数理解析研究所講究録
第 740 巻 1991 年 269-280

Nonlinear eddies and waves in planetary fluids

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1. Introduction

In recent years a concept of nonlinear Rossby modes has received much attention pursuant to an explanation of the longevity of various coherent structures in planetary fluids such as atmospheric blocks, Jovian eddies, various ocean eddies and the Kuroshio steady meander. The concept itself is not new and can be traced back to Scott-Russell's discovery of solitary water waves more than 150 years ago. The recent progress in planetary fluid dynamics, however, enriched the classical field and catalogued several important coherent structures with possible applications even in other fields such as plasma physics. Those are, for example, planetary solitary waves (cf. MALANOTTE-RIZZOLI, 1982), modons (STERN, 1975; LARICHEV and REZNIK, 1976; FLIERL et al., 1980; MCWILLIAMS, 1980) and IG eddies (cf. WILLIAMS and YAMAGATA, 1985). One important property to be noted here is that the nonlinear modes are distinct from finite amplitude planetary waves in a sense that they have no linear counterparts.

When we are interested in their life cycle, forcing and dissipation of the potential vorticity q become very important. This is because q plays a fundamental role for those planetary structures (cf. HOSKINS et al.,1985; RHINES, 1986). In particular, it is very important to clarify how q is imparted to the fluid when generation of the nonlinear coherent structures is concerned. There are several ways to impart q to planetary fluids Those are wind stresses, weak frictional or eddy-driven stresses, mass and/or buoyancy fluxes, interactions with rotating planets (the "Jebar" effect), etc.

Recently, YAMAGATA and UMATANI (1987) discussed the problem of the bimodal behavior of the Kuroshio path, south of Japan by use of the Korteweg-de Vries equation with forcing and dissipation of the potential vorticity. They showed that a localized, large meander with a shape of a solitary wave may be produced by coastal step-like geometry when the upstream current is faster than the long Rossby wave speed. Even if the forcing due to the geometry torque is weak, the dynamical system has a chance to jump from a small meander state to the large meander state by capturing a large disturbance.

This is due to the existence of multiple equlibria under the supercritical condition. They demonstrated subsequently using the QG equation that the model Kuroshio can actually take a localized, large meander path as a result of direct interaction between the current and the step-like coastal geometry (YAMAGATA and UMATANI, 1989). They also found that the cyclonic eddy associated with the large meander is in the "almost-free" limit of the nonlinear QG equation.

The above work prompted us to slightly extend the idea on locally-induced nonlinear modes and multiple equilibria by considering Modons and IG eddies. In the present article we first show how modons are excited by a continuous supply of the potential vorticity. We then proceed to excitation of IG vortices by either a continuous source of mass or potential vorticity together with some comments on the recent experiment by DAVEY and KILLWORTH (1989). The final section gives a brief summary of the present work.

2. *Evolution of Modons*

We consider the barotropic quasi-geostrophic equation in the presence of forcing of relative vorticity ζ_b with a time constant λ_1^{-1} and Ekman-type dissipation with a time constant λ_2^{-1} Then the equation may be written

$$\zeta_t + J(\psi, \zeta) + \beta\, \psi_x = \lambda_1 \zeta_b - \lambda_2 \zeta, \qquad (2.1)$$

where $J(a,b) = a_x b_y - a_y b_x$, $\psi(x,y)$ is the geostrophic streamfuction, ζ $(= \Delta\, \psi)$ is the relative vorticity and β denotes the meridional gradient of the the Coriolis parameter f at its mean value f_0. It is now well-known that the homogeneous form of (2.1) has the exact solution called Stern's stationary modon, which takes the form

$$\psi_b = -\,U \sin\theta\, \{r - R\, J_1(r(\beta/U)^{1/2})/J_1(R(\beta/U)^{1/2})\} \qquad \text{for } 0< r < R$$

and

$$= 0 \qquad \text{elsewhere}, \qquad (2.2)$$

where $r^2=x^2+y^2$ and $\theta= \tan^{-1}[y/x]$, R is the modon radius, J_n is the n-th order Bessel function of the first kind and U satisfies the condition $J_2(R(\beta/U)^{1/2})= 0$ (hereafter we adopt the lowest zero point that satisfies $R(\beta/U)^{1/2} = 5.136...$).

We adopt the relative vorticity associated with the Stern's modon as the forcing ζ_b. Therefore

$$\zeta_b = -\beta R \sin\theta \, J_1(r(\beta/U)^{1/2})/J_1(R(\beta/U)^{1/2}. \qquad (2.3)$$

In the present section we discuss two cases, distinguished by relative importance between forcing and dissipation.

a. Forcing Balanced with Dissipation ($\lambda_1 = \lambda_2$)

Here we consider the case in which forcing is balanced with dissipation (cf. PIERREHUMBERT and MALGUZZI, 1984). Equation (2.1) may be written with using the same time constant λ

$$\zeta_t + J(\psi, \zeta) + \beta \psi_x = \lambda (\zeta_b - \zeta), \qquad (2.4)$$

Since we adopt (2.3) as the forcing ζ_b, $\zeta = \zeta_b$ ($\psi = \psi_b$) is one steady solution of (2.4). Under weak forcing and dissipation, i.e. for λ small, however, we might have a case in which the advection of relative vorticity can be neglected at the lowest order of approximation. The steady , linear version of (2.1) is then

$$\lambda \Delta \psi + \beta \psi_x = \lambda \zeta_b. \qquad (2.5)$$

Equation (2.5) is quite well-known in physical oceanography (cf. STOMMEL, 1948). By replacing the right hand side with two-dimensional Dirac δ-function $\delta(x)\delta(y)$, the Green's function of (2.5) is easily obtained as

$$G = -(2\pi)^{-1} e^{-(\beta/2\lambda)x} K_0((\beta/2\lambda) r), \qquad (2.6)$$

where K_0 is the modified Bessel function of the second kind of order zero. The complete soulution is formally written as

$$\psi = \lambda \int G(x-x^*, y-y^*) \zeta_b(x^*, y^*) \, dx^* dy^*. \qquad (2.7)$$

There are two remarkable features of the linear solution. Firstly, the magnitude of the solution is proportional to that of forcing. Secondly, it is asymmetric in the zonal direction. The second feature is clearly seen in the asymptotic form of G (YAMAGATA, 1976; RHINES 1983). As $r >> 2\lambda/\beta$, it follows that

$$G \approx - (4\pi r\beta/\lambda)^{-1/2} e^{-(\beta/2\lambda)r(1+\cos\theta)}. \qquad (2.8)$$

Currents decay algebraically to the west of the forcing but decay exponentially within the Stommel boundary layer of order λ/β in any other direction.

In order to check the above possibility of multiple steady states, we integrated Eq. (2.4) using ARAKAWA (1966)'s formulation for the Jacobian term with a leapfrog scheme (cf. YAMAGATA and UMATANI, 1989). The model ocean is a channel (2000 km × 1000 km) with a cyclic condition in the zonal direction. The grid spacings are $\Delta x = \Delta y = 10$ km. Since R is assumed to be 150 km, the number of grid spacings per a modon diameter is 30. This number gives a reasonable resolution of the modon structure (cf. MCWILLIAMS et al., 1981). The parameter β (= 1.92×10^{-11} $cm^{-1}s^{-1}$) is evaluated at a reference latitude of 33 °N.

The results are summarized in Figure 1, where the normalized maximum magnitude of ζ for realized steady states is shown as a function of λ. It is seen that the distinct high and low amplitude states exist when λ is smaller than 0.3×10^{-1} /day. The criterion may be interpreted in the following way. Since the mean square vorticity of quasi-geostrophic Stern's modon is $\beta^2R^2/2$ (cf. STERN,1975), the time for a particle to circulate about the eddy once will be given by $2\pi\sqrt{2}/(\beta R)$, which corresponds to about 36 days in the present model. If a time scale (λ^{-1}) of forcing the modon is less than the characteristic time scale given above, a fully nonlinear solution will be excited. The magnitude of this high amplitude state is now independent of λ, whereas the magnitude of the low amplitude state increases almost linearly with increasing λ for a sufficiently small λ. It should be noted that Figure 1 resembles Figure 2 of YAMAGATA and UMATANI (1987), in which excitation of a planetary shear soliton was discussed as a conceptual model of the Kuroshio large meander. This suggests the existence of a generalized theory for the present type of problems*.

* The simplest example will be a swing with a thrust against friction. If the thrust exceeds a certain critical magnitude, the swing will begin to rotate around the axis. Once it rotates, a weak

Figure 2 shows streamfunctions for the two distinct states for $\lambda=0.1 \times 10^{-1}$ /day. The linear solution shows a remarkable east-west asymmetry as expected from the linear theory.

b. Inviscid Response versus Viscous Response

In general characteristic time of forcing the nonlinear structure is not always equal to that of dissipation. One typical example is an inviscid problem ($\lambda_2= 0$) , for which a steady response is not realizable any longer. Figures 3 and 4 demonstrate how such an inviscid system evolves from the initial condition of no motion. It is seen that the inviscid model sheds modons propagating eastward intermittently for $\lambda_1= 0.2 \times 10^{-1}$ /day (Figure 3). A similar phenomenon is also observed for $\lambda_1= 0.1 \times 10^{-1}$ /day (not shown). For even smaller value of λ_1 such as $\lambda_1= 0.5 \times 10^{-2}$ /day, however, only the low amplitude disturbance spreads west of the forcing as a long Rossby wave (Figure 4). Those experiments show that there exists a critical magnitude of forcing which divides the response between the low amplitude state consisting of long Rossby waves propagating westward and the high amplitude state consisting of shed modons which propagate eastward.

Increasing the dissipation rate λ_2 leads to suppression of the above shedding process as demonstrated in Figure 7, in which various streamfunction patterns at day 300 are shown for $\lambda_1 = 0.2 \times 10^{-1}$ /day and λ_2 from zero through 0.2×10^{-1} /day. Another noticeable effect of dissipation is obviously the reduction of eddy amplitude.

3. Evolution of IG Eddies

Quite recently, UMATANI and YAMAGATA (1989) have demonstrated, using the eddy-resolving limited area OGCM, that the warm nonlinear ocean eddies are excited off Costa Rica by strong northers in winter. Those eddies not only resemble observed ones but also appear to be governed by the singular dynamical process--IG dynamics--as anticipated by MATSUURA and YAMAGATA (1982) using a one-layer reduced gravity model. In particular, UMATANI and YAMAGATA (1989) have suggested that those nonlinear coherent structures may be successively generated under the steady supply of

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thrust may keep it going. The same weak thrust may also excite an ordinary oscillation from a state of no motion. Therefore two states may exist under the same weak forcing.

potential vorticity from the atmosphere. We will discuss this problem in the present section.

For the present purpose we adopt a one-layer reduced gravity model with a rigid lid. It is well-known that the shallow water equations work well when the active layer is confined within the upper part of the ocean by a steep thermocline. Let L, $L/(\beta L_R^2)$, V and $g^{*-1} f_0 VL$ denote scale factors for horizontal coordinates (x, y), time t, velocity (u, v) and interface depression η from the mean depth H, where $L_R = C_g/f_0$ is the deformation radius and C_g($=\sqrt{g^*H}$) the internal long-wave speed. Then, introducing three nondimensional parameters β^* ($=\beta L/f_0$: beta parameter), ε^* ($=V/(f_0 L)$: Rossby number) and s^*($=L_R^2/L^2$: stratification parameter), we have

$$\beta^* s^* \frac{Du}{Dt} - (1 + \beta^* y)v = - \eta_x,$$
$$\beta^* s^* \frac{Dv}{Dt} - (1 + \beta^* y)u = - \eta_y,$$
$$\beta^* \frac{D\eta}{Dt} + (1 + \frac{\varepsilon^*}{s^*}\eta)(u_x + v_y) = 0,$$
$$\frac{D(\ )}{Dt} \equiv (\ )_t + \frac{\varepsilon^*}{\beta^* s^*}[u(\ )_x + v(\ )_y]. \qquad (3.1)$$

To derive the IG equation from the shallow water equations, we need to introduce the following relations among the three parameters:

$$\beta^* << O(1), \ \ \varepsilon^* = E\,\beta^{*2}, \ \ s^* = S\,\beta^*, \qquad (3.2)$$

where E and S are numbers of O(1) (cf. YAMAGATA, 1982: WILLIAMS and YAMAGATA, 1985). Then we find

$$\eta_t - \eta_x - \beta^* \left(ES^{-1}\eta\eta_x + S\,\Delta\eta_x - 2y\eta_x - EJ(\Delta\eta,\eta)\right) = W, \qquad (3.3)$$

where W is the forcing due to either direct mass source or Ekman pumping of the wind stress. A remarkable property of the above IG equation is that only anticyclonic eddies are long-lived due to the balance between the scalar nonlinearity and the planetary dispersion.

DAVEY and KILLWORTH (1989) have recently shown using a shallow water system that a sufficiently strong constant mass source generates a chain of discrete anticyclonic eddies. LINDEN (1989, personal communication) also reported a similar phenomenon observed in laboratory experiments with the planetary β-effect. According to DAVEY and KILLWORTH (1989), a necessary condition for successive formation of eddies can be reduced to

$$\varepsilon^* >> \beta^{*2} s^*. \qquad (3.4)$$

It is immediately seen that the condition (3.1) for the IG dynamics certainly satisfies the above inequality. Furthermore, the three nondimensional parameters in their experiments suggest that the anticyclonic eddies may actually be dominated by the IG dynamics.

Therefore we report here some results using Eq. (3.2) with the forcing similar to the one adopted by DAVEY and KILLWORTH (1989). The forcing function W is then

$$W = \begin{cases} \frac{1}{2}[1+\cos(\pi r/r_0)], & r < r_0 \\ 0. & r > r_0 \end{cases} \qquad (3.5)$$

The parameter β^* is assumed to be 0.13 (corresponding to the Costa Rica eddies) with E = S = 1 and $r_0 = 1$ in our experiment. The method to solve the forced IG equation is exactly the same with the one adopted in MATSUURA and YAMAGATA (1982). The evolution of η shows clearly how the anticyclonic IG eddies are shed west of the forcing (Figure 5). As expected, this sequence is quite similar to Figure 9 of DAVEY and KILLWORTH (1989). Changing the sign of the forcing (a sink of mass), however, leads to a totally different result as shown in Figure 5, in which long Rossby waves excited by the sink propagate westward.

If the nondimensional amplitude of the forcing (which is equivalent to a reciprocal of forcing time scale) is reduced by a factor of β^*, the solution becomes rather linear so that changing the sign of the forcing does not affect the response except for the sign of η (not shown). In other words, the nonlinear IG eddies cannot be excited for such a weak forcing. This is quite reasonable since the nondimensional time for a particle to circulate about the quasi-geostrophic eddy is $O(\beta^{*-1})$ as shown for the Stern's modon in the previous section.

4. Summary

We have shown that nonlinear Rossby modes (modons and IG eddies as examples) can be excited by a sufficiently strong constant forcing of potential vorticity. In the case of IG eddies the forcing must be a positive one. When the time scale of forcing the nonlinear modes is equal to that of dissipation, two (linear and nonlinear) equilibrium states can be produced, depending on the initial condition, for a sufficiently weak forcing. This has been demonstrated for the Stern's modon in the present paper.

When the system is inviscid, a sufficiently strong, steady forcing may generate a sequence of propagating nonlinear coherent structures. One typical example seems to be provided by the successive formation of warm eddies off Costa Rica as demonstrated by UMATANI and YAMAGATA (1989). A weak forcing, however, generates linear long Rossby waves which propagate westward. This is generally believed to occur in tropical oceans.

The criterion which divides the high amplitude (nonlinear) state and the low amplitude (linear) state may be interpreted in terms of a simple measure, which is a ratio of a time scale of forcing the nonlinear structure to a time for a particle to circulate about the nonlinear eddy once. If the ratio exceeds unity, a linear Rossby wave response will be dominant. If the ratio is smaller than unity, nonlinear Rossby modes will be excited. The latter means a strong kick to the planetary fluid.

A simple concept developed here may be generalized to any forced nonlinear evolution equation with a nonlinear coherent structure as a free solution. One way to excite such structures externally is to apply a sufficiently strong forcing to a fluid as SCOTT-RUSSEL (1844) described: ",when the boat suddenly stopped--not so the mass of water in the channel which it had put into motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled downward with great velocity, assuming the form of a large solitary elevation..."

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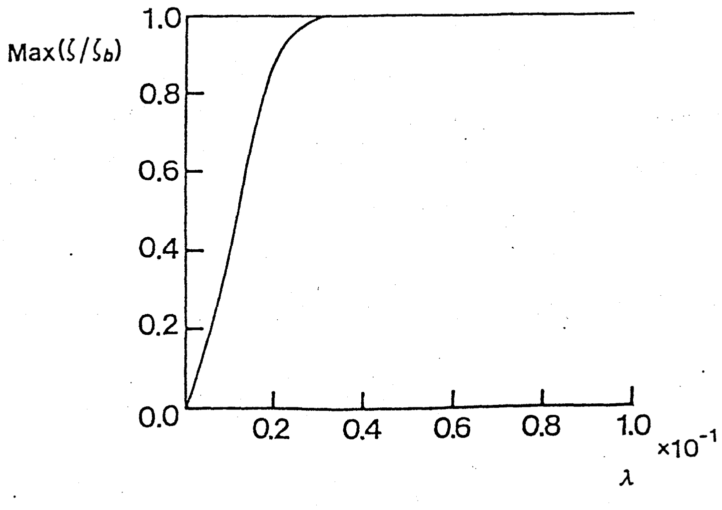

Figure 1. Normalized magnitude of maximum ζ for the final steady state as a function of λ.

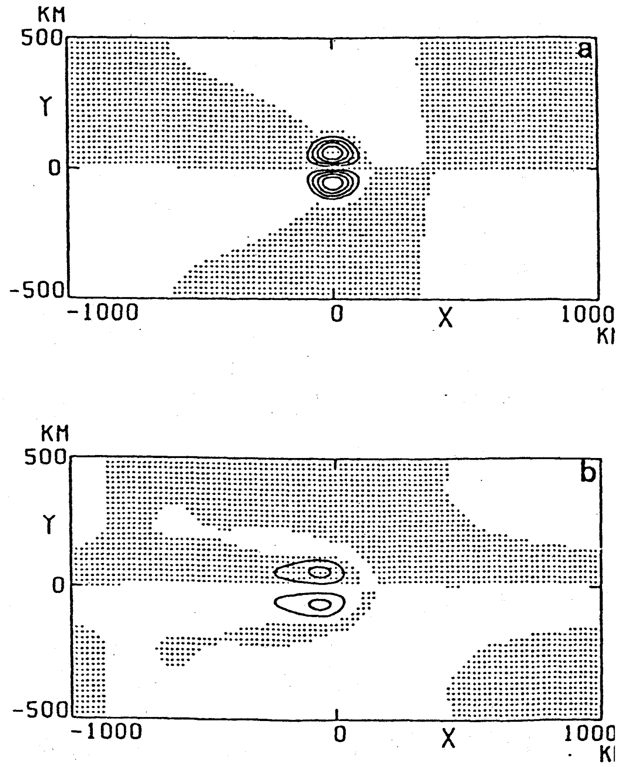

Figure 2. Streamfunction patterns for two distinct states for $\lambda = 0.1 \times 10^{-1}$ /day. (a) High amplitude state of Stern's modon. (b) Low amplitude state of damped long Rossby waves. The stippled areas represent negative values. The contour interval is 1.025×10^7 cm^2/s, which corresponds to one fifth of the maximum value of ψ_b.

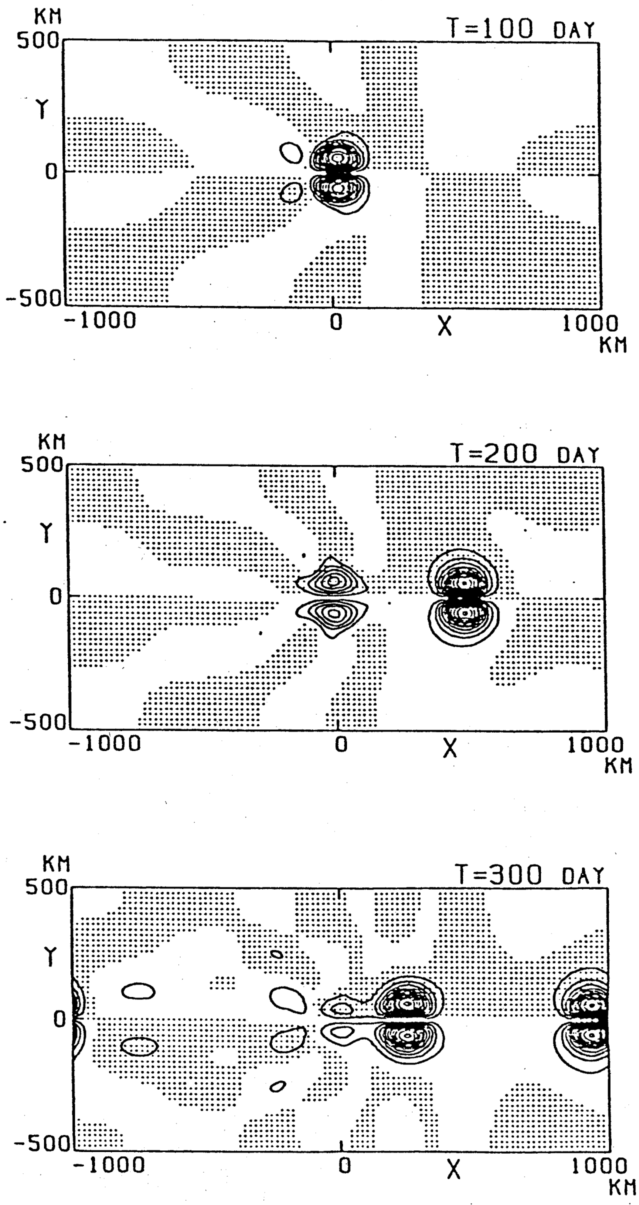

Figure 3. Evolution of streamfuction patterns for $\lambda_1 = 0.2 \times 10^{-1}$ /day and $\lambda_2 = 0$.

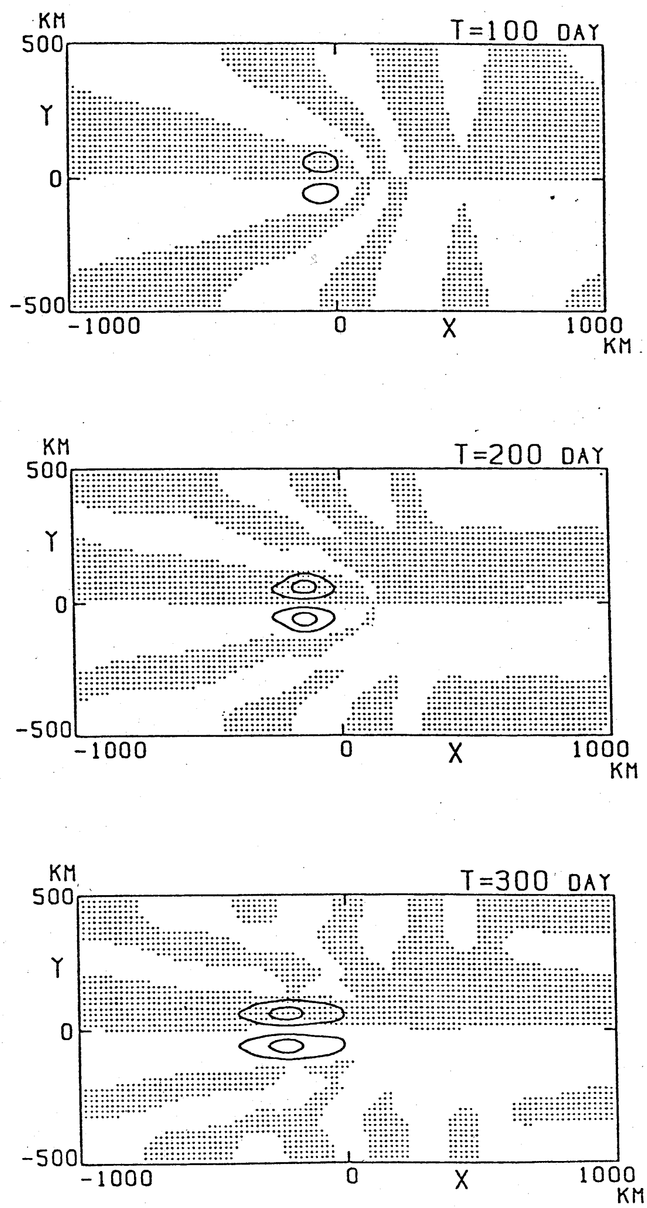

Figure 4. As in Figure 3 but for $\lambda_1 = 0.5 \times 10^{-2}$ /day.

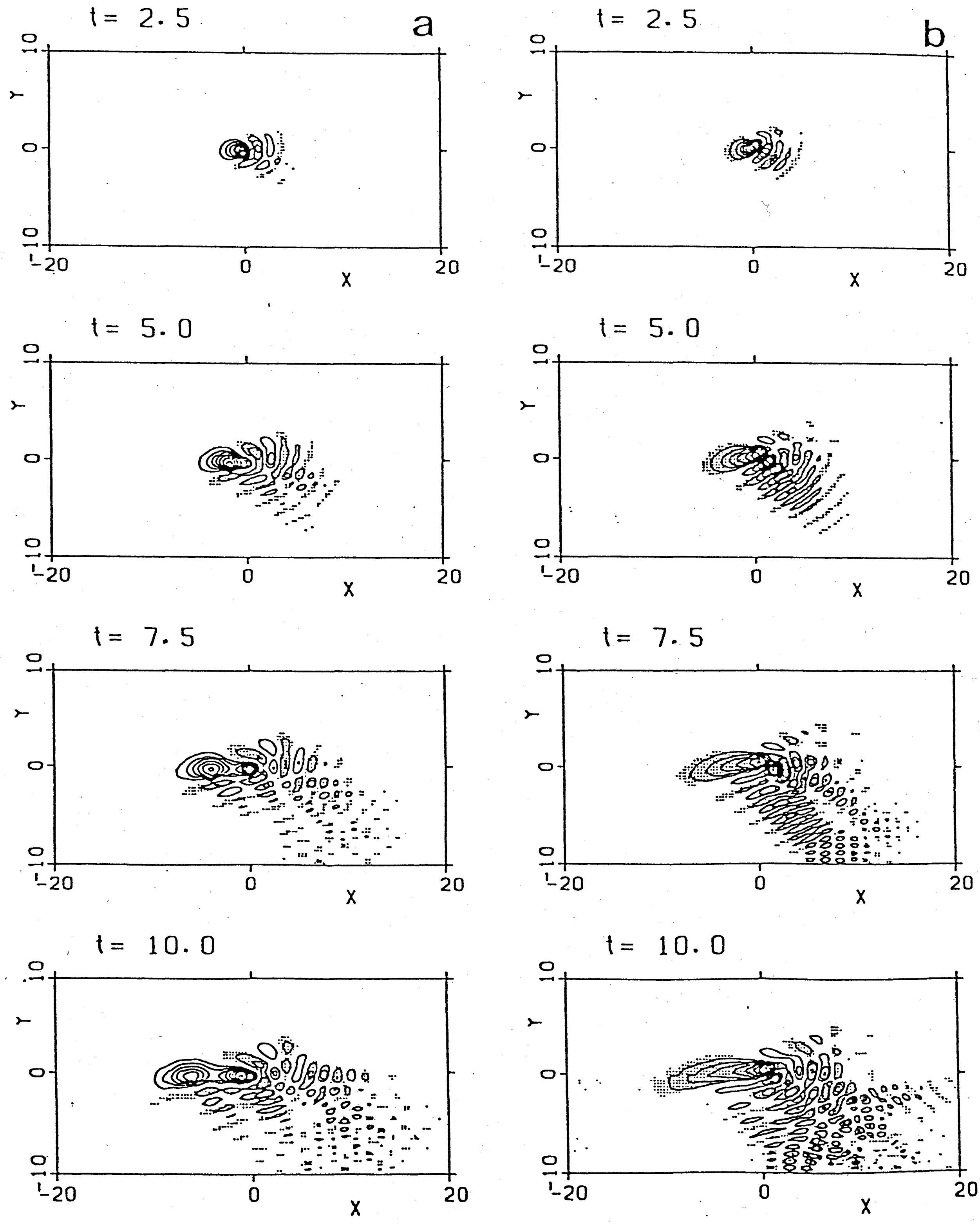

Figure 5. Evolution of the interface depression η. The forcing function is given by (3.5) in the text. (a) A case with positive forcing. (b) A case with negative forcing. The stippled areas represent negative values. The contour interval is 0.15.