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GENERALIZED VECTOR MEASURES AND FEYNMAN PATH INTEGRALS

お茶の水女大・理　竹尾富貴子(Fukiko TAKEO)

§1. Introduction.

Let us consider the following Cauchy problem

$$(1.1)\qquad \begin{cases} \frac{\partial}{\partial t}\Psi(t,x) = (-iH + V(t,x))\Psi(t,x) \\ \Psi(0,x) = g(x) \qquad 0 < t < T,\ x \in \mathbf{R}^d \end{cases}$$

where $0 < T < \infty$, $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$, $V(t,\cdot) = \overline{V}(t)$ with $\overline{V} \in C^1([0,T];\ \mathbf{B}(\mathbf{R}^d))$ and H is a self-adjoint operator on a Hilbert space $L^2(\mathbf{R}^d;\mathbf{C}^N)$.

Feynman[3] introduced the idea of path integral to make an intuitive representation of the Schroedinger equation. Various approaches to the "Feynman integral" have been taken by many mathematicians. In [1,2,6] they treated the Feynman integral by considereing the analytic extension. K.Ito [5] gave the mathematical formulation of the Feynman integral by considereing the Gauss measure in the Hilbert space. But those integrable functions are limited to a Fourier transform of a bounded complex measure or so on. In [7], I. Kluvanek defined the space of integrable functions which is complete with respect to the integrating seminorm depending on the norm of image of an operator μ_t. In a special case of a hyperbolic system which includes the Dirac equation in two space-time dimensions, T. Ichinose [4] constructed a countably additive measure by using the L^{∞} well-posedness of the Cauchy problem and gave the solution of the Cauchy problem by the Feynman integral with this measure.

In [8], we have constructed a $\mathfrak{L}(L^2(\mathbf{R}^d;\mathbf{C}^N))$-valued generalized measure μ_t on the path space $\hat{\mathbf{X}}_t$ and in case that $V(t,x)$ is independent of t, i.e. $V(t,x) = V(0,x)$, we gave the solution of $\Psi(t)$ of (1.1) by the Feynman integral.

In this paper, we shall examine the space $L(\mathcal{X},\beta)$ of integrable functions with respect to μ_t, which is defined as an extension of a tensor product space and is complete with respect to a seminorm β which does not depend on μ_t [Theorem 1]. $L(\mathcal{X},\beta)$ includes the function $F(X) = \exp\{\int_0^t V(s,X(s))\,ds\}$ with time-dependent potential $V(t,x)$. We shall also show that there is a kind of dominated convergence theorem with respect to μ_t [Theorems 2,3] though it is not countably additive. By using this measure μ_t, we shall give the solution $\Psi(t)$ of (1.1) by the Feynman integral [Theorem 4]

$$\Psi(t) = \int_{\hat{X}_t} d\mu_t(X) \exp\{\int_0^t V(s,X(s))\,ds\}\, g(X(0)).$$

§2. Generalized vector measures μ_t on $\hat{\mathbf{X}}_t$.

For $0 < t < \infty$, let $\mathbf{X}_t = \prod_{[0,t]}\mathbf{R}^d$ be the product of the uncountably many copies of $\mathbf{R}^d$. Let Δ_n be a finite partition of the interval $[0,t]$ such that

$$\Delta_n\colon\ 0 = t_{0,n} < t_{1,n} < \cdots < t_{2^n,n} = t, \quad \text{where } t_{j,n} = \frac{j}{2^n}t$$

and let σ_n be a mapping of $\mathbf{X}_t$ into itself such that

$$\sigma_n(X)(s) = \begin{cases} X(t_{j,n}) & \text{for } t_{j-1,n} < s \leq t_{j,n} (j=1,2,\cdots,2^n) \\ X(0) & \text{for } s = 0 \end{cases}$$

for any $X \in \mathbf{X}_t$. Let $\hat{\mathbf{X}}_t$ be the subset of $\mathbf{X}_t$ such that

$$\hat{\mathbf{X}}_t = \{ X \in \mathbf{X}_t;\ X \in C([0,t];\mathbf{R}^d) \text{ or } X \in \cup_{n=1}^{\infty}\sigma_n(\mathbf{X}_t)\}.$$

For $F\colon \hat{\mathbf{X}}_t \to \mathbf{C}$, define $F_{\sigma(n)}\colon \hat{\mathbf{X}}_t \to \mathbf{C}$ by

$$F_{\sigma(n)}(X) = F(\sigma_n(X)) \quad \text{for any } X \in \hat{\mathbf{X}}_t.$$

Let $\mathcal{B}$ be the set of Borel subsets of $\mathbf{R}^d$. For $n \in \mathbf{N}$ and $B_j \in \mathcal{B}(j=0,1,\cdots,2^n)$, put $J(B_0,B_1,\cdots,B_{2^n}) := \{X \in \hat{\mathbf{X}}_t; X(t_{j,n}) \in B_j (j=0,1,\cdots,2^n)\}$. Let $\mathcal{J} = \{J(B_0,B_1,\cdots,B_{2^n});\ n \in \mathbf{N},\ B_j \in \mathcal{B}\}$ and $\mathcal{F}$ be the field generated by $\mathcal{J}$.

Let $\{U_t\}_{t\in\mathbf{R}}$ be a C_0-group of unitary operators on $L^2(\mathbf{R}^d;\mathbf{C}^N)$. For $J = J(B_0,B_1,\cdots,B_{2^n}) \in \mathcal{J}$, we shall define an operator $\mu_t(J) \in \mathcal{L}(L^2(\mathbf{R}^d;\mathbf{C}^N))$ by

$$(\mu_t(J))g := U_{\delta_n}\chi_{B_{2^n}}U_{\delta_n}\cdots U_{\delta_n}\chi_{B_1}U_{\delta_n}\chi_{B_0}g \quad \text{for } g \in L^2(\mathbf{R}^d;\mathbf{C}^N),$$

where χ_B is a multiplicative operator on $L^2(\mathbf{R}^d;\mathbf{C}^N)$ by the characteristic function of the set B and $\delta_n = \frac{t}{2^n}$. Then μ_t can be considered as a finitely additive $\mathcal{L}(L^2(\mathbf{R}^d;\mathbf{C}^N))$-valued measure defined on $\mathcal{J}$.

Now we shall consider the integral with respect to this measure μ_t.

Put $\chi_m(x) = \begin{cases} 1 & \|x\| \leq m \\ 0 & \|x\| > m \end{cases}$ for $x \in \mathbf{R}^d$.

For $\bar{\mathbf{a}} \in \mathbf{C}^N$ and $J \in \mathcal{J}$, we shall write

(2.1) $\quad \mu_t(J)\bar{\mathbf{a}} := \operatorname*{s-lim}_{m\to\infty} \mu_t(J)(\bar{\mathbf{a}}\chi_m)$

if the limit of the right-hand side exists and we shall naturally use the integral as follows

(2.2) $\quad \mu_t(J)\bar{\mathbf{a}} = \int_{\hat{X}_t} d\mu_t(X)\ \chi_J(X)\bar{\mathbf{a}}.$

For $J = J(B_0, B_1, \cdots, B_{2^n}) \in \mathcal{J}$ and relatively compact set $C \in \mathcal{B}$, put $J \circ C := J(B_0 \cap C, B_1, \cdots, B_{2^n})$. Then we have

(2.3) $\quad \mu_t(J \circ C)\bar{\mathbf{a}} = \operatorname*{s-lim}_{m\to\infty} \mu_t(J \circ C)(\bar{\mathbf{a}}\chi_m) = \mu_t(J)(\bar{\mathbf{a}}\chi_C).$

Let $\mathcal{S}_0$ be the space of $\mathcal{J}$-measurable simple functions on $\hat{\mathbf{X}}_t$.

For $g = \Sigma_{k=1}^r \bar{\mathbf{a}}_k\chi_{C_k} \in L^2(\mathbf{R}^d;\mathbf{C}^N)$ ($\bar{\mathbf{a}}_k \in \mathbf{C}^N$ and $C_k \in \mathcal{B}$ is relatiely compact) and $\Psi = \Sigma_{j=1}^{q}\alpha_j\chi_{J_j} \in \mathcal{S}_0$ ($\alpha_j \in \mathbf{C}$ and $J_j \in \mathcal{J}$), we have

$$\int_{\hat{X}_t} d\mu_t(X)\ \Psi(X)\ g(X(0)) = \Sigma_{j=1}^{q}\alpha_j\Sigma_{k=1}^{r}\mu_t(J_j)(\bar{a}_k\chi_{C_k})$$

by using (2.2) and (2.3).

Let $\mathbf{B}(\mathbf{R}^d)$ be the space of complex-valued bounded Borel measurable functions on $\mathbf{R}^d$ and $\mathbf{B}(\hat{\mathbf{X}}_t:\otimes_\pi,\Delta_n)$ be the space of complex-valued functions F on $\hat{\mathbf{X}}_t$ for which there exist $m \in \mathbf{N}$ and functions $f_{j,k} \in \mathbf{B}(\mathbf{R}^d)(j=0,1,\cdots,2^n$ and $k=1,2,\cdots,m)$ such that $F(X) = \Sigma_{k=1}^{m} \prod_{j=0}^{2^n} f_{j,k}(X(t_{j,n}))$ for any $X \in \hat{\mathbf{X}}_t$, equipped with π-norm:

$$\|F\|_\pi := \inf \Sigma_{k=1}^{m} \prod_{j=0}^{2^n} \|f_{j,k}\|_\infty,$$

where the infimum is taken over all representations of F and $\|f\|_\infty = \sup\{|f(x)|;\ x \in \mathbf{R}^d\}$. Let $\mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi,\Delta_n)$ be the completion of $\mathbf{B}(\hat{\mathbf{X}}_t:\otimes_\pi,\Delta_n)$ with respect to π-norm.

For $F \in \mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi,\Delta_n)$ and $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$, there exist sequences $\{\Psi_n\} \subset \mathcal{S}_0$ and $\{g_n\}$ of $\mathbf{C}^N$-valued simple functions on $\mathbf{R}^d$ such that $\lim_{n\to\infty}\|\Psi_n - F\|_\pi = 0$ and $\lim_{n\to\infty}\|g - g_n\|_2 = 0$.

So we shall define the integral of $F \in \mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi,\Delta_n)$ by

$$(2.4)\quad \int_{\hat{X}_t} d\mu_t(X)\ F(X)\ g(X(0)) = \operatorname*{s-lim}_{n\to\infty} \int_{\hat{X}_t} d\mu_t(X)\ \Psi_n(X)\ g_n(X(0)).$$

for $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$.

This is well-defined since the right hand side of (2.4) does not depend on sequences $\{\Psi_n\}$ and $\{g_n\}$ but only on g and F. We shall define the space $\mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$ as the space of complex-valued functions F on $\hat{\mathbf{X}}_t$ such that $F_{\sigma(n)}$ belongs to $\mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi,\Delta_n)$ for each $n \in \mathbf{N}$ and $\sup_n \|F_{\sigma(n)}\|_\pi < \infty$. We shall define the seminorm β on $\mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$ by

$$\beta(F) = \sup_n \|F_{\sigma(n)}\|_\pi$$

for $F \in \mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$.

A subset C of $\hat{\mathbf{X}}_t$ is said to be β-null if $\chi_C \in \mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$ and $\beta(\chi_C) = 0$, where χ_C is the characteristic function of the set C. For functions f, g on $\hat{\mathbf{X}}_t$, $f(X) = g(X)$ β-a.e. means that the set $\{X \in \hat{\mathbf{X}}_t;\ f(X) \neq g(X)\}$ is β-null.

<u>DEFINITION.</u> We shall call a function $F \in \mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$ to be <u>integrable with respect to μ_t</u> if for any $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$, there exists a sequence $\{\Psi_n\}$ of $\mathcal{S}_0$ satisfying $F(X) = \lim_{n\to\infty}\Psi_n(X)$ β-a.e. and there exists $\operatorname{s-lim}_{n\to\infty} \int_{\hat{X}_t} d\mu_t(X)\ \Psi_n(X)\ g(X(0))$, which does not depend on $\{\Psi_n\}$ but only on F.

So we shall write

$$\int_{\hat{X}_t} d\mu_t(X)\ F(X)\ g(X(0)) = \underset{n\to\infty}{\text{s-lim}} \int_{\hat{X}_t} d\mu_t(X)\ \Psi_n(X)\ g(X(0)).$$

Let $\mathbf{B}([0,t]\times\mathbf{R}^d)$ be the space of bounded Borel measurable functions θ on $[0,t]\times\mathbf{R}^d$ such that $\tilde{\theta}(s) = \theta(s,\cdot) \in \mathbf{B}(\mathbf{R}^d)$ is piecewise continuous on $[0,t]$.

Let S be the set of those functions Ψ on $\hat{\mathbf{X}}_t$ for which there exist $m \in \mathbf{N}$, $C_k \in \mathcal{B}([0,t]\times\mathbf{R}^d)$ (= set of Borel subsets of $[0,t]\times\mathbf{R}^d$) $(k=1,2,\cdots,m)$ such that $\Psi(X) = \prod_{k=1}^m \int_0^t \chi_{C_k}(s,X(s))\, ds$.

Let $\mathcal{S}$ be the linear span of $\mathcal{S}_0 \cup S$.

Let $L(\mathcal{S},\beta)$ be the space of functions F of $\mathbf{B}(\hat{\mathbf{X}}_t:\hat{\otimes}_\pi)$ for which there exists a sequence $\{F_j\} \subset \mathcal{S}$ such that $\lim_{j\to\infty} \beta(F - F_j) = 0$. Then we have

Proposition. For $F \in L(\mathcal{S},\beta)$ and $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$, there exists a sequence $\{h_n\}$ of $\mathcal{S}_0$ such that

i) $F(X) = \lim_{n\to\infty} h_n(X)$ β-a.e. and

ii) $\underset{n\to\infty}{\text{s-lim}} \int_{\hat{X}_t} d\mu_t(X)\ h_n(X)\ g(X(0))$ exists.

Proof. For $F \in L(\mathcal{S},\beta)$, there exists a sequence $\{F_j\} \subset \mathcal{S}$, such that $\beta(F - F_j) < \frac{1}{2^j}$. For $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$ and $F_j \in \mathcal{S}$,

$\operatorname{s-lim}_{n\to\infty} \int_{\hat{X}_t} d\mu_t(X)\ (F_j)_{\sigma(n)}(X)\ g(X(0))$ exists. For any $\varepsilon > 0$, there exists $h_{j,n} \in \mathcal{S}_0$ such that $\|(F_j)_{\sigma(n)} - h_{j,n}\|_\pi < \varepsilon$. So we can find $\{h_n\} \subset \mathcal{S}_0$ satisfying the desired conditions. //

The above proposition shows that the space $L(\mathcal{S},\beta)$ consists of integrable functions with respect to μ_t and we have

Theroem 1. A C_0-group $\{U_t\}_{t\in\mathbf{R}}$ of unitary operators on $L^2(\mathbf{R}^d;\mathbf{C}^N)$ induces a $\mathcal{L}(L^2(\mathbf{R}^d;\mathbf{C}^N))$-valued generalized measure μ_t on $\hat{\mathbf{X}}_t$ such that the space $L(\mathcal{S},\beta)$ consisting of an integrable function with respect to μ_t is complete with respect to the seminorm β.

§3. The property of the generalized measure μ_t.

The generalized measure μ_t defined at §2 is not countably additive but it has a kind of convergence theorem as shown below.

DEFINITION. We shall call a sequence $\{f_n\} \subset \mathbf{B}(\mathbf{R}^d)$[resp. $\mathbf{B}([0,t]\times\mathbf{R}^d)$] to be (*)-sequentially compact if for any subsequence $\{f_{n(j)}\}$ of $\{f_n\}$, there exists a subsequence

$\{f_{n(j(k))}\}$ of $\{f_{n(j)}\}$ such that $f_{n(j(k))}(x)$[resp. $f_{n(j(k))}(s,x)$] converges to some function $g(x) \in \mathbf{B}(\mathbf{R}^d)$ for any $x \in \mathbf{R}^d \setminus N$ [resp. $g(s,x) \in \mathbf{B}([0,t]\times\mathbf{R}^d)$ for any $(s,x) \in [0,t] \times (\mathbf{R}^d \setminus N)$] with $\nu(N) = 0$ as $k \to \infty$, where ν is the Lebesgue measure on $\mathbf{R}^d$.

Then we have the following convergence theorems.

<u>**Theorem 2**</u>. For k, m $\in \mathbf{N}$ and $\{F_n\}_{n=0}^{\infty} \subset \mathbf{B}(\hat{\mathbf{X}}_t:\otimes_\pi,\Delta_m)$ with $F_n(X) = \Sigma_{\ell=1}^{K} \prod_{j=0}^{2^m} f_{j,\ell,n}(X(t_{j,m}))$, suppose

$\sup_n \|F_n\|_\pi < \infty$, $\lim_{n\to\infty} F_n(X) = F_0(X)$ a.e. on $\mathbf{R}^{(2^m+1)d}$ and $\{f_{j,\ell,n};\ 1\le j\le m,\ 1\le \ell\le K,\ n \in \mathbf{N}\}$ is (*)-sequentially compact.

Then we have

$$\int_{\hat{X}_t} d\mu_t(X)\ F_0(X)\ g(X(0)) = \operatorname*{s-lim}_{n\to\infty} \int_{\hat{X}_t} d\mu_t(X)\ F_n(X)\ g(X(0))$$

for any $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$

<u>Proof</u>. By the assumption, there exist a subsequence $\{n(k)\}$ and sequences $\{\tilde{f}_{j,\ell,n(k)}\} \subset \mathbf{B}(\mathbf{R}^d)$ and $\{h_{j,\ell}\} \subset \mathbf{B}(\mathbf{R}^d)$ satisfying

$\sup\{\|\tilde{f}_{j,\ell,n(k)}\|_\infty;\ 1\le j\le m,\ 1\le \ell\le K,\ k \in \mathbf{N}\} < \infty$,

$F_{n(k)}(X) - F_0(X) = \Sigma_{\ell=1}^{K} \prod_{j=0}^{2^m} \tilde{f}_{j,\ell,n(k)}(X(t_{j,m}))$

$\lim_{k\to\infty} \tilde{f}_{j,\ell,n(k)}(x) = h_{j,\ell}(x)$ a.e. on $\mathbf{R}^d$ and for any $\ell=1,\cdots,K$, there exists $j_\ell \in \{1,\cdots,m\}$ such that $\lim_{k\to\infty} \tilde{f}_{j_\ell,\ell,n(k)}(\dot{x}) = 0$ a.e. on $\mathbf{R}^d$. By using the property of μ_t, we have $\underset{n\to\infty}{\text{s-lim}} \int_{\hat{X}_t} d\mu_t(X)\ (F_{n(k)}(X) - F_0(X))g(X(0)) = 0$. By the property of (*)-sequential compactness, we have

$$\underset{n\to\infty}{\text{s-lim}} \int_{\hat{X}_t} d\mu_t(X)\ (F_n(X) - F_0(X))g(X(0)) = 0. \qquad //$$

For a function $F \in \mathbf{B}(\hat{\mathbf{X}}_t : \hat{\otimes}_\pi)$ we shall call $F(X) = 0$ α-a.e. if $F_{\sigma(n)}(X) = 0$ a.e. on $\mathbf{R}^{(2^n+1)d}$ for any $n \in \mathbf{N}$.

Then we have

<u>**Theorem 3**</u>. For $\{\theta_n\}_{n=0}^{\infty} \subset \mathbf{B}([0,t]\times\mathbf{R}^d)$, put $F_n(X) = \exp \int_0^t \theta_n(s,X(s))\,ds$ for any $X \in \hat{\mathbf{X}}_t$ and $n = 0,1,\cdots$. Suppose $\lim_{n\to\infty} F_n(X) = F_0(X)$ α-a.e. and $\{\theta_n\}_{n=0}^{\infty}$ is (*)-sequentially compact.

Then we have $F_n \in L(\mathcal{S},\mathcal{B})$ and

$$\int_{\hat{X}_t} d\mu_t(X)\ F_0(X)\ g(X(0)) = \underset{n\to\infty}{\text{s-lim}} \int_{\hat{X}_t} d\mu_t(X)\ F_n(X)\ g(X(0))$$

for any $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$.

Proof. By the property of (*)-sequential compactness, there exist a subsequence $\{n_j\}$ and $\tilde{\theta} \in \mathbf{B}([0,t]\times\mathbf{R}^d)$ such that

(3.1) $\lim_{j\to\infty}\theta_{n_j}(s,x) = \tilde{\theta}(s,x)$ for any $(s,x) \in [0,t] \times (\mathbf{R}^d \setminus N)$

with $\nu(N) = 0$. Then we have $\lim_{j\to\infty}F_{n_j}(X) = \exp\int_0^t\tilde{\theta}(s,X(s))\,ds$ α-a.e. , which implies $F_0(X) = \exp\int_0^t\tilde{\theta}(s,X(s))\,ds$ α-a.e. Put $G_{n,k}(X) = \Sigma_{\ell=0}^k \frac{1}{\ell!}\left(\int_0^t\theta_n(s,X(s))\,ds\right)^\ell$ and $\tilde{G}_k(X) = \Sigma_{\ell=0}^k \frac{1}{\ell!}\left(\int_0^t\tilde{\theta}(s,X(s))\,ds\right)^\ell$. By the definitions of the integral and μ_t, we have $\int_{\hat{X}_t} d\mu_t(X)\,\tilde{G}_k(X)\,g(X(0)) = \Sigma_{\ell=1}^k\int_0^t\int_0^{s_\ell}\cdots\cdots\int_0^{s_2} U_{t-s_\ell}\tilde{\theta}(s_\ell)$ $U_{s_\ell-s_{\ell-1}}\tilde{\theta}(s_{\ell-1})\cdots\cdots U_{s_2-s_1}\tilde{\theta}(s_1)U_{s_1}g\,ds_1ds_2\cdots ds_m + g$ for $g \in L^2(\mathbf{R}^d;\mathbf{C}^N)$. So by (3.1), we have

$\operatorname*{s-lim}_{n\to\infty}\int_{\hat{X}_t} d\mu_t(X)\,G_{n,k}(X)\,g(X(0)) = \int_{\hat{X}_t} d\mu_t(X)\,\tilde{G}_k(X)\,g(X(0))$. By using the relation $\lim_{k\to\infty}\beta(F_n - G_{n,k}) = 0$, we have the desired result. //

§4. The Feynman path integral.

Now we shall consider the Cauchy problem described at §1. By using the above theorem, we have

Theorem 4. Let H be a self-adjoint operator on a Hilbert space $L^2(\mathbf{R}^d;\mathbf{C}^N)$ and $0 < T < \infty$. Suppose $\overline{V} \in C^1([0,T];\mathbf{B}(\mathbf{R}^d))$, $V(t,\cdot) = \overline{V}(t)$ and g belongs to the domain of iH.
Then the solution $\Psi(t,\cdot)$ of the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} \Psi(t,x) = (-iH + \theta(t,x))\Psi(t,x) \\ \Psi(0,x) = g(x) \qquad 0 < t < T,\ x \in \mathbf{R}^d \end{cases}$$

is expressed as follows;

$$\Psi(t, \cdot) = \int_{\hat{\mathbf{X}}_t} d\mu_t(X) \exp\{\int_0^t V(s,X(s))\, ds\}\, g(X(0)).$$

Proof. iH generates a C_0-group $\{U_t\}_{t\in\mathbf{R}}$ of unitary operators on $L^2(\mathbf{R}^d;\mathbf{C}^N)$ such that $U_t = e^{iHt}$. By Theorem 1, the C_0-group induces a $\mathfrak{L}(L^2(\mathbf{R}^d;\mathbf{C}^N))$-valued generalized measure μ_t on $\hat{\mathbf{X}}_t$. Since $F(X) = \exp\{\int_0^t V(s,X(s))\, ds\}$ belongs to $L(\mathcal{S},\mathcal{B})$, there exists its integral with respect to μ_t and put

$$\overline{\Psi}(t) = \int_{\hat{\mathbf{X}}_t} d\mu_t(X)\, F(X)\, g(X(0)).$$

Then it holds that $\overline{\Psi}(t)= U_t\overline{\Psi}(0) + \int_0^t U_{t-s}\overline{V}(s)\overline{\Psi}(s)\, ds$ and $\overline{\Psi}(t)$ is the solution of the above Cauchy problem. //

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