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Classification of toric singularities with simple K3 singularities as hypersurface sections

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Introduction. Yonemura[8] classified the weights of non-degenerate quasi-homogeneous polynomials on $\mathbb{C}^4$ which define simple K3 singularities. On the other hand, to each quasi-homogeneous polynomial $f = \sum_{v\in(\mathbb{Z}_{\geq 0})^4} c_v z^v$ there exists an element u_0 in $(\mathbb{Q}_{>0})^4$ such that $\langle v,u_0\rangle = 1$ if $c_v \neq 0$, where $z^{(m_1,m_2,m_3,m_4)} = z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4}$. Then we may regard the point u_0 as the weight of f. Let Δ^* be the convex hull of $\{ v \in (\mathbb{Z}_{\geq 0})^4 \mid \langle v,u_0\rangle = 1 \}$. Then $\dim \Delta^* = 3$ and $(1,1,1,1) \in \mathrm{Int}(\Delta^*)$, if f defines a simple K3 singularity (see [8]). As a generalization of this fact, we obtain:

Theorem. Let f be a non-degenerate holomorphic function on the toric singularity $Y = \mathrm{Spec}\,\mathbb{C}[\sigma^* \cap (\mathbb{Z}^4)^*]$ with $f(y) = 0$ and let $X = \{f = 0\}$, where σ^* is the dual cone of a 4-dimensional strongly convex cone σ in $\mathbb{R}^4$ generated by primitive elements u_1, u_2, ... and u_s in $\mathbb{Z}^4$ and $\{y\} = \{ x \in Y \mid z^v(x) = 0$ for any $v \in (\sigma^*\cap(\mathbb{Z}^4)^*)\setminus\{0\} \}$. If (X,y) is a simple K3 singularity, then the following two conditions are satisfied.

(1) Y is Gorenstein, i.e., there exists an element $v_0 \in (\mathbb{Z}^4)^*$ such that $\langle v_0, u_i \rangle = 1$, if $\mathbb{R}_{\geq 0} u_i$ is a 1-dimensional face of σ for $i = 1$ through s.

(2) There exists an element $u_0 \in \mathrm{Int}(\sigma)$ such that $\dim \Delta^* = 3$ and that $v_0 \in \mathrm{Int}(\Delta^*)$, where Δ^* is the convex hull of $\{ v \in \sigma^* \cap (\mathbb{Z}^4)^* \mid \langle v, u_0 \rangle = 1 \}$.

The purpose of this paper is to show that the pairs (σ, u_0) satisfying the conditions of the above theorem are finite modulo $GL(4,\mathbb{Z})$. We can obtain all representatives of them, using a computer. However, they are too many to make a list here.

In §1, we prove the above theorem and show that there exists a partial order on the set of the pairs satisfying the conditions of the above theorem such that for a pair (σ, u_0), all the pairs $(\tau, u_0) \geq (\sigma, u_0)$ are finite and obtained by a simple algorithm (see Proposition 1.6 and its proof).

In §2, we give a list of representatives of all the minimal pairs (σ, u_0). We omit the proof of the completeness of the list, for the sake of its length.

§1 Toric singularities and their hypersurface sections.

Let $N = \mathbb{Z}^{n+1}$ be a free $\mathbb{Z}$-module of rank $n + 1 \geq 3$ and let N^* be its dual module with canonical pairing $\langle \ , \ \rangle$:

$N^* \times N \to \mathbb{Z}$. Let $\sigma = \mathbb{R}_{\geq 0}u_1 + \mathbb{R}_{\geq 0}u_2 + \ldots + \mathbb{R}_{\geq 0}u_s$ be an $(n+1)$-dimensional strongly convex rational cone in $N_{\mathbb{R}}$ generated by primitive elements u_i in N. Here we may assume that $\mathbb{R}_{\geq 0}u_i$ is a 1-dimensional face of σ for each $i = 1$ through s. Let $Y = \mathrm{Spec}\mathbb{C}[\sigma^* \cap N^*]$ and let $z^v : Y \to \mathbb{C}$ be the character of v, which is the natural extension of $v \otimes 1_{\mathbb{C}^\times}$: $\mathrm{Spec}\mathbb{C}[N^*] \simeq (\mathbb{C}^\times)^{n+1} \to \mathbb{C}^\times$, for each v in $\sigma^* \cap N^*$. Then the set $\{ x \in Y \mid z^v(x) = 0 \text{ for all } v \in (\sigma^* \cap N^*) \setminus \{0\} \}$ consists of only one point y and any holomorphic function f on Y with $f(y) = 0$ is expressed as the power series:

$$f := \Sigma_{v \in (\sigma^* \cap N^*) \setminus \{0\}} c_v z^v.$$

Let X be a hypersurface section of Y containing y, i.e., X $= \{f = 0\}$, for a holomorphic function f on Y with $f(y) = 0$. Here we note that if (X,y) is an isolated singularity, then the dimension of the singular locus Sing(Y) of Y is not greater than 1, i.e., any $(n-1)$-dimensional face of σ is non-singular. Assume that X is normal and that $X \setminus \{y\}$ has only rational singularities. Then by [6] and [1], we obtain:

Proposition 1.1. The following three conditions are equivalent.

(1) (X,y) is Gorenstein.

(2) (Y,y) is Gorenstein.

(3) (G) There exists an element $v_0 \in N^*$ such that $\langle v_0, u_i \rangle = 1$ for $1 \leq i \leq s$.

We denote the above v_0, by $v(\sigma)$.

Definition 1.2. The Newton polyhedron $\Gamma_+(f)$ of f is the convex hull of $\cup_{c_v \neq 0}(v + \sigma^*)$ and the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$.

Definition 1.3. We call f non-degenerate, if

$\partial f_{\Delta^*}/\partial z_1 = \dots = \partial f_{\Delta^*}/\partial z_{n+1} = 0$

has no solutions in $T := \mathrm{Spec}\mathbb{C}[N^*] \subset Y$ for each face Δ^* of $\Gamma(f)$, where $f_{\Delta^*} = \sum_{v\in\Delta^*\cap N^*} c_v z^v$ and $(z_1, z_2, \dots, z_{n+1})$ is a global coordinate of T, i.e., $z_i = z^{v_i}$ for a basis $\{v_1, v_2, \dots, v_{n+1}\}$ of N^*.

Proposition 1.4. ([5, Theorem 2.2]) Assume that the condition (G) in Proposition 1.1 is satisfied and that f is non-degenerate. Then (X,y) is purely elliptic if and only if $v(\sigma) \in \Gamma(f)$. (See [7], for the definition of a purely elliptic singularity.)

Remark. If $v(\sigma) \in \partial\Gamma_+(f) \setminus \Gamma(f)$, then $X \setminus \{y\}$ has worse singularities than rational singularities.

Proposition 1.5. Under the assumption of Proposition 1.4, (X,y) is of $(0,n-1)$-type if and only if $\dim \Delta^* = n$, where Δ^* is a face of $\Gamma(f)$ with $v(\sigma) \in \mathrm{Int}(\Delta^*)$. (See [2], for the definition of (0,i)-type of a purely elliptic singularity.)

Proof. Let Σ be a subdivision of the dual Newton boundary $\Gamma^*(f)$ of $\Gamma(f)$ consisting of non-singular cones (see [4] and [5], for the definition of $\Gamma^*(f)$). Then $\hat{Y} := T_N\mathrm{emb}(\Sigma)$ and $\hat{X}$ are non-singular, where $\hat{X}$ is the proper transformation of X under the holomorphic map $p : \hat{Y} \to Y$ obtained by the morphism of r.p.p. decompositions $(N,\Sigma) \to (N,\{\text{faces of } \sigma\})$. Let Σ_1 be the set of the 1-dimensional cones in Σ whose generators are contained in $\mathrm{Int}(\sigma)$ and let E_τ be the intersection of the closure of $\mathrm{orb}(\tau)$ and $\hat{X}$, for each τ in Σ_1. Then $\sum_{\tau\in\Sigma_1} E_\tau$ is the exceptional set of the resolution $p_{|\hat{X}} : \hat{X} \to X$ and we can express $K_{\hat{X}} = (p_{|\hat{X}})^*K_X + \sum_{\tau\in\Sigma_1} a_\tau E_\tau$. Here we note that $a_\tau = \langle v(\sigma),u_\tau\rangle - d(u_\tau) - 1$, by [5, Lemma 2.1], where u_τ is the primitive element in N generating τ and $d(u_\tau) = \min\{ \langle v,u_\tau\rangle \mid v \in \Gamma_+(f) \}$. Hence $a_\tau \geq -1$, for each τ in Σ_1. Assume that $\dim \Delta^* = n$. Then there exists only one 1-dimensional cone τ in Σ_1 with $a_\tau = -1$ and E_τ is irreducible. Hence (X,y) is of (0,n-1)-type. Next, assume that $\dim \Delta^* \leq n-1$. Then we easily see that there exist at least two 1-dimensional cones τ in Σ_1 such that $a_\tau = -1$ and that $E_\tau \neq \emptyset$. Hence (X,y) is not of (0,n-1)-type. q.e.d.

Assume that f is a non-degenerate holomorphic function on Y with $f(y) = 0$ and let $X = \{f = 0\}$. When $n = 3$, (X,y) is a simple K3 singularity (, i.e., (X,y) is Gorenstein purely elliptic of $(0,2)$-type [3]), if and only if (Y,y) is Gorenstein, $v(\sigma)$ is contained in the interior of a 3-dimensional face of $\Gamma(f)$, by Propositions 1.1, 1.4 and 1.5. Assume that (X,y) is Gorenstein purely elliptic of $(0,n-1)$-type. Then there exists a unique element u_0 in $\mathrm{Int}(\sigma)$ such that $\langle v,u_0\rangle = 1$ for any element v in the face Δ^* of $\Gamma(f)$ whose interior contains $v(\sigma)$. Hence Δ^* is contained in

$$\Delta^*_\sigma(u_0) := \text{convex hull of } \{ v \in \sigma^* \cap N^* \mid \langle v,u_0\rangle = 1 \}.$$

Therefore, the pair (σ,u_0) satisfies the following condition:

(E) $\dim \Delta^*_\sigma(u_0) = n$ and $v(\sigma) \in \mathrm{Int}(\Delta^*_\sigma(u_0))$.

Thus we obtain the theorem in Introduction. Conversely, assume that (σ,u_0) satisfies the conditions (G) and (E), and let $X = \{f = 0\}$, where $f = \sum_{v\in\Delta^*_\sigma(u_0)\cap N^*} c_v z^v$ + higher terms, for certain non-zero complex numbers c_v. Then (X,y) is Gorenstein purely elliptic of $(0,n-1)$-type, if (X,y) is an isolated singularity.

Problem 1. Is the set of the pairs (σ,u_0) satisfying the conditions (G) and (E), finite modulo $GL(N)$?

Let

$\tilde{\mathcal{E}}^n = \{ (\sigma,u_0) \mid \sigma$ is an $(n+1)$-dimensional strongly convex rational cone satisfying (G), $u_0 \in \mathrm{Int}(\sigma)$ and u_0 satisfies (E) $\}$

and let $\mathcal{E}^n = \tilde{\mathcal{E}}^n/\sim$, where $(\sigma,u_0) \sim (\sigma',u_0')$ if and only if there exists an element g in $GL(N)$ such that $g(\sigma) = \sigma'$ and that $g(u_0) = u_0'$. We define a partial order on $\tilde{\mathcal{E}}^n$ as follows: $(\sigma,u_0) \geq (\sigma',u_0')$ if and only if $\sigma \supset \sigma'$, $v(\sigma) = v(\sigma')$ and $u_0 = u_0'$. Let

$\tilde{\mathcal{E}}_0^n = \{ (\sigma,u_0) \in \tilde{\mathcal{E}}^n \mid (\sigma,u_0)$ is minimal $\}$

and let $\mathcal{E}_0^n = \tilde{\mathcal{E}}_0^n/\sim$, where we call (σ,u_0) minimal, if $(\sigma,u_0) \geq (\tau,u_0)$ implies $(\sigma,u_0) = (\tau,u_0)$, for any $(\tau,u_0) \in \tilde{\mathcal{E}}^n$.

Remark. (1) Assume that $(\sigma,u_0) \in \tilde{\mathcal{E}}^n$. If the cone τ generated by a subset of $L := \{ u \in \sigma \cap N \mid \langle v(\sigma),u\rangle = 1 \}$ is $(n+1)$-dimensional strongly convex and contains u_0 in the interior, then $(\tau,u_0) \in \tilde{\mathcal{E}}^n$, because $\tau^* \supset \sigma^*$.
(2) Since $\#L < +\infty$, for any pair (σ,u_0) in $\tilde{\mathcal{E}}^n$, we have $\#\{ (\tau,u_0) \in \tilde{\mathcal{E}}^n \mid (\sigma,u_0) \geq (\tau,u_0) \} < +\infty$. Hence for any pair (σ,u_0) in $\tilde{\mathcal{E}}^n$, there exists a pair (τ,u_0) in $\tilde{\mathcal{E}}_0^n$ with $(\sigma,u_0) \geq (\tau,u_0)$.

Let $C(\sigma,u_0) = \{ (\tau,u_0) \in \tilde{\mathcal{E}}^n \mid (\tau,u_0) \geq (\sigma,u_0) \}$, for a pair (σ,u_0) in $\tilde{\mathcal{E}}^n$. Then by the above remark, we have $\tilde{\mathcal{E}}^n = \bigcup_{(\sigma,u_0)\in\tilde{\mathcal{E}}_0^n} C(\sigma,u_0)$. Hence if $\mathcal{E}_0^n$ is a finite set, then so is

$\widetilde{\mathcal{E}}^n$, by the following proposition.

Proposition 1.6. $C(\sigma,u_0)$ is a finite set, for any pair (σ,u_0) in $\widetilde{\mathcal{E}}^n$.

Proof. Since for any pair (τ,u_0) in $C(\sigma,u_0)$, $\Delta^*_\tau(u_0)$ is the convex hull of a subset of the finite set $L^* := \{ v \in \sigma^* \cap N^* \mid \langle v,u_0\rangle = 1 \}$, we have $\#\{ \Delta^*_\tau(u_0) \mid (\tau,u_0) \in C(\sigma,u_0) \} < +\infty$. Conversely, let Δ^* be the convex hull of a subset of L^* such that $v(\sigma) \in \mathrm{Int}(\Delta^*)$ and that $\dim \Delta^* = n$. Then $\#\{ u \in (\mathbb{R}_{\geq 0}\Delta^*)^* \cap N \mid \langle v(\sigma),u\rangle = 1 \} < +\infty$. Hence $C' := \{ (\tau,u_0) \in C(\sigma,u_0) \mid \Delta^*_\tau(u_0) = \Delta^* \}$ is a finite set, because $\tau \subset (\mathbb{R}_{\geq 0}\Delta^*)^*$ for any pair (τ,u_0) in C'. Therefore, $C(\sigma,u_0)$ is a finite set. q.e.d.

Next, we show that for a cone σ satisfying the condition (G), all the elements u_0 in $\mathrm{Int}(\sigma)$ satisfying the condition (E) are finite. Let $W_\sigma(v_0) = \{ u \in \mathrm{Int}(\sigma) \mid \dim \Delta^*_\sigma(u) = n,\ v_0 \in \mathrm{Int}(\Delta^*_\sigma(u)) \}$, for an $(n+1)$-dimensional strongly convex rational cone σ and for an element v_0 in $N^*_{\mathbb{R}}$.

Theorem 1.7. $W_\sigma(v_0)$ is a finite set, for any $v_0 \in \mathrm{Int}(\sigma^*)$.

Proof. For $v_1, v_2, \ldots, v_j \in \sigma^* \cap N^*$, let

$W(v_1, v_2, \ldots, v_j) = \{ u \in W_\sigma(v_0) \mid \langle v_1, u\rangle = \ldots = \langle v_j, u\rangle = 1 \}$. For $u \in N_{\mathbb{R}}$, let $W^*(u) = \{ v \in \sigma^* \cap N^* \mid \langle v,u\rangle < 1 \}$. Here we note that if $u \in \mathrm{Int}(\sigma)$, then $W^*(u)$ is a finite set. First, take an element $u_0 \in \mathrm{Int}(\sigma)$ with $\langle v_0, u_0\rangle = 1$. Then for any element u in $W_\sigma(v_0)$ with $u \neq u_0$, we see that $\{ v \in W^*(u_0) \mid \langle v,u\rangle = 1 \} \neq \emptyset$. Hence $W_\sigma(v_0) \subset \{u_0\} \cup \bigcup_{v_1 \in W^*(u_0)} W(v_1)$. Here we note that if $W(v_1) \neq \emptyset$, then v_0 and v_1 are linearly independent. Next, if $W(v_1) \neq \emptyset$, then we can take an element $u_1 \in \mathrm{Int}(\sigma)$ with $\langle v_0, u_1\rangle = \langle v_1, u_1\rangle = 1$, for each $v_1 \in W^*(u_0)$. Then we have $W(v_1) \subset \{u_1\} \cup \bigcup_{v_2 \in W^*(u_1)} W(v_1, v_2)$. Proceeding similarly, we finally obtain $W(v_1, \ldots, v_{n-1}) \subset \{u_{n-1}\} \cup \bigcup_{v_n \in W^*(u_{n-1})} W(v_1, \ldots, v_n)$. Then $\#W(v_1, \ldots, v_n) \leq 1$, because v_0, v_1, ... and v_n are linearly independent, if $W(v_1, \ldots, v_n) \neq \emptyset$. Hence $\#W(v_1, \ldots, v_{n-1}) < +\infty$ and $\#W_\sigma(v_0) < +\infty$. q.e.d.

Now, Problem 1 is reduced to:

Problem 2. Is the set of the cones σ satisfying the condition (G) such that $\{ u_0 \in W_\sigma(v(\sigma)) \mid (\sigma, u_0) \in \tilde{\mathcal{E}}_0^n \} \neq \emptyset$, finite modulo $GL(N)$?

Proposition 1.8. If $W_\sigma(v_0) \neq \emptyset$ for an element $v_0 \in \mathrm{Int}(\sigma^*)$, then $\#IL_\sigma(v_0) \leq 1$, where $IL_\sigma(v_0) = \{ u \in \mathrm{Int}(\sigma) \cap N \mid \langle v_0, u\rangle = 1 \}$. Conversely, if $IL_\sigma(v_0) = \{u_0\}$, then $W_\sigma(v_0) \subset$

$\{u_0\}$.

Proof. If $IL_\sigma(v_0) \neq \emptyset$, then for each element u_0 in $IL_\sigma(v_0)$, we have $\langle v,u_0\rangle > 0$ for any v in $\sigma^* \setminus \{0\}$ and hence $\langle v,u_0\rangle \geq 1$ for any v in $(\sigma^* \setminus \{0\}) \cap N^*$. Therefore, $W_\sigma(v_0) \subset \{u_0\}$, as we see in the proof of Theorem 1.7. Hence if $\# IL_\sigma(v_0) \geq 2$, then $W_\sigma(v_0) = \emptyset$. q.e.d.

§2 A list of representatives of all elements in $\mathcal{E}_0^3$.

1. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,0) + \mathbb{R}_{\geq 0}(0,1,0,0) + \mathbb{R}_{\geq 0}(0,0,1,0) + \mathbb{R}_{\geq 0}(0,0,0,1)$. See [8, Table 2.2], for u_0.
2. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1)$ and $u_0 = \frac{1}{2}(1,1,1,2)$, $\frac{1}{3}(1,1,1,3)$, $\frac{1}{12}(5,3,4,12)$, $\frac{1}{8}(3,2,2,8)$, $\frac{1}{20}(7,5,4,20)$, $\frac{1}{16}(5,4,2,16)$, $\frac{1}{9}(3,2,1,9)$, $\frac{1}{5}(2,1,1,5)$, $\frac{1}{13}(4,3,1,13)$, $\frac{1}{11}(5,2,3,11)$, $\frac{1}{14}(5,3,2,14)$, $\frac{1}{16}(7,3,4,16)$, $\frac{1}{19}(4,7,3,19)$ or $\frac{1}{24}(5,9,4,24)$.
3. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,3,1)$ and $u_0 = \frac{1}{4}(2,1,2,4)$, $\frac{1}{7}(3,2,1,7)$, $\frac{1}{5}(3,1,2,5)$, $\frac{1}{9}(5,2,3,9)$, $\frac{1}{6}(2,3,3,6)$, $\frac{1}{9}(3,4,3,9)$ or $\frac{1}{2}(1,1,2,2)$.
4. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,4,1)$ and $u_0 = \frac{1}{2}(1,1,2,2)$.
5. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,5,1)$ and $u_0 = \frac{1}{4}(2,3,5,4)$, $\frac{1}{5}(2,3,5,5)$, $\frac{1}{2}(1,1,1,2)$ or $\frac{1}{3}(1,1,1,3)$.

6. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,7,1)$ and $u_0 = \frac{1}{2}(1,1,2,2)$.

7. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,3,7,1)$ and $u_0 = \frac{1}{2}(1,2,3,2)$.

8. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,3,8,1)$ and $u_0 = \frac{1}{2}(1,2,4,2)$.

9. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,3,10,1)$ and $u_0 = \frac{1}{2}(1,2,5,2)$.

10. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(1,0,1,1)$ and $u_0 = \frac{1}{2}(0,0,1,2)$, $\frac{1}{3}(-1,-1,1,3)$, $\frac{1}{4}(-1,-1,2,4)$ or $\frac{1}{6}(-1,-1,2,6)$.

11. $\sigma = \mathbb{R}_{\geq 0}(2,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(1,0,1,1)$ and $u_0 = \frac{1}{2}(1,0,1,2)$.

12. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(1,2,3,1)$ and $u_0 = \frac{1}{3}(2,2,3,3)$.

13. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,0,1,1) + \mathbb{R}_{\geq 0}(-1,0,-1,1)$ and $u_0 = \frac{1}{3}(1,1,0,3)$, $\frac{1}{4}(1,1,0,4)$ or $\frac{1}{6}(1,2,0,6)$.

14. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,0,1,1) + \mathbb{R}_{\geq 0}(0,0,-1,1)$ and $u_0 = \frac{1}{3}(1,1,0,3)$ or $\frac{1}{4}(1,2,0,4)$.

15. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(0,0,-1,1)$ and $u_0 = \frac{1}{4}(1,1,0,4)$.

16. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(-1,-1,-2,1)$ and $u_0 = \frac{1}{4}(1,1,0,4)$.

17. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,3,1) + \mathbb{R}_{\geq 0}(1,-1,-3,1)$ and $u_0 = \frac{1}{3}(1,1,0,3)$.

18. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(0,2,-1,1)$ and $u_0 = \frac{1}{2}(0,1,0,2)$.

19. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(-1,1,-2,1)$ and $u_0 = \frac{1}{2}(0,1,0,2)$.

20. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1)$ and $u_0 = \frac{1}{2}(0,0,1,2)$.

21. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1)$ and $u_0 = \frac{1}{2}(1,1,2,2)$.

22. $\sigma = \mathbb{R}_{\geq 0}(1,1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(0,0,-1,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1)$ and $u_0 = \frac{1}{2}(1,1,0,2)$.

23. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,2,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1)$ and $u_0 = \frac{1}{2}(0,1,1,2)$, $\frac{1}{3}(0,3,1,3)$ or $\frac{1}{4}(0,4,1,4)$.

24. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,3,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1)$ and $u_0 = \frac{1}{2}(0,2,1,2)$.

25. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,2,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1)$ and $u_0 = \frac{1}{2}(1,1,2,2)$.

26. $\sigma = \mathbb{R}_{\geq 0}(0,0,0,1) + \mathbb{R}_{\geq 0}(1,1,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(1,1,-1,1)$ and $u_0 = \frac{1}{2}(1,1,0,2)$.

27. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(-1,-1,-1,1)$ and $u_0 = (0,0,0,1)$.

28. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(-1,-1,-1,1)$ and $u_0 = (0,0,0,1)$.

29. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,1,3,1) + \mathbb{R}_{\geq 0}(-1,-1,-2,1)$ and $u_0 = (0,0,0,1)$.

30. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,5,1) + \mathbb{R}_{\geq 0}(-1,-1,-1,1)$ and $u_0 = (0,0,0,1)$.

31. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,5,1) + \mathbb{R}_{\geq 0}(-1,-1,-2,1)$ and $u_0 = (0,0,0,1)$.

32. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,5,1) + \mathbb{R}_{\geq 0}(-2,-3,-5,1)$ and $u_0 = (0,0,0,1)$.

33. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,2,7,1) + \mathbb{R}_{\geq 0}(-1,-1,-2,1)$ and $u_0 = (0,0,0,1)$.

34. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,3,7,1) + \mathbb{R}_{\geq 0}(-1,-2,-3,1)$ and $u_0 = (0,0,0,1)$.

35. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,2,1)$ and $u_0 = (0,0,1,1)$.

36. $\sigma = \mathbb{R}_{\geq 0}(2,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(-1,0,2,1)$ and $u_0 = (0,0,1,1)$.

37. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(-1,0,3,1)$ and $u_0 = (0,0,1,1)$.

38. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(0,0,-1,1)$ and $u_0 = (0,0,0,1)$.

39. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0},(1,1,-1,1)$ and $u_0 = (0,0,0,1)$.

40. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(2,0,-1,1)$ and $u_0 = (0,0,0,1)$.

41. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(-1,-1,0,1) +$

$\mathbb{R}_{\geq 0}(1,2,3,1) + \mathbb{R}_{\geq 0}(-1,-2,-3,1)$ and $u_0 = (0,0,0,1)$.

42. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}-1,0,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(0,4,-1,1)$ and $u_0 = (0,1,0,1)$.

43. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(-1,3,-2,1)$ and $u_0 = (0,1,0,1)$.

44. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,2,1)$ and $u_0 = (0,0,1,1)$.

45. $\sigma = \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(2,0,-1,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1)$ and $u_0 = (0,0,0,1)$.

46. $\sigma = \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(1,-1,-2,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1)$ and $u_0 = (0,0,0,1)$.

47. $\sigma = \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(2,0,-1,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(1,-1,0,1)$ and $u_0 = (0,0,0,1)$.

48. $\sigma = \mathbb{R}_{\geq 0}(0,-1,2,1) + \mathbb{R}_{\geq 0}(0,2,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1)$ and $u_0 = (0,0,1,1)$.

49. $\sigma = \mathbb{R}_{\geq 0}(0,0,2,1) + \mathbb{R}_{\geq 0}(0,3,0,1) + \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1)$ and $u_0 = (0,1,1,1)$.

50. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1) + \mathbb{R}_{\geq 0}(0,0,1,1) + \mathbb{R}_{\geq 0}(0,0,-1,1)$ and $u_0 = (0,0,0,1)$.

51. $\sigma = \mathbb{R}_{\geq 0}(1,0,0,1) + \mathbb{R}_{\geq 0}(-1,0,0,1) + \mathbb{R}_{\geq 0}(0,1,0,1) + \mathbb{R}_{\geq 0}(0,-1,0,1) + \mathbb{R}_{\geq 0}(1,1,2,1) + \mathbb{R}_{\geq 0}(-1,-1,-2,1)$ and $u_0 = (0,0,0,1)$.

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