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On the integral representation of $\zeta(3)$ in terms of elliptic modular forms

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§0. Introduction

In this paper the authors show the integral representation of $\zeta(3)=\sum_{n=1}^{\infty} n^{-3}$. Namely $\zeta(3)$ is represented by the Mellin transformation of a modular form relative to $\Gamma_1(6)$. Our result is based on the theory developed by Apéry [A], Beukers [B1] and others.

In 1978 Apéry introduced his sequences $\{a_n\}$, $\{b_n\}$. Those sequences satisfy the recurrence relation

$$n^3u_n + (n-1)^3u_{n-2} = (34n^3-51n^2+27n-5)u_{n-1} \qquad (0\text{-}1)$$

with initial conditions $a_0= 1$, $a_1=5$ and $b_0=0$, $b_1=6$. And he showed the irrationality of $\zeta(3)$ by the approximation

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \zeta(3). \qquad (*)$$

Beukers-Peters[B-P] studied the generating functions $A(t) = \Sigma\, a_nt^n$, $B(t) = \Sigma\, b_nt^n$ for these sequences. As easily shown those two functions satisfy the following Fuchsian type differential equation with singularities $t = 0$, $\lambda = (1-\sqrt{2})^4$, $\lambda' = (1+\sqrt{2})^4$, ∞;

$$D:\ L(y) = 0, \qquad (0\text{-}2)$$

$$L(y) = 6, \qquad (0\text{-}3)$$

respectively, where L indicates the differential operator

$$L=(t^4-34t^3+t^2)(\tfrac{d}{dt})^3+(6t^3-153t^2+3t)(\tfrac{d}{dt})^2+(7t^2-112t+1)\tfrac{d}{dt}+(t-5). \quad (0\text{-}4)$$

As Peters[P] pointed out the differential equation (0-2) is closely related with the Picard-Fuchs equation

$$D_6:\ s(s-1)(9s-1)\frac{d^2z}{ds^2} + (27s^2-20s+1)\frac{dz}{ds} + (9s-3)z = 0$$

for the modular family of elliptic curves

$$\mathcal{F}_6:\ y^2+ (1+s)xy - (s^2-s)y = x^3- (s^2-s)x^2$$

relative to a congruence subgroup of $\Gamma = SL(2,\mathbb{Z})$:

$$\Gamma_1(6) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}):\ a \equiv 1,\ c \equiv 0 \pmod 6\right\}$$

So we proceed our study as the following way

(1) We construct the solutions $\omega(s)$, $\tilde{\omega}(s)$ of D_6 using a modular form $\omega(\tau)$ of weight 1 and the uniformizing function $s=s(\tau)$ relative to $\Gamma_1(6)$, where τ is a variable on $H = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$.

(2) We show the exact cnnection between D and D_6, and describe A(t) and B(t)in terms of $\omega(s),\tilde{\omega}(s)$. During this procedure we find the relation $t = s(9s-1)/(s-1)$ between the two variables t and s.

(3) We observe the behavior of A(t) and B(t) around $t=\lambda$. Then we can find a linear combination of A(t) and B(t)

$$\Phi(t) = C\, A(t) + B(t)$$

with trivial monodromy around $t=\lambda$ (the coefficient C is given by (4-3)). Hence $\Phi(t)$ is single valued holomorphic on $\{t\in\mathbb{C} : |t|< \lambda'\}$. By this property we can deduce

$$C + \frac{b_n}{a_n} \longrightarrow 0 \qquad (n \longrightarrow \infty),$$

by virtue of Apéry's relation (*) it indicates the equality

$$C + \zeta(3) = 0.$$

If we write down the coefficient c, we obtain the representation of $\zeta(3)$.

§1. Statement of the result

At first we have the properties of $\Gamma_1(6)$:

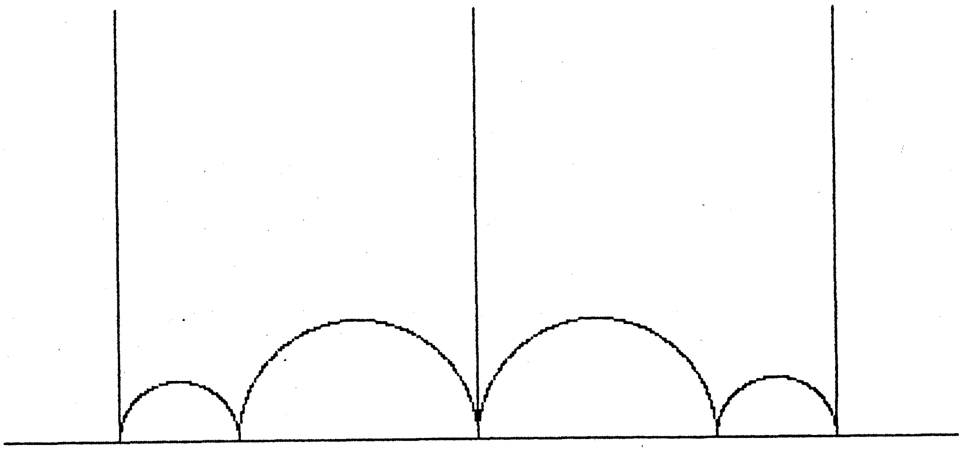
(i) $[\Gamma:\Gamma_1(6)] = 24$,

(ii) the quatient space $H/\Gamma_1(6)$ has 4 cusp points thier representatives are given by $\tau = 0,\ 1/3,\ 1/2,\ \infty$.

(iii) the genus of $\widehat{H/\Gamma_1(6)}$, the compactification of $H/\Gamma_1(6)$, is equal to 0,

(iv) the fundamental domain of $\Gamma_1(6)$ is pictured in fig.1.

FIG.1



According to the result of Klein (see [K] p.391) we can take $s(\tau)=\left\{\frac{\vartheta_2(0,3\tau)}{\vartheta_2(0,\tau)}\right\}^4$, where ϑ_2 denotes the Jacobi's theta function, as the uniformizing function of $\Gamma_1(6)$ (namely $s(\tau)$ is the generator of the field $M(\widehat{H/\Gamma_1(6)})$ of meromorphic modular functions relative to $\Gamma_1(6)$). Using the infinite product expression of ϑ_2:

$$\vartheta_2(0,\tau) = 2\, q^{1/4} \prod_{n=1}^{\infty} (1-q^n) \prod_{n=1}^{\infty} (1+q^n)^2 \qquad (q = e^{2\pi i\tau})$$

we obtain the q-expansion of $s(\tau)$:

$$s(\tau)= q - 4q^2+ 10q^3- 20q^4+\cdots \quad (q=e^{2\pi i\tau}). \qquad (1\text{-}1)$$

Using the transformation formula for ϑ_2 we obtain

$$(1\text{-}2) \qquad \begin{cases} s(\infty) = 0 \\ s(0) = 1/9 \\ s(1/3) = 1 \\ s(1/2) = \infty . \end{cases}$$

Next let $G_{N,k,\vec{a}}(\tau)$ denote an Eisenstein series of level N and dimension -k, where k is a positive integer and $\vec{a}$ is an element of $\mathbb{Z}^2$, namely

$$G_{N,k,\vec{a}}(\tau) = \sum_{(m_1,m_2)\equiv \vec{a}\ (\mathrm{mod}.N)} \frac{1}{(m_1\tau + m_2)^k} \qquad \text{for } k \geq 3.$$

It is known that $G_{6,1,\vec{a}}(\tau)$ can be defined and is a modular form of weight 1 for $\Gamma(6)$ (see [S] chap. VII). If we make the linear combination $\sum_{k=0}^{5} G_{6,1,(1,k)}(\tau)$, then we can show that it is a modular form of weight 1 for $\Gamma_1(6)$ with the only zero at $\tau = 1/2$ (cf.[S]). We set

$$\omega(\tau) = \frac{3}{2\pi i}\sum_{k=0}^{5} G_{6,1,(1,k)} = 1-3q-3q^2-3q^3-3q^4-3q^6-6q^7-\cdots \qquad (1\text{-}3)$$

Now we can state our result.

Proposition.

It holds

$$\zeta(3) = -3\ (2\pi i)^3 \int_0^{i\infty} s(9s^2-18s+1)(9s-1)\tau^2\omega^4\ d\tau.$$

§2. The solution of the modular differential equation.

In this section we perform the process (1) in the introduction. Set $\tilde{\omega}(\tau) = \tau\omega(\tau)$. We can consider ω and $\tilde{\omega}$ as multivalued function on the s-space via the mapping $\tau \to s(\tau)$. We denote them by $\omega(s)$ and $\tilde{\omega}(s)$ Since $s(\tau)$ gives the universal covering map of $\mathbb{C}-\{0,\ 1/9,\ 1\}$, it induces the isomorphism $\pi_1(\mathbb{C}\setminus\{0,\ 1/9,\ 1\}) \simeq \Gamma_1(6)$. Let γ be a closed path in $\mathbb{C}\setminus\{0,\ 1/9,\ 1\}$, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the corresponding element of $\Gamma_1(6)$. After an analytic continuation along γ, ω and $\tilde{\omega}$ are changed by the transformation

$$\begin{pmatrix} \tilde{\omega}(s) \\ \omega(s) \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{\omega}(s) \\ \omega(s) \end{pmatrix}$$

Next we determine the differential equation for ω and $\tilde{\omega}$. It is easy to see that ω and $\tilde{\omega}$ satisfy the equation

$$\begin{vmatrix} \omega & \omega' \\ \tilde{\omega} & \tilde{\omega}' \end{vmatrix} F'' - \begin{vmatrix} \omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}'' \end{vmatrix} F' + \begin{vmatrix} \omega' & \omega'' \\ \tilde{\omega}' & \tilde{\omega}'' \end{vmatrix} F = 0 \qquad (2\text{-}1)$$

By a little bit of calcuation we obtain

$$\begin{vmatrix} \omega & \omega' \\ \tilde{\omega} & \tilde{\omega}' \end{vmatrix} = \frac{\omega^2}{ds/d\tau} = \frac{1}{s}\frac{\omega^2}{\frac{1}{s}\frac{ds}{d\tau}},$$

$$\begin{vmatrix} \omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}'' \end{vmatrix} = d\begin{vmatrix} \omega & \omega' \\ \tilde{\omega} & \tilde{\omega}' \end{vmatrix}/ds, \qquad (2\text{-}2)$$

$$\begin{vmatrix} \omega' & \omega'' \\ \tilde{\omega}' & \tilde{\omega}'' \end{vmatrix} = \frac{1}{(ds/d\tau)^3}\{2(d\omega/d\tau)^2 - \omega(d/d\tau)^2\omega\}.$$

Here we note that $\frac{1}{s}\frac{ds}{d\tau}$ is a modular form of weight 2 for $\Gamma_1(6)$. The values of ω^2, s and $\frac{1}{s}\frac{ds}{d\tau}$ at cusp points are given as the following.

τ	0	1/3	1/2	∞
ω^2	—	—	double zero	—
s	1/9	1	simple pole	simple zero
$\frac{1}{s}\frac{ds}{d\tau}$	simple zero	simple zero	—	—

Thus we obtain $\frac{\omega^2}{ds/d\tau} = \frac{c}{s(s-1)(9s-1)}$, where c is a certain constant. As for the third term of (2-2) it has double poles at s=0, 1/9, 1 and s=∞ is its zero of oder 5. Then it takes the form $\frac{as+b}{s^2(s-1)^2(9s-1)^2}$.

We can determine the constants a and b by comparing with the s-expansion of ω :

$$\omega(s) = 1 + 3s + 15s^2 + 93s^3 + 639s^4 + \cdots.$$

By this caluculation we find that (2-1) coincides with D_6.

Here we determine the constant c from the relation $s\frac{\omega^2}{ds/d\tau} = \frac{c}{(s-1)(9s-1)}$. In fact (1-1) and (1-3) induce the equality

$$\frac{c}{(s-1)(9s-1)} = (q-4q^2+\cdots)\frac{(1-3q-\cdots)^2}{(2\pi i q+\cdots)}.$$

By substituting s=0 (namely q=0) we get $c = \frac{1}{2\pi i}$.

§3. The relation between A(t), B(t) and $\omega(s)$, $\tilde{\omega}(s)$.

Let us consider the intermediate differential equation

$$D_6^*: t^2(t^2-34t+1)y'' + (2t^2-51t+1)y' + \frac{1}{4}(t-10)y = 0.$$

If we substitute $t=s(9s-1)/(s-1)$ in D_6^* and perform the gauge transformation $y=(s-1)^{1/2}z$, we obtain D_6. On the other hand D is the symmetric tensor product of D_6^*. Namely the vector space of solutions for D is given by the symmetric tensor of the one for D_6^*. Then we obtain three solutions of D:

$$\left.\begin{aligned} \varphi_1(t) &= (s(t)-1)\omega^2(t) = \varphi \\ \varphi_2(t) &= (s(t)-1)\omega(t)\tilde{\omega}(t) = \tau\varphi \\ \varphi_3(t) &= (s(t)-1)\tilde{\omega}^2(t) = \tau^2\varphi \end{aligned}\right\} \qquad (3\text{-}1)$$

Obviously φ_1 is holomorphic at t=0.

Let us find a solution g(t) of L(y)=1 in terms of φ_i. Set $g(t)= c_1\varphi_1+ c_2\varphi_2+ c_3\varphi_3$, where $c_i (i=1,2,3)$ is a function of t. We assume the following

$$\left.\begin{aligned} \Sigma\, c_i'\varphi_i &= 0 \\ \Sigma\, c_i'\varphi_i' &= 0 \end{aligned}\right\} \qquad (3\text{-}2)$$

Then we have

$$g' = \Sigma\, c_i\varphi_i'$$

$$g'' = \Sigma\, c_i\varphi_i''$$

$$g''' = \Sigma\, c_i\varphi_i''' + \Sigma\, c_i'\varphi_i''$$

Because $\varphi_i (i=1,2,3)$ satisfies L(y) = 0 we have

$$\begin{aligned} L(g) &= P_0(t)g''' + P_1(t)g'' + P_2(t)g' + P_3(t)g \\ &= \Sigma\, c_i(P_0(t)\varphi_i''' + P_1(t)\varphi_i'' + P_2(t)\varphi_i' + P_3(t)\varphi_i) \\ &\qquad + P_0(t)\, \Sigma\, c_i'\varphi_i'' \\ &= P_0(t)\, \Sigma\, c_i'\varphi_i'', \end{aligned}$$

where $P_i(t)(i=0,1,2,3)$ is the coefficient of $(d/dt)^i$ in (0-4). So we request that

$$P_0(t)\, \Sigma\, c_i'\varphi_i'' = 1. \qquad (3\text{-}3)$$

From (3-2) and (3-3) we get the required condition :

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/P_0 \end{pmatrix}.$$

Hence we obtain the solution $g(t)$ in a neighborhood of $t=0$ of $L(y) = 1$ by putting

$$g(t) = \varphi\int_0^t f\tau'\tau^2\varphi^2 dt - 2\tau\varphi\int_0^t f\tau'\tau\varphi^2 dt + \tau^2\varphi\int_0^t f\tau'\varphi^2 dt, \qquad (3\text{-}4)$$

where $f(t) = \dfrac{1}{P_0(t)\cdot W(t)}$, $W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix}$ and $\tau' = d\tau/dt$.

In (3-4) the path of integral is supposed to be a line segment from 0 to a point t.

We can show that $W(t) = 2\varphi^3(d\tau/dt)^3$. If we note $\dfrac{dt}{ds} = \dfrac{9s^2-18s+1}{(s-1)^2}$ and the first equality of (2-2), we have

$$W(t(s)) = 2(2\pi i)^{-3}\,\frac{(s-1)^6}{\{s(9s-1)(9s^2-18s+1)\}^3}.$$

As we will show in the next section φ and g are holomorphic in the neiborhood of $t=0$. It is easy to see the relations:

$$(3\text{-}5)\qquad \begin{cases} A(t) = -\varphi(t), \\ B(t) = 6g(t). \end{cases}$$

§4. Monodromy trick.

The differential equations (0-2) and (0-3) have same singularities $t=0$, λ, λ', ∞. Here we calculate the monodromy of the solutions $\varphi_i(t)$ $(i=1,2,3)$ and $g(t)$ around $t=0$ and $t=\lambda$.

At first we examine the singularity $t=0$. Let γ_0 be a closed arc in $\mathbb{C}\setminus\{0, \lambda, \lambda'\}$ going around $t=0$ in the positive sense. The mapping $t = s(9s-1)/(s-1)$ gives a biholomorphic correspondence between a neighborhood of s=0 and that of t=0. When t moves along γ_0, s varies around s=0 in the same sense also. By observing the correspondence $\tau \longrightarrow s(\tau)$ we know that γ_0 induces the translation $\tau \longrightarrow \tau+1$. Because $\omega(t)$ has a trivial monodromy around t=0, γ_0 induces the monodromy

$$M_0: \begin{pmatrix} \tilde{\omega} \\ \omega \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \omega \end{pmatrix}. \qquad (4\text{-}1)$$

Next let us study the singularity $t=\lambda$. This point corresponds to $\sigma = \frac{3-2\sqrt{2}}{3}$ on the s-plane. Let γ_λ be a closed arc in $\mathbb{C} \setminus \{0, \lambda, \lambda'\}$ going around $t=\lambda$ in the positive sense. We supporse γ_λ starts from a point near t=0. This loop corresponds to an arc from a point $s=s_0$ near s=0 to a point near s=1/9. Because we have $s|_{\tau=\infty} = 0$ and $s|_{\tau=0} = 1/9$, γ_λ should carry $\tau=\infty$ to $\tau=0$. The composite loop $\gamma_\lambda \cdot \gamma_\lambda$ corresponds to a loop starting from s_0 and goes around $s=\sigma$ in the positive sense. The point σ is not a singularity of D_6. Hence this loop induces a trivial monodromy. So the monodromy M_λ (relative to ${}^t(\tilde{\omega}, \omega)$) must be of order 2. By a little bit of observation we know that M_λ maps $\tau=1/3$ and $\tau=1/2$ to $\tau=-1/2$ and $\tau=-1/3$, respectively. Hence γ_λ induces the transformation $\tau \longrightarrow -\frac{1}{6\tau}$. If we calculate the values $\omega(i)$ and $\tilde{\omega}(i/6)$, then we have

$$M_\lambda = \frac{i}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}. \qquad (4\text{-}2)$$

Let ℓ be a line segment from 0 to t. When t varies along γ_0, ℓ is deformed to the composite arc $\ell \cdot \gamma_0$. Then

$$\varphi \int_0^t f(t)\tau'\tau^2\varphi^2 dt$$

is changed to

$$\tilde{\varphi} \int_0^t \tilde{f}(t)\tilde{\tau}'\tilde{\tau}^2\, \tilde{\varphi}^2 dt,$$

where ~ indicates the result of the monodromy along γ_0. Using (4-1) we have

$$\tilde{f}(t) = f(t)$$

$$\tilde{\tau}(t) = \tau(t) + 1$$

$$\tilde{\tau}'(t) = \tau'(t)$$

$$\tilde{\varphi}(t) = \varphi(t)$$

By the same way we can calculate the monodromy of other terms in (3-5). As a consequence γ_0 induces a monodromy relative to ${}^t(\varphi_1, \varphi_2, \varphi_3, g)$:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ g \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ g \end{pmatrix}.$$

Thus we know that g(t) has a trivial monodromy around t=0 and it is single valued holomorphic there.

By a similar method (4-2) induces the monodromy along γ_λ:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ g \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & -6 & 0 \\ 0 & 1 & 0 & 0 \\ -1/6 & 0 & 0 & 0 \\ -C/6 & 0 & -C & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ g \end{pmatrix}, \qquad (4-4)$$

where $C = \int_0^\lambda f\tau'\varphi^2 dt + 6\int_0^\lambda f\tau'\tau^2\varphi^2 dt.$ (4-3)

This indicates that $\Phi(t) = -C\,\varphi(t) + 6\,g(t)$ is single valued holomorphic in the neighborhood of $t=\lambda$. Namely $\Phi(t)$ is single valued holomorphic on $\{t \in \mathbb{C} \mid |t| < \lambda'\}$. If we recall (3-5), then we have

$$\limsup_{n\to\infty} \sqrt[n]{|C\,a_n + b_n|} = \frac{1}{\lambda'} = \lambda.$$

If we consider that $\Sigma\, a_n t^n$ has the radius of convergence λ and $\{a_n\}$ satisfies (0-1), we can show that

$$|a_n| \sim (1/\lambda)^n.$$

Then we have

$$C + \frac{b_n}{a_n} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

By changing the variable from t to s in C we obtain

$$\zeta(3) = -3\ (2\pi i)^2 \int_0^{1/9} \frac{9s^2-18s+1}{s-1}\, \tilde{\omega}^2(s)\, ds. \qquad (4\text{-}4)$$

If we rewrite it in terms of the variable τ, we have the required form in the Proposition.

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