + \hat{c}s + g_{10})\zeta - \xi^3,$$

where $g_9, g_{10} \in \mathcal{M}^2 \cap \langle u, v, s\rangle$. We choose ϕ such that

$$\frac{\lambda + \epsilon^{-1}\lambda^2\eta_1(\lambda)}{1 + \phi(\lambda)} = \frac{\lambda + \delta^{-1}\lambda^2\eta_2(\lambda)}{(1+\phi(\lambda))^3}.$$

Namely we define ϕ by

$$\phi(\lambda) = \sqrt{\frac{1+\delta^{-1}\lambda\eta_2}{1+\epsilon^{-1}\lambda\eta 1}} - 1.$$

We then write $(\lambda + \epsilon^{-1}\lambda^2\eta_1)/(1+\phi)$ as λ. After these computations, we may assume that G fas the following form:

$$G_1 = (\epsilon\lambda + au + bv + cs + g_{12})\xi + \overline{\xi}^2\zeta,$$
$$G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}s + g_{12})\zeta - \xi^3,$$

where $g_{11}, g_{12} \in \mathcal{M}^2 \cap \langle u, v, s\rangle$.

We next consider the following change of variables, which also preserves O(2)-equivariance:

$$(\xi, \zeta) \to \Big(\xi + \alpha\overline{\xi}^2\zeta,\ \zeta + \beta\xi^3\Big),$$

where α and β are real constants. By this change, G is transformed to

$$G_1' = (\epsilon\lambda + au + bv + (c + 2a\alpha + 2b\beta)s + g_{13})\xi + (1+\mu_1)\overline{\xi}^2\zeta, \tag{5.3}$$
$$G_2' = (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\hat{a}\alpha + 2\hat{b}\beta)s + g_{15})\zeta - (1+\mu_2)\xi^3, \tag{5.4}$$

where $g_{13}, g_{15} \in \mathcal{M}^2 \cap \langle u, v, s\rangle$ and $\mu_1, \mu_2 \in \langle u, v, s\rangle$. If

$$a\hat{b} - \hat{a}b \neq 0, \tag{H2}$$

then we can make the coefficient of s vanish in (5.3,4). Accordingly we obtain after dividing (5.3) by $1+\mu_1$ and (5.4) by $1+\mu_2$,

$$G_1'' = (\epsilon\lambda + au + bv + g_{15})\xi + \bar{\xi}^2\zeta, \tag{5.5}$$

$$G_2'' = (\delta\lambda + \hat{a}u + \hat{b}v + g_{16})\zeta - \xi^3, \tag{5.6}$$

where $g_{15}, g_{16} \in \mathcal{M}^2 \cap \langle u, v, s\rangle$. Our next simplification is obtained by the following change of variables:

$$(\xi, \zeta) \to \Big(\ell_1\xi,\ \ell_2\zeta\Big),$$

where ℓ_1 and ℓ_2 denote $1+\alpha_1 u+\alpha_2 v+\alpha_3 s$ and $1+\beta_1 u+\beta_2 v+\beta_3 s$, respectively, with real constants α_j and β_j. Applying this transformation and dividing (5.5) by $\ell_1^2\ell_2$, (5.6) by ℓ_1^3, we have

$$H_1 = \frac{1}{\ell_1^2\ell_2}(\epsilon\lambda + au + bv + g_{17})\xi + \bar{\xi}^2\zeta,$$

$$H_2 = \frac{\ell_2}{\ell_1^3}(\delta\lambda + \hat{a}u + \hat{b}v + g_{18})\zeta - \xi^3,$$

where $g_{17}, g_{18} \in \mathcal{M}^2 \cap \langle u, v, s\rangle$.

We wish to eliminate some of the second order terms. Let

$$g_{17} = A_1\lambda u + A_2\lambda v + A_3\lambda s + g_{19}, \quad g_{18} = B_1\lambda u + B_2\lambda v + B_3\lambda s + g_{20},$$

where g_{19} and g_{20} belong to $\langle u, v, s\rangle^2 + \mathcal{M}^3 \cap \langle u, v, s\rangle$. Then the coefficients of $\lambda u, \lambda v, \lambda s$ in G_1 and G_2 are, respectively,

$$A_1 - \epsilon(\alpha_1 + \beta_1), \quad A_2 - \epsilon(\alpha_2 + \beta_2), \quad A_3 - \epsilon(\alpha_3 + \beta_3)$$

and

$$B_1 - \delta(3\alpha_1 - \beta_1), \quad B_2 - \delta(3\alpha_2 - \beta_2), \quad B_3 - \delta(3\alpha_3 - \beta_3).$$

Choosing α_j and β_j appropriately, we can make these coefficients vanish. This is possible if

$$\epsilon \neq 0, \qquad \delta \neq 0. \tag{H3}$$

Thus we have proved that the original G in (5.1,2) is O(2)-equivalent to the following new G:

$$G_1 = (\epsilon\lambda + au + bv + c_1u^2 + c_2uv + c_3v^2 + g_{21})\xi + \bar{\xi}^2\zeta, \tag{5.7}$$

$$G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + c_4u^2 + c_5uv + c_6v^2 + g_{22})\zeta - \xi^3, \tag{5.8}$$

where $g_{21}, g_{22} \in \mathcal{M}^3 \cap \langle u, v, s\rangle + \langle us, vs, s^2\rangle$. We wish to prove that (5.7,8) is O(2)-equivalent to itself with $g_{21} \equiv 0, g_{22} \equiv 0$. If this is possible, we have a normal form in some sense and our task is reduced to consider

$$G_1 = (\epsilon\lambda + au + bv + c_1u^2 + c_2uv + c_3v^2)\xi + \bar{\xi}^2\zeta, \tag{5.9}$$

$$G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + c_4u^2 + c_5uv + c_6v^2)\zeta - \xi^3, \tag{5.10}$$

However, the results in [**11**] is limited to the case of mode (1,2) and no ready-made result seems to be available concerning the elimination of g_{21} and g_{22}. Accordingly we give up proving that (5.7,8) is O(2)-equivalent to (5,9,10). We only show that (5.9,10) explains the figures in [**18**] well.

We now assume that $\epsilon = \delta = 1$. Setting $y = w = 0$, we have

$$G_1 = (\lambda + ax^2 + bz^2 + c_1x^4 + c_2x^2z^2 + c_3z^4)x + x^2z, \tag{5.11}$$

$$G_2 = (\lambda + \hat{a}x^2 + \hat{b}z^2 + c_4x^4 + c_5x^2z^2 + c_6z^4)z - x^3. \tag{5.12}$$

Numerical Results for mode (1,3). Let us see the numerical bifurcation diagrams. Fig.14-19 are cited from [**18**]. Notation is the same as in the case of mode (1,2). As the figure number increases, the value of κ increases. Let us explain Fig. 17, for instance. The diagrams are viewed in three directions. The figure in the right above is a view in the direction of the μ-axis. The figure in the left below is a view in the A_3direction. Wave profiles are drawn in the right below. The index a, b, etc. correspond to the points marked in the figure in the left above. We can observe a pair of closed loops, which are symmetric with respect to the λ-axis. As the parameter κ increases, they intersect, are tangent to or separate from the primary pitchfork branch. Fig. 19 shows that the loops shrink to points. Here we observe a very important fact about the connectedness of the branch. Some of the physical solutions on the loops are connected to the mode 3 branch through unphysical solutions (see Fig. 18) . Thus the physical solutions on the loops are disconnected from other physical solutions if we neglect the unphysical solutions. In computing the branch, [**18**] used Keller's path-tracing method as is common in the bifurcation problem. Therefore, disconnected part may well be missed, when you look for only physical solutions. If, however, we consider all the solutions, both physical and unphysical, then all the solutions are traced.

We introduce two unfolding parameters α and β in (5.11, 12):

$$G_1 = (\lambda + \alpha + ax^2 + bz^2 + c_1x^4 + c_2x^2z^2 + c_3z^4)x + x^2z, \tag{5.13}$$

$$G_2 = (\lambda + \hat{a}x^2 + (b + \beta)z^2 + c_4x^4 + c_5x^2z^2 + c_6z^4)z - x^3. \tag{5.14}$$

Here we assumed that $\hat{b}$ is close to b and introduced a small β. In other words, we assumed that there is a ν at which $b = \hat{b}$ and we unfolded the bifurcation equation there.

The equation (5.13) implies $x = 0$ or

$$\lambda = -\alpha - ax^2 - bz^2 - c_1x^4 - c_2x^2z^2 - c_3z^4 - xz. \tag{5.15}$$

The former relation and (5.14) give a pitchfork in (λ, z) plane: $x = 0, \quad \lambda z = -(b+\beta)z^3 - c_6 z^5$. We assume that $b > 0$ and $c_6 > 0$. Substituting (5.15) into (5.14), we have

$$(5.16) \qquad -\alpha z + (\hat{a} - a)x^2 z + \beta z^3 - x^3 - xz^2 + d_1 x^4 z + d_2 x^2 z^3 + d_3 z^5. = 0.$$

where we have put $d_1 = c_4 - c_1, d_2 = c_5 - c_2, d_3 = c_6 - c_3$. This equation does not contain λ.

The roots of (5.13, 14) constitute of either the pitchfork ($x = 0 \quad \& \quad \lambda z + \hat{b} z^3 + c_6 z^4 = 0$) or the curves given by (5.15, 16).

Let us check whether the normal form above can reproduce these figures. Choosing small parameters α and β appropriately, we can draw various diagrams. Among those, Fig.20-23 suit our purpose. In these figures, we fixed the following parameters:

$$a = 5.25, \quad b = 0.2, \quad c_1 = 0 \quad c_2 = 0 \quad c_3 = 0.101$$
$$\hat{a} = 1.0, \quad \hat{b} = 1.0, \quad c_4 = 0 \quad c_5 = 0 \quad c_6 = 0.001$$

The values of unfolding parameter α in Fig.20 - 23 are, $1.0, 1.6, 1.8, 1.85$, respectively. These abstract diagrams explains Fig.16-19 completely. Fig.14,15 are more complex and we can not reproduce them by the normal form.

Remark. As in §4, it is necessary to prove that Fig. 20-23 can be drawn in any neighborhood of the degenerate bifurcation point. So far, we can not prove this.

Concluding Remark. We consider two pairs of algebraic equations (4.12,13) and (5.13,14). These " normal forms " reproduce quite a large class of the numerical solutions in [**18**]. The success of our equations suggests that another appropriate parameter exists and that the degeneracy which we assumed exists. Unfortunately we can not specify the parameter. We showed that the aspect ratio parameter is not the one.

Appendix.

In this Appendix, we give an outline of the derivation of (2.2). As Stokes did, we regard z as a function of f rather than regarding f as a function of z. Following Levi-Civita [**10**], we introduce independent variable

$$\zeta = \exp\left(-\frac{2\pi i f}{cL}\right)$$

and dependent variable $\omega = i \log(c^{-1} df/dz)$. By the relation $\zeta \leftrightarrow f \leftrightarrow z \leftrightarrow \omega$, we regard ω as a function of ζ. Note that ζ runs in $0 < |z| < 1$ and that ω is analytic in ζ. As $\zeta \to 0$, ω approaches zero. Thus the origin is a removable singularity and ω is an analytic function in the whole disk $|\zeta| < 1$. Let θ and τ denote, respectively, the real and the imaginary part of ω. Let (ρ, σ) be the polar coordinates for ζ, i.e., $\zeta = \rho e^{\sigma}$. We now rewrite (2.1) by means of $\theta(\rho, \sigma)$ and $\tau(\rho, \sigma)$: On the free boundary, we have $V=0$. Therefore it holds that $dz/df = \frac{\partial x}{\partial U} + i\frac{\partial y}{\partial U}$. On the other hand, by

$$\frac{df}{dz} = ce^{-i\theta+\tau},$$

we have

$$\sigma = \frac{-2\pi U}{cL}, \quad \frac{\partial x}{\partial U} = \frac{e^{-\tau}}{c}\cos\theta, \quad \frac{\partial y}{\partial U} = \frac{e^{-\tau}}{c}\sin\theta, \quad \frac{dh}{dx}(x) = \tan\theta$$

and

$$\left|\frac{df}{dz}\right|^2 = c^2e^{2\tau}, \quad \frac{\partial}{\partial x} = \frac{\partial\sigma}{\partial x}\frac{\partial}{\partial\sigma} = -\frac{2\pi e^{\tau}}{L\cos\theta}\frac{\partial}{\partial\sigma}.$$

By these formula we easily get to

$$\text{(A.1)} \qquad e^{2\tau}\frac{\partial\tau}{\partial\sigma} - pe^{-\tau}\sin\theta + q\frac{\partial}{\partial\sigma}\left(e^{\tau}\frac{\partial\theta}{\partial\sigma}\right) = 0 \quad \text{on} \quad \rho = 1,$$

where $p = \frac{gL}{2\pi c^2}, q = \frac{2\pi T}{c^2 Ld}$.

We thus obtain the following reformulation:

Find a function $\omega = \omega(\zeta)$ which is continuous on $\{|\zeta| \le 1\}$, is analytic in $\{|\zeta| < 1\}$ and satisfy (A.1) and $\omega(0) = 0$.

Levi-Civita considered the problem in this formulation when $q = 0$.

Since an analytic function is completely determined by its boundary value, a further reduction of the equation (A.1) is possible. In fact we can write (A.1) only by $\theta(1,\sigma)$ $(0 \le \sigma < 2\pi)$. This is done by utilizing (2.3). We have $\tau(1,\sigma) = H(\theta^*)$, where $\theta^*(\sigma) = \theta(1,\sigma)$. The equation (A.1) is now written as (2.2).

References

1. B. Chen and P.G. Saffman, Studies in Appl. Math. **60** (1979), 183–210.
2. B. Chen and P.G. Saffman, Studies in Appl. Math. **62** (1980), 95–111.
3. G.D. Crapper, J. Fluid Mech. **2** (1957), 532–540.
4. H. Fujii, M. Mimura and Y. Nishiura, Physica D **5** (1982), 1–42.
5. H. Fujii, Y. Nishiura and Y. Hosono, in " Patterns and Waves ", eds. T. Nishida, M. Mimura and H. Fujii (1986), 157–216.
6. M. Golubitsky and D. Schaeffer, Comm. Math. Phys. **67** (1979), 205–232.
7. M. Golubitsky and D. Schaeffer, "Singularity and Groups in Bifurcation Theory, vol. I," Springer Verlag, New York, 1985.
8. M. Golubitsky, I. Stewart and D. Schaeffer, "Singularity and Groups in Bifurcation Theory, vol. II," Springer Verlag, New York, 1988.
9. M. Jones and J.Toland, Arch. Rat. Mech. Anal. **96** (1986), 29–53.
10. T. Levi-civita, Math. Ann. **93** (1925), 264–314.
11. H. Okamoto, Sci. Papers of College of Art and Sci., Univ. of Tokyo **39** (1989), 1–43.
12. H. Okamoto and S.J. Tavener, to appear in Japan J. Appl. Math.
13. H. Okamoto, Nonlinear Anal., Theory, Meth. and Appl. **14** (1990), 469–481.
14. H. Okamoto, Kokyu-roku, Research Institute for Mathematical Sciences (in press).

15. W.J. Pierson and P.H. Fife, J. Geophys. Res. **99** (1961), 163–179.
16. J. Reeder and M. Shinbrot, Arch. Rat. Mech. Anal. **77** (1981), 321–347.
17. L.W. Schwartz and J.-M. Vanden-Broeck, J. Fluid Mech. **95** (1979), 119–139.
18. M. Shōji, J. Fac. Sci. Univ. Tokyo, Sec. IA **36** (1989), 571–613.
19. G.G. Stokes, Trans. Cambridge Phil. Soc. **8** (1849), 441–455.
20. J.F. Toland and M.C.W. Jones, Proc. R. Soc. London A **399** (1985), 391–417.
21. J.R. Wilton, Phil. Mag. (6) **29** (1915), 688–700.

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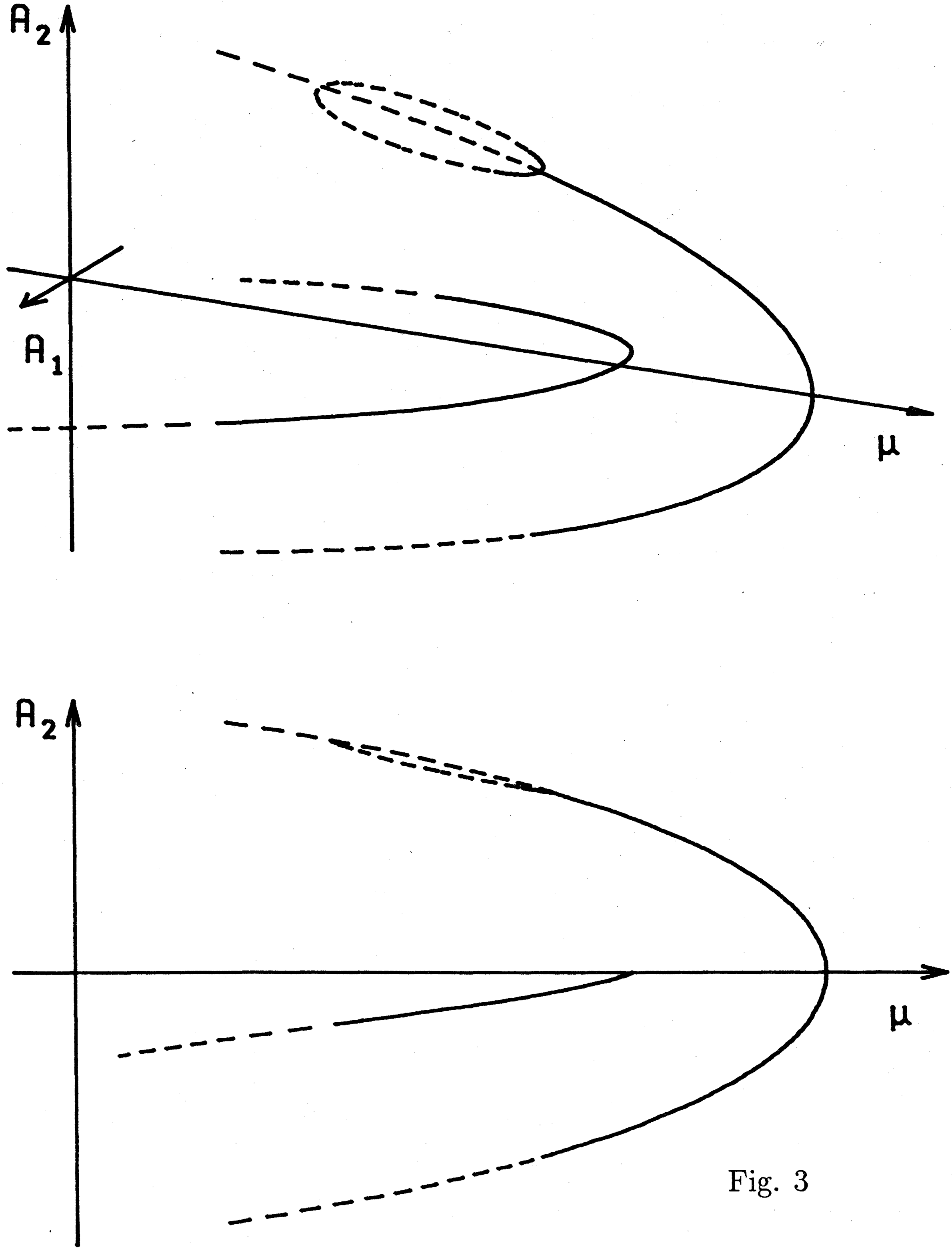


Fig. 3

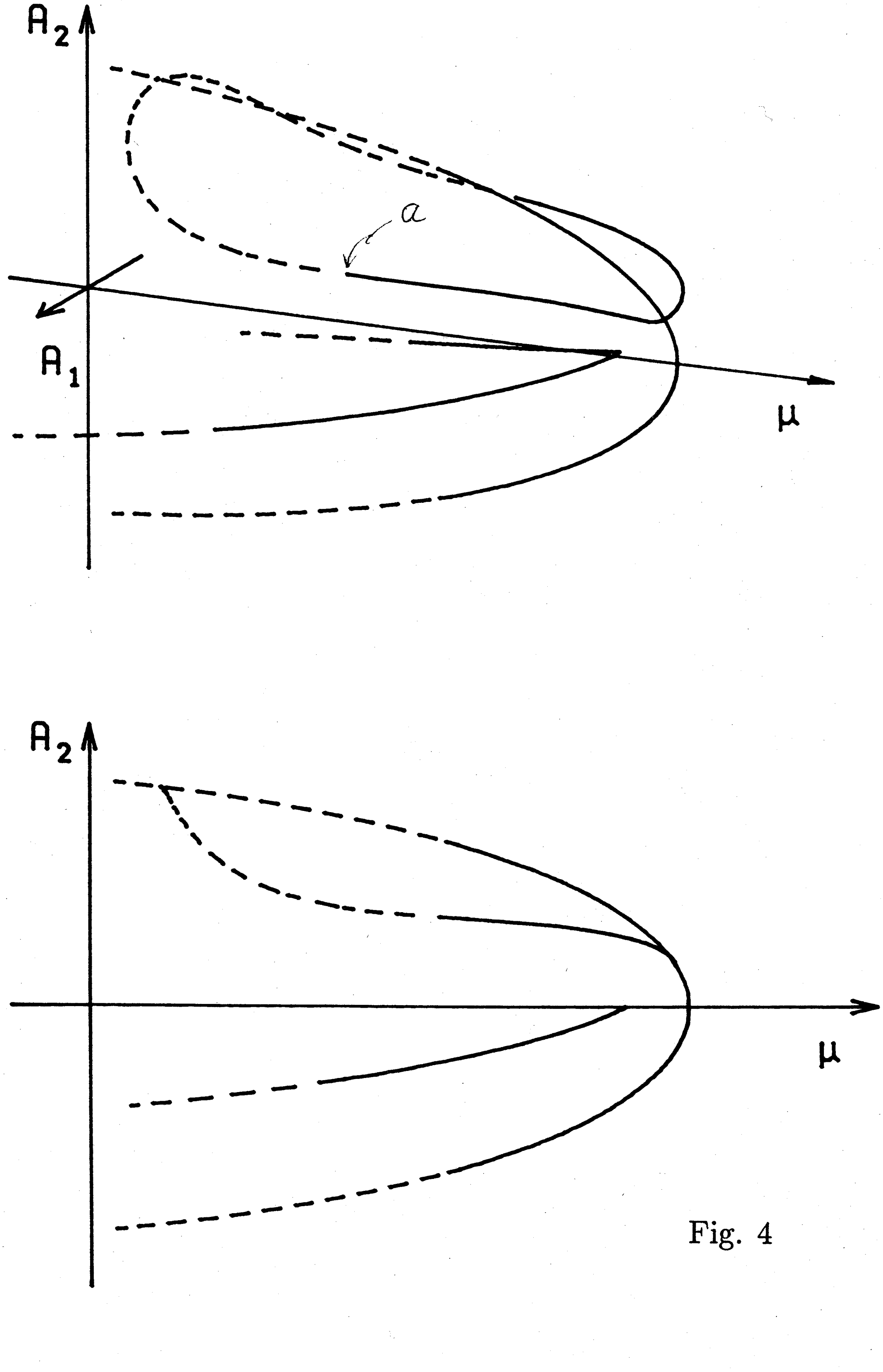
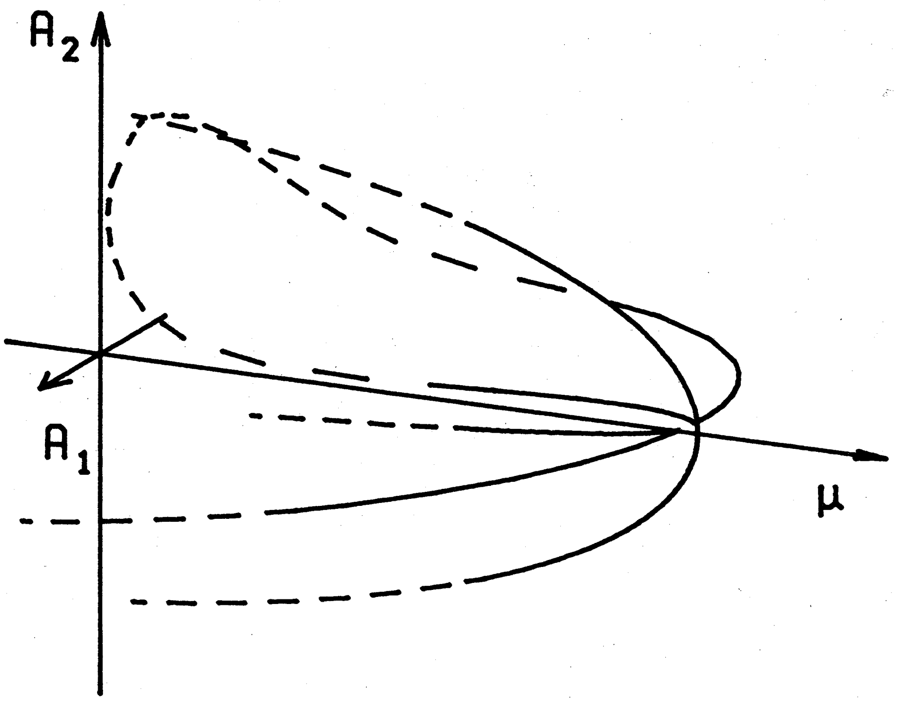


Fig. 4



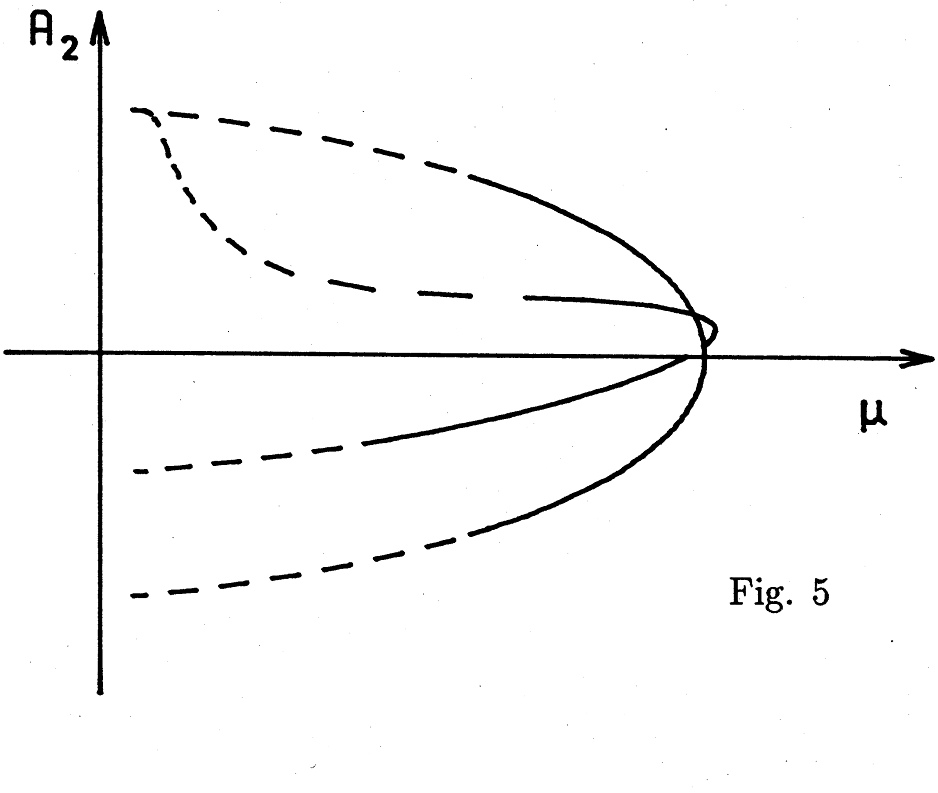


Fig. 5

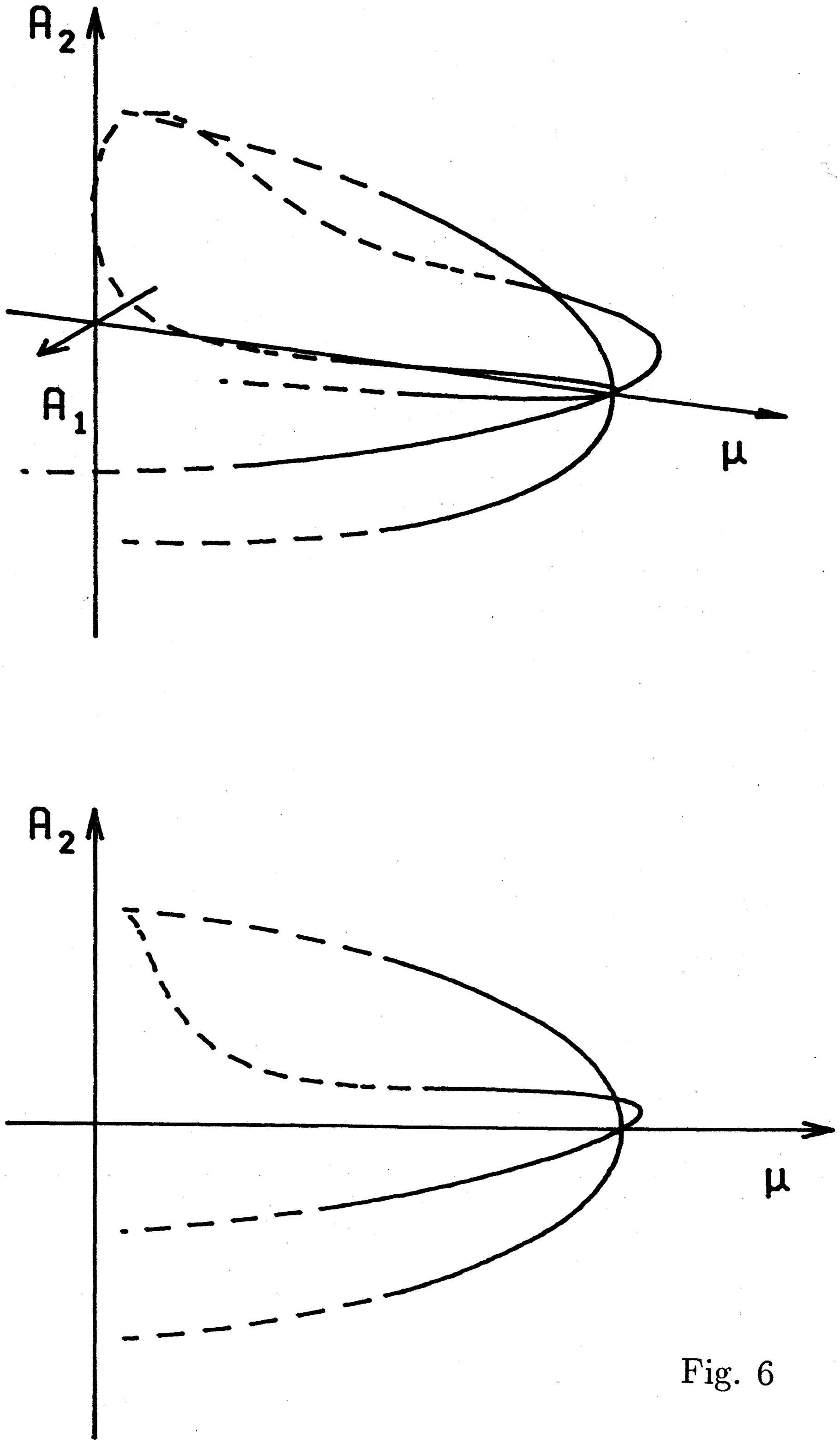
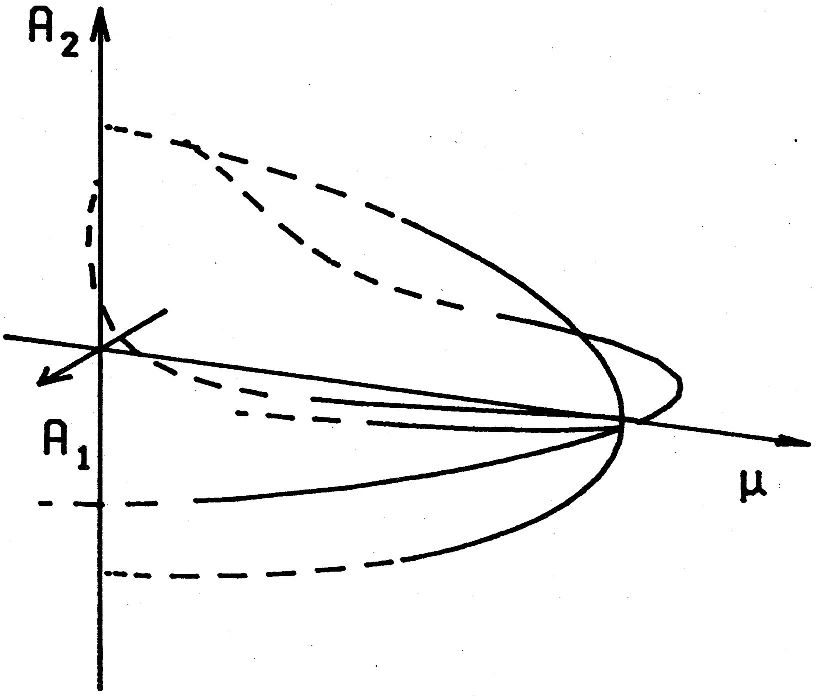


Fig. 6



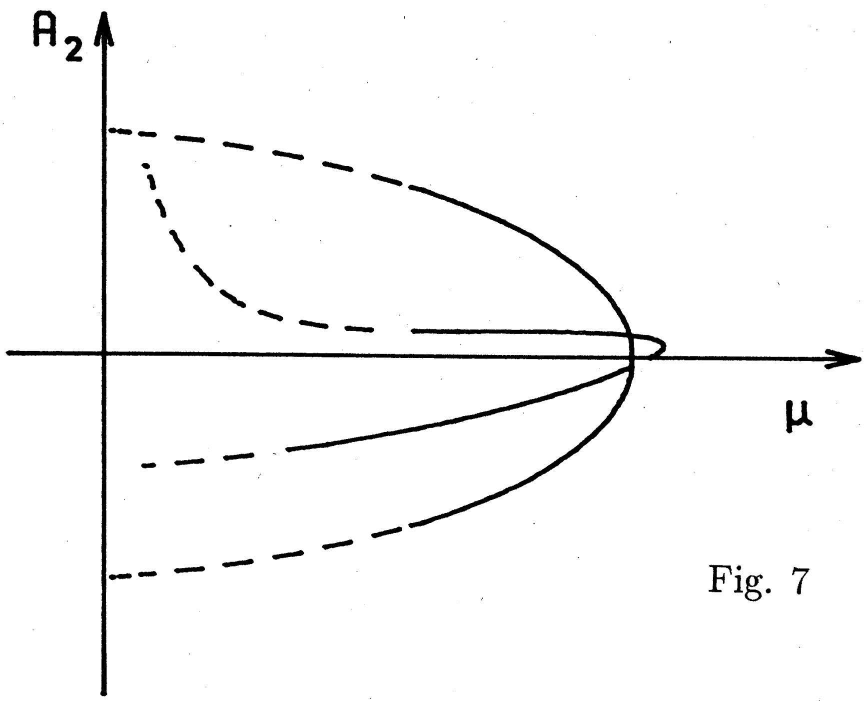


Fig. 7

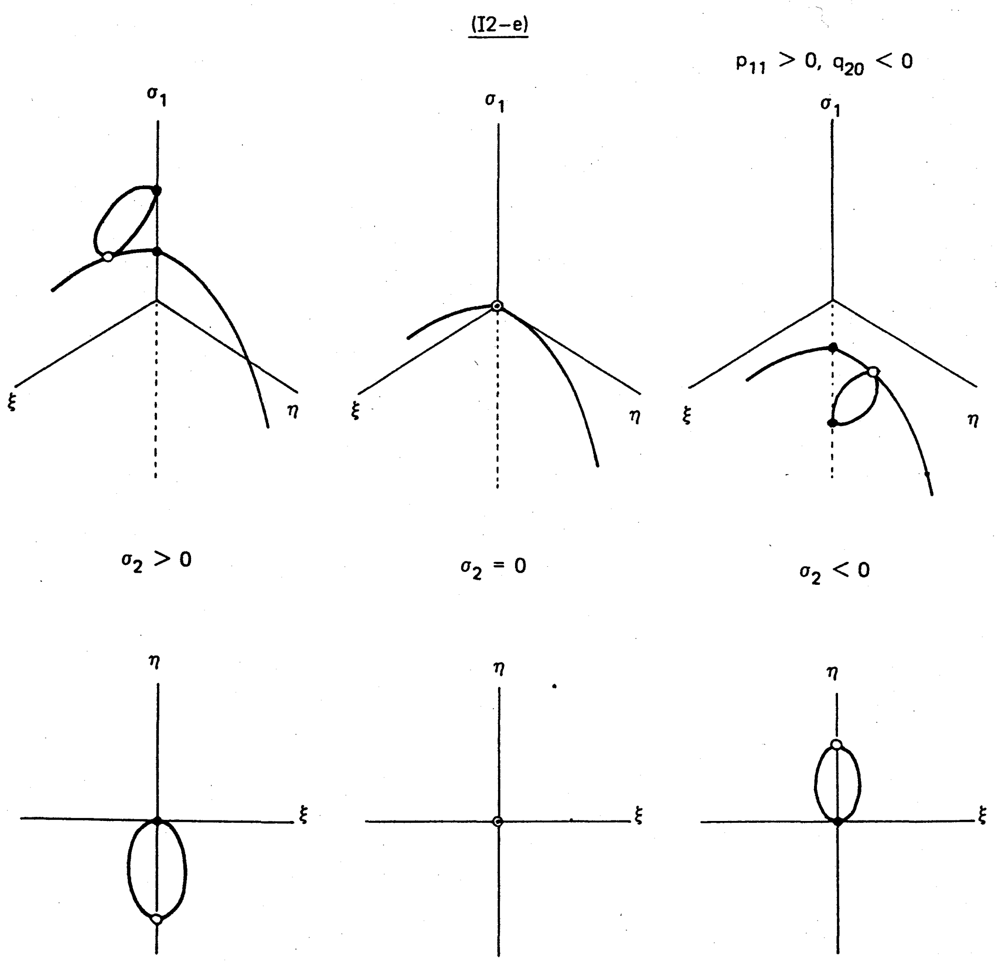


Fig. 8

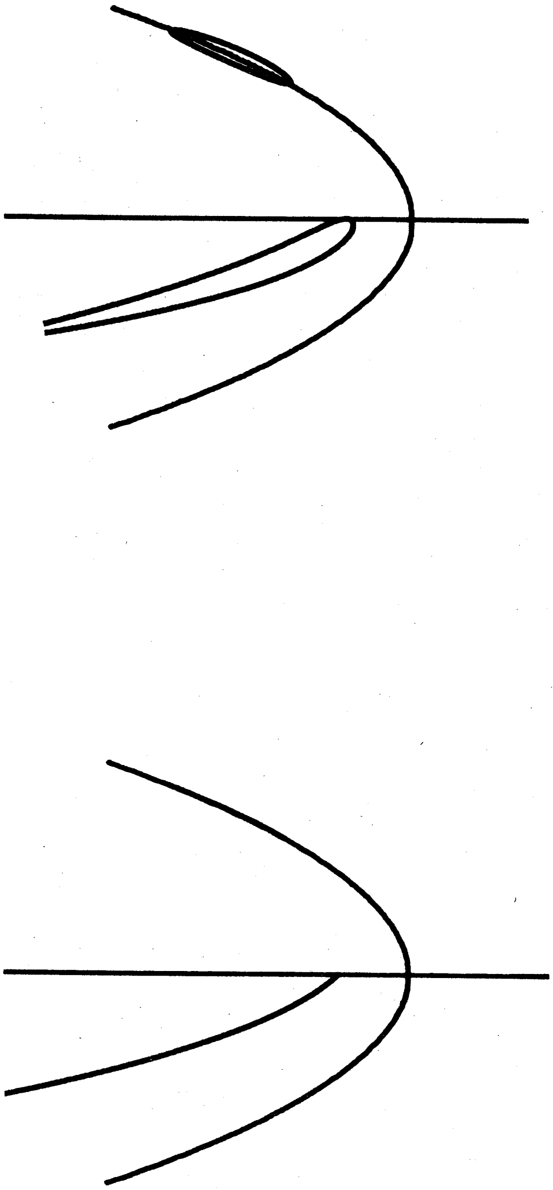


Fig. 9

Fig. 10

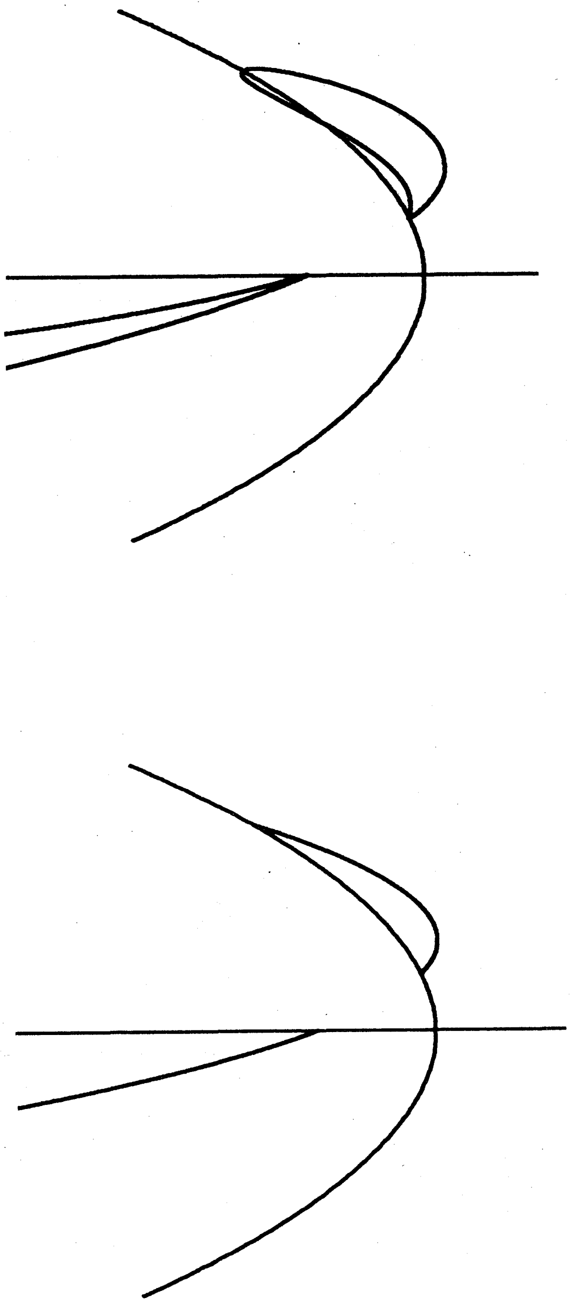


Fig. 11

Fig. 12

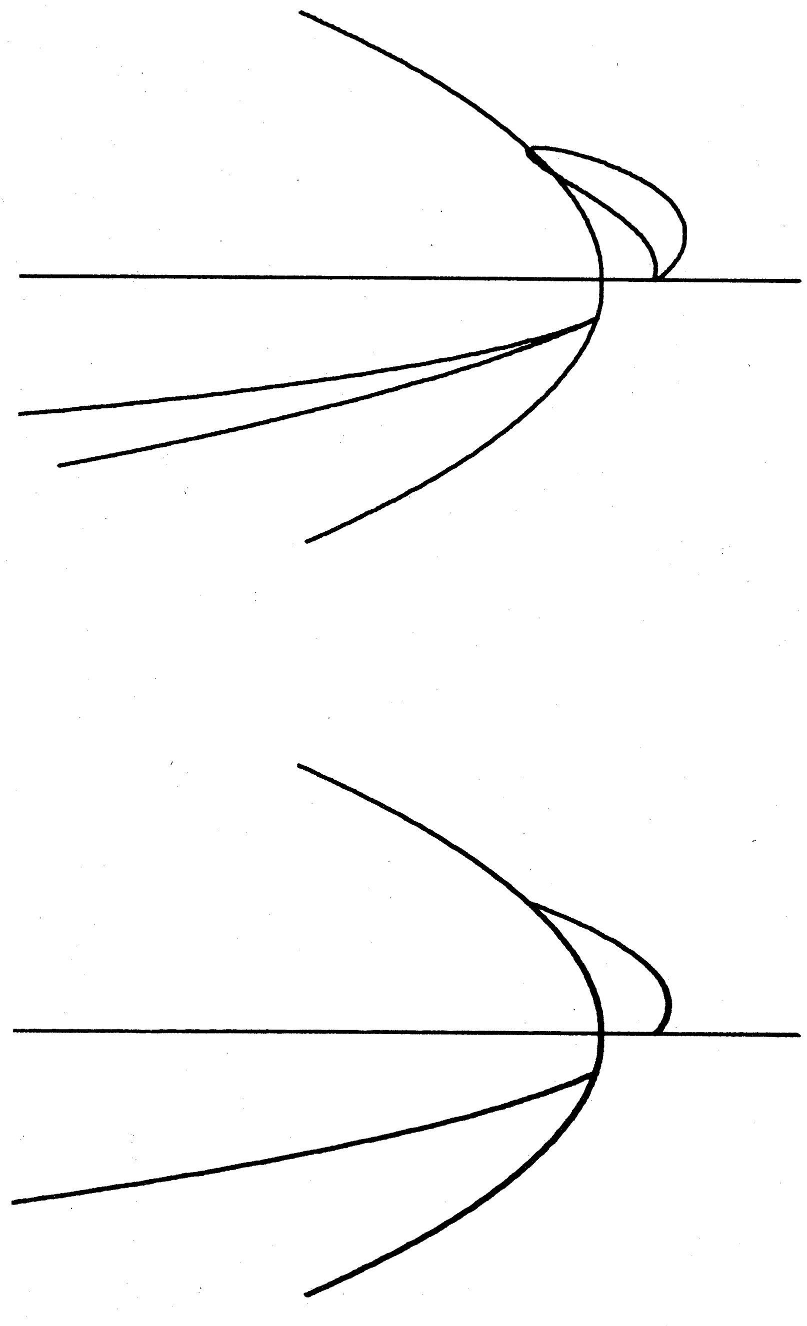


Fig. 13

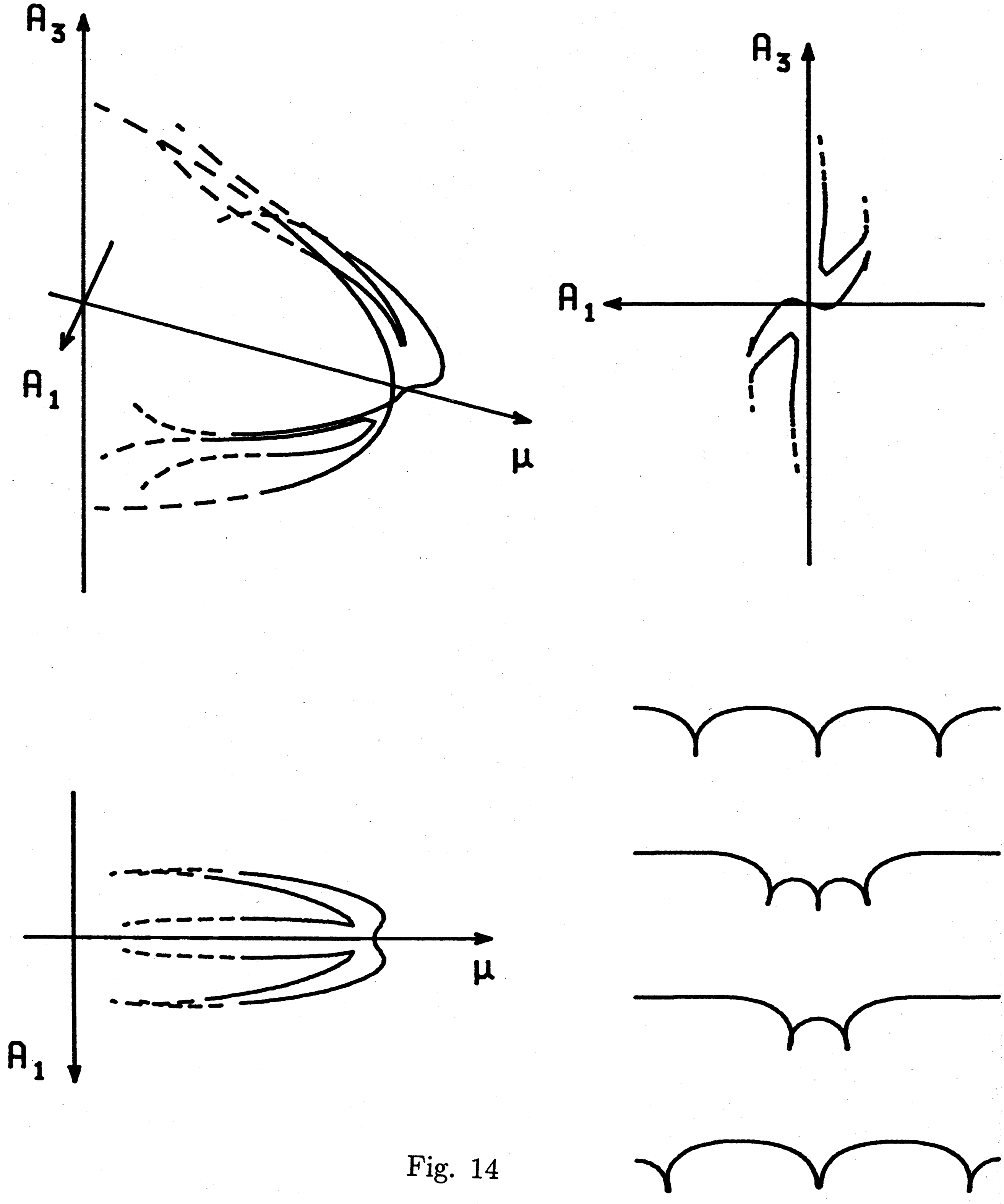


Fig. 14

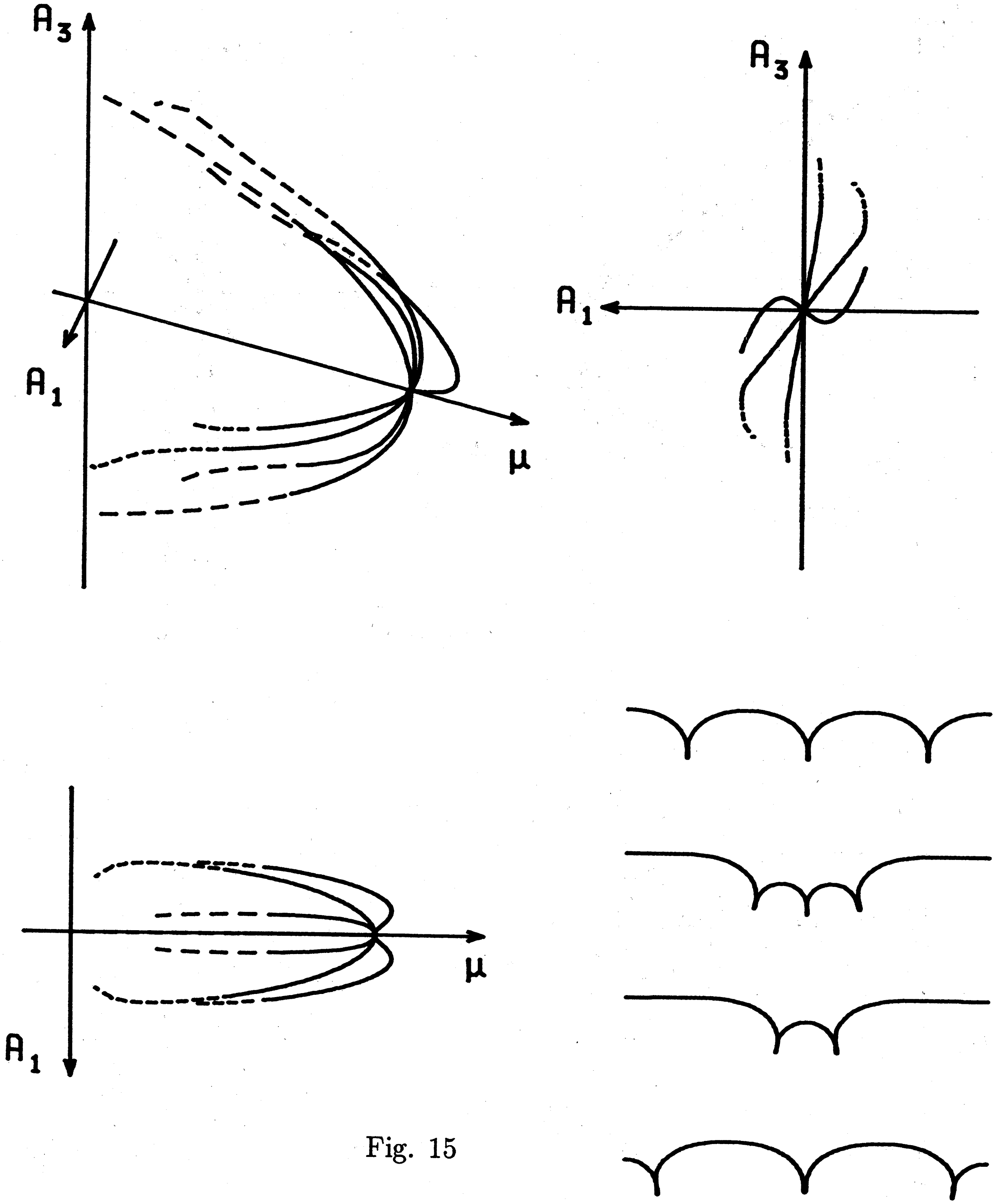
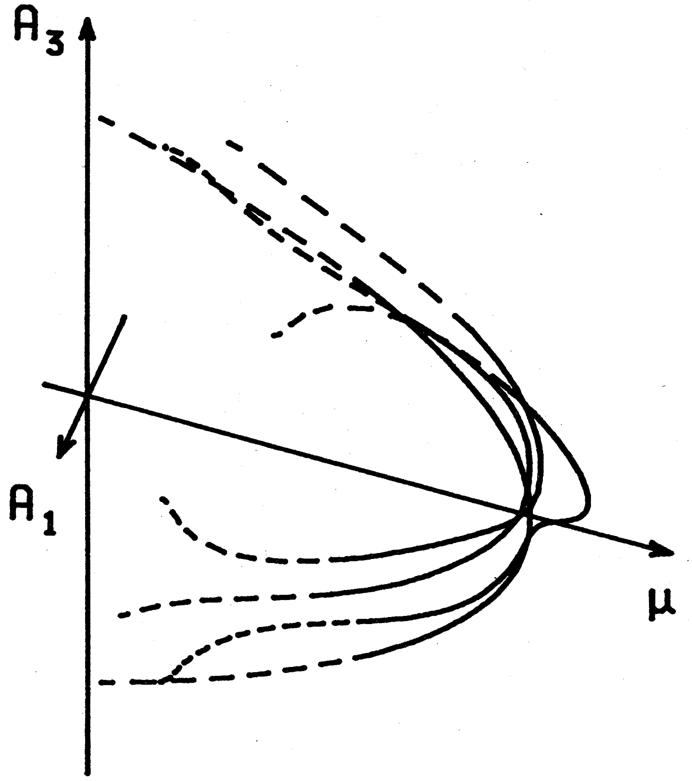
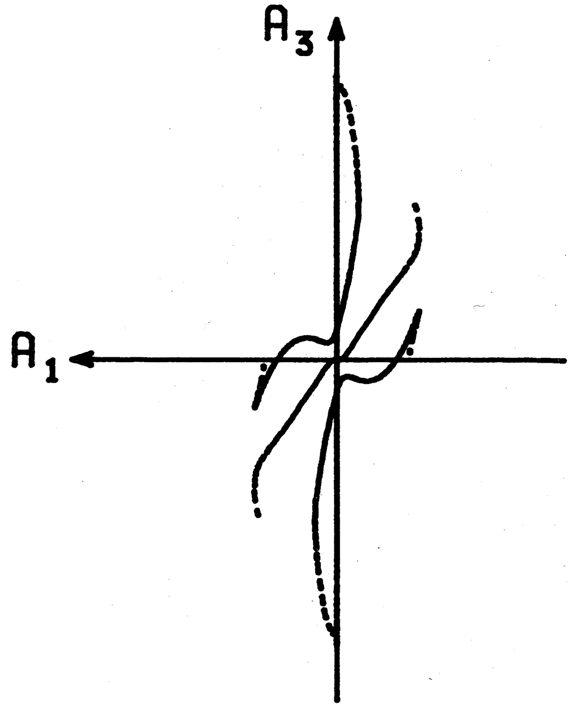
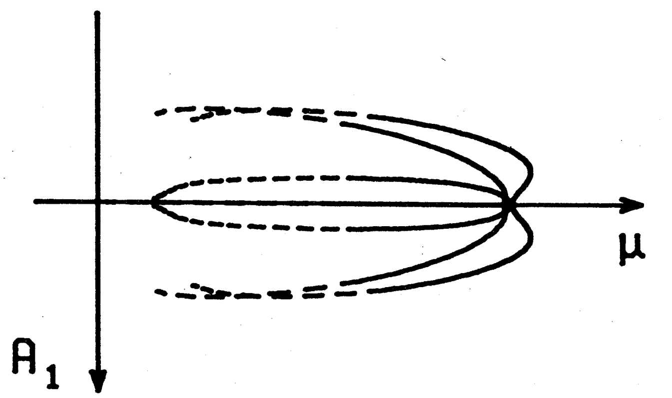


Fig. 15







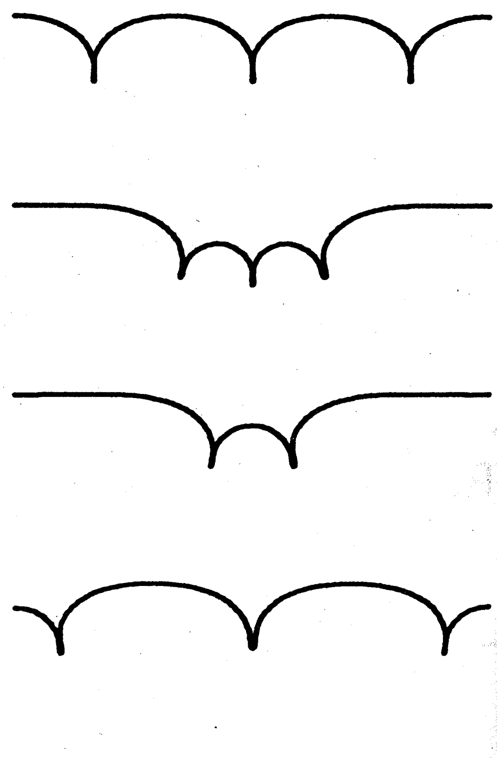


Fig. 16

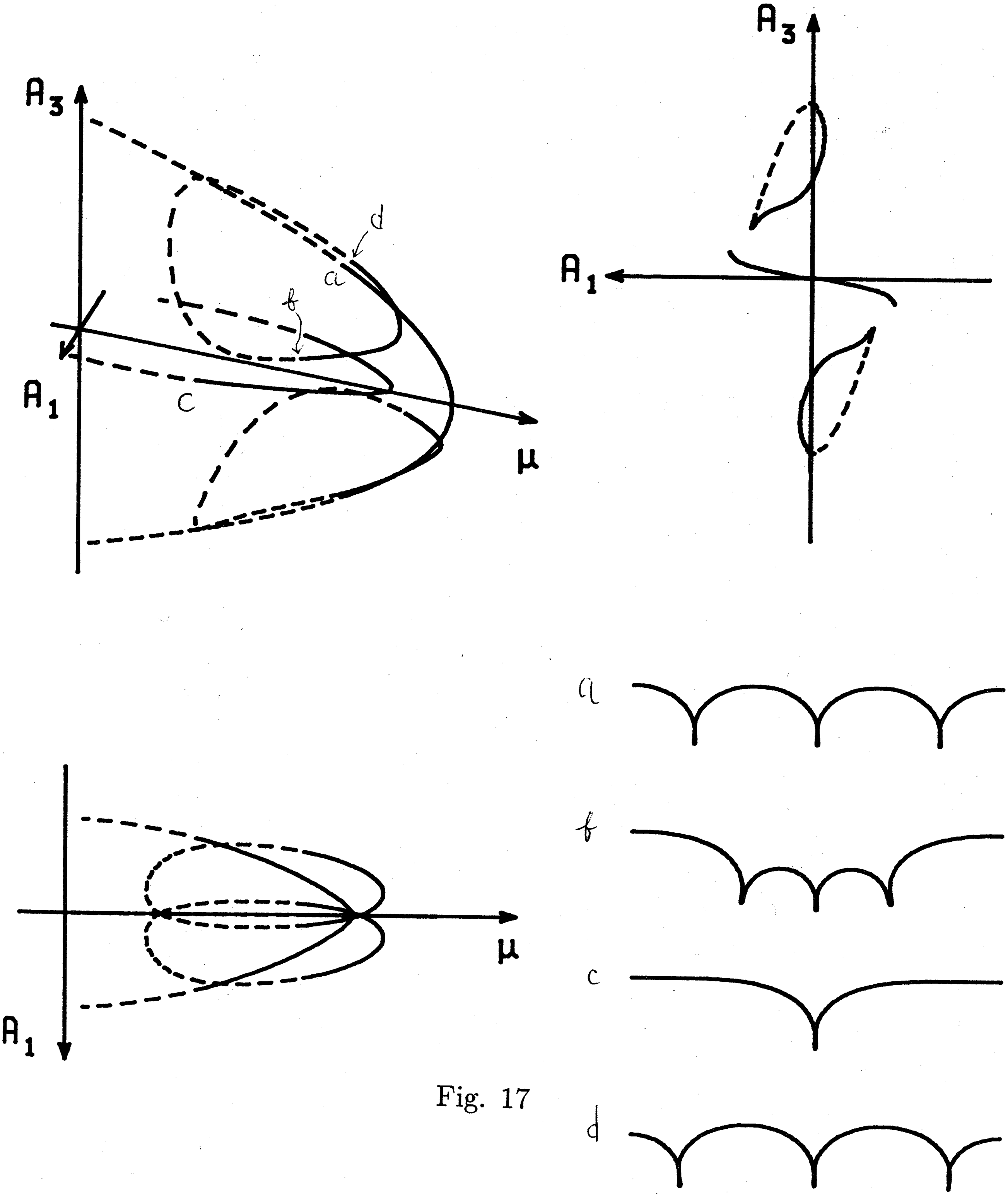


Fig. 17

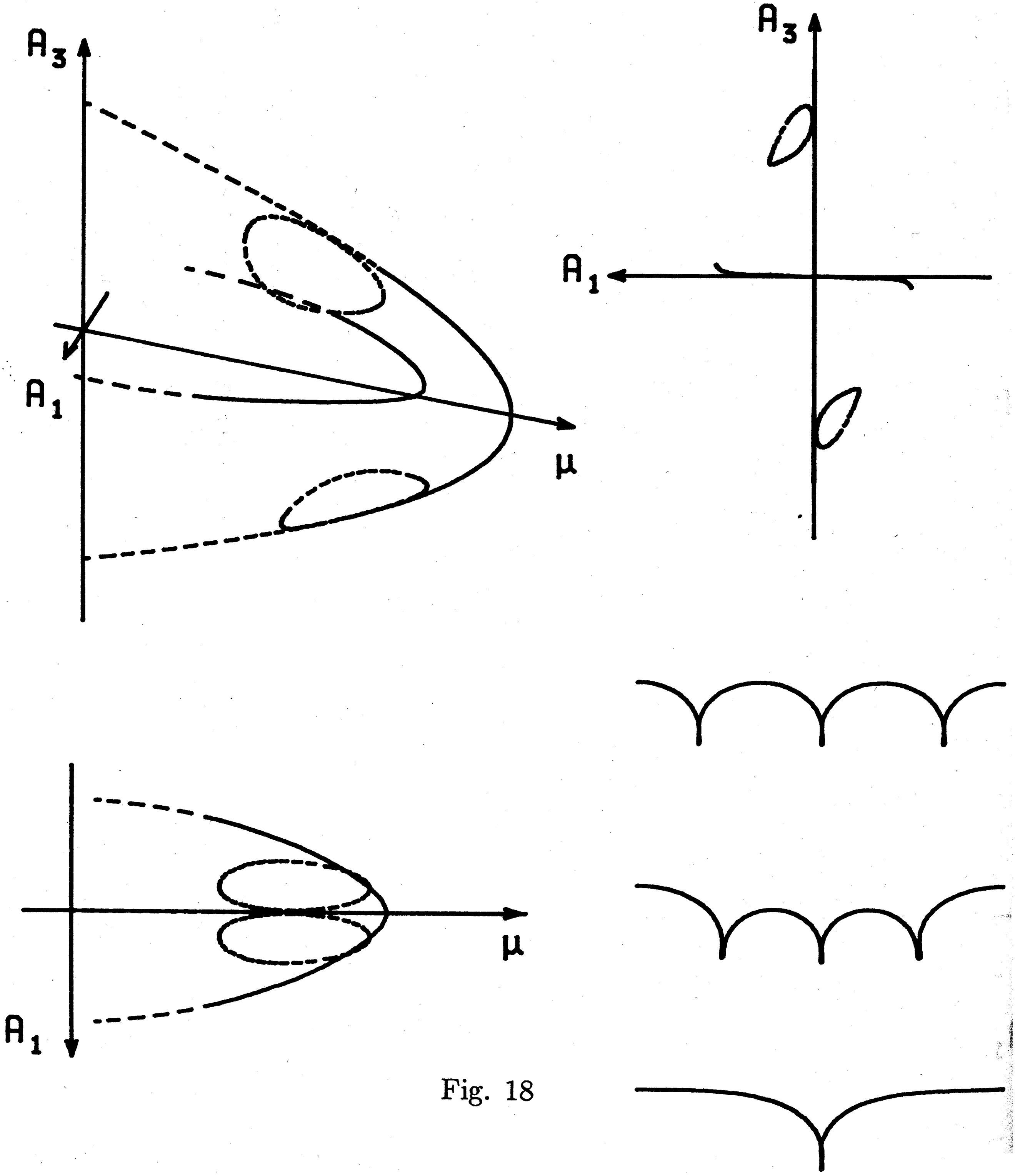


Fig. 18

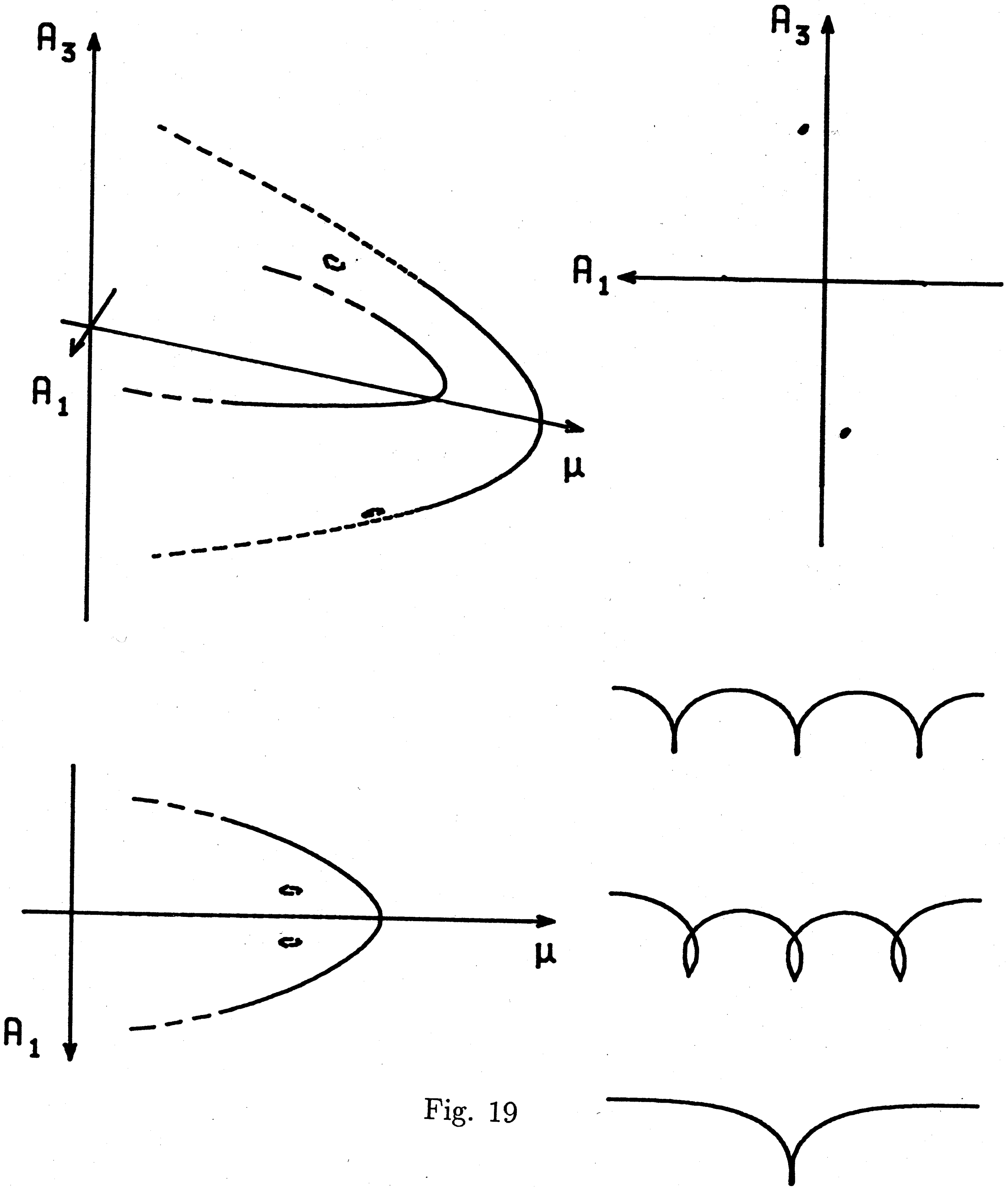


Fig. 19

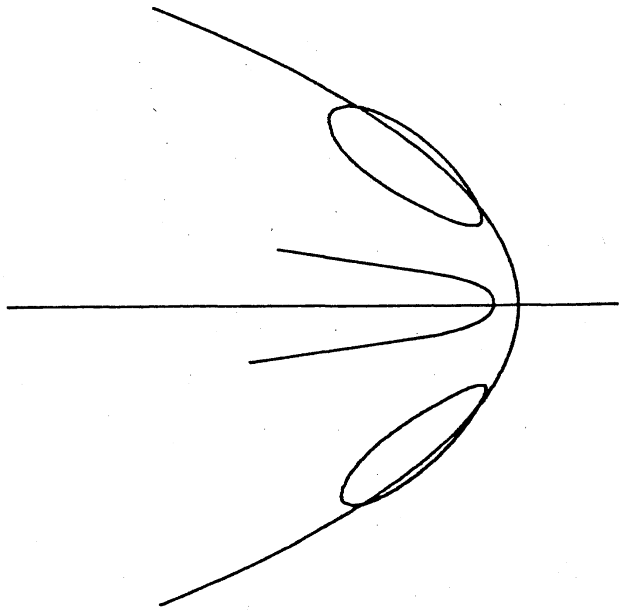


Fig. 20

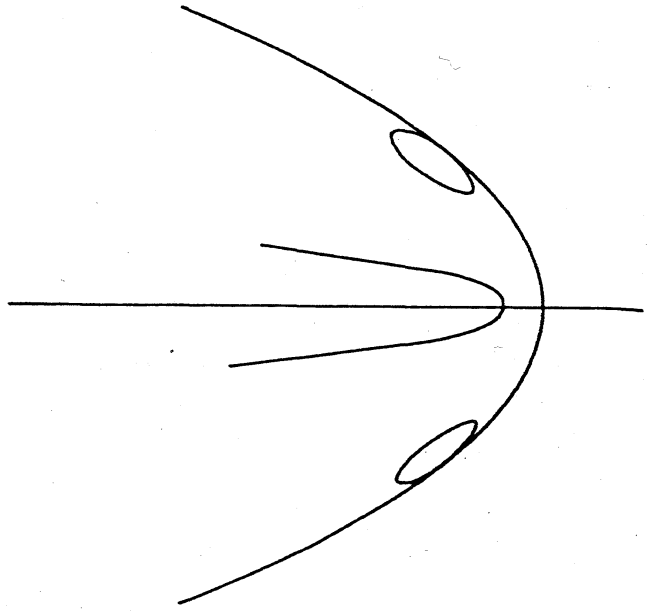


Fig. 21

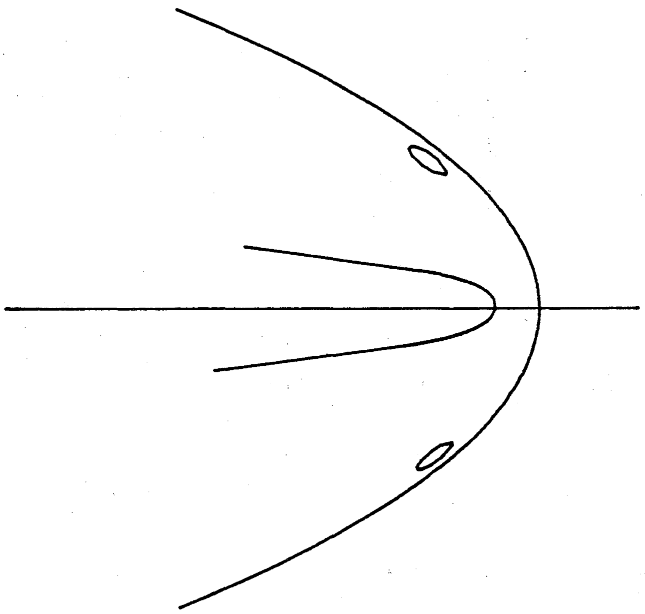


Fig. 22

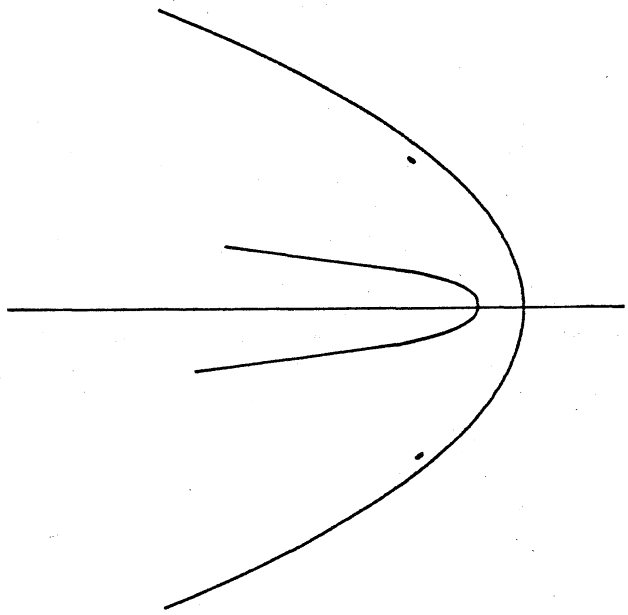


Fig. 23