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Selberg inequality

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§1. Introduction.

Very recently, K. and F. Kubo [4] discussed the Selberg inequality and gave it an elegant proof by using diagonal matrix which dominatesa positive semidefinite matrix.

The Selberg inequality. Let $x_1, \ldots, x_n$ be nonzero vectors in a Hilbert space H with inner product $(\ ,\ )$. Then, for all $x \in H$,

$$\sum_{i=1}^{n} \frac{|(x, x_i)|^2}{\sum_{j=1}^{n} |(x_i, x_j)|} \leq \|x\|^2. \tag{1}$$

It is easily seen that the Schwarz inequality is nothing but the case $n = 1$, and the Bessel one is also the case where $X = \{x_1, \ldots, x_n\}$ is mutually orthogonal. Therefore it might be supposed that if the equality in (1) holds, then x is a linear combination of X and X is mutually orthogonal. Clearly the former is necessary. However, not so is the latter. In fact, we can easily give counterexamples, one of which is given by

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \tag{2}$$

and $x = x_1 + x_2 + x_3$. Another is as follows :

$$y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ y_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ y_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3}$$

and $y = y_1 + y_2 + y_3$. These are typical examples in our discussion, but are essentially different as seen in the below. In sucession with them, Furuta [1] posed an elementary proof of the Selberg inequality and conditions enjoying the equality.

Theorem A. The equality in (1) holds if and only if $x = \sum_{i=1}^{n} a_i x_i$ for some complex numbers $a_1, ..., a_n$ such that $(x_i, x_j) = 0$ or $|a_i| = |a_j|$ with $(a_i x_i, a_j x_j) \geq 0$ for all $i \neq j$.

The purpose of this note is to continue the discussion due to Furuta [1]. To do this, we consider an operator corresponding to it. This is defined by

$$S_X = \sum_i \frac{x_i \otimes x_i}{\sum_j |(x_i, x_j)|}, \tag{4}$$

where $(w \otimes w)v = (v, w)w$ for $v, w \in H$, and it is called the Selberg operator for $X = \{x_1, ..., x_n\}$. Then the Selberg inequality says that every Selberg operator is a positive contraction. First of all, we give a refinement to Theorem A by the graph theoretic consideration for given vectors $x_1, ..., x_n$, by which we can see an ammusing relation between the Selberg inequality and the Schwarz one. That is, if S_X is a projection, then the subspace spanned by X is one dimensional.

§2. The Selberg inequality.

Suppose that $x_1, ..., x_n \in H$ is given. We consider a graph G with vertices $\{1, 2, ..., n\}$ such that

(i, j) is an arc if $(x_i, x_j) \neq 0$.

In other words, i and j are adjacent if $(x_i, x_j) \neq 0$. Then G is decomposed into connected components;

$G = G_1 \cup ... \cup G_m$

and so we have, by the unicity of the decomposition,

Lemma 1. $X = \{x_1, ..., x_n\}$ is uniquely decomposed into $X_1, ... X_m$ such that

(a) $X_1, ... X_m$ are mutually orthogonal, i.e., for $k \neq l$, $(v, w) = 0$ for all $v \in X_k$ and $w \in X_l$, and

(b) Each X_k is not decomposed into nontrivial subsets which are orthogonal. $v \neq w \in X_k$.

Proof. For the completeness, we give a proof. Take another decomposition $\{Y_1, ..., Y_l\}$. For each $x = x_i$, if $x \in X_j$ and $x \in Y_k$, then we have to show $X_j = Y_k$. Suppose that $Z_1 = X_j \backslash Y_k \neq \phi$ and $Z_2 = X_j \cap Y_k \neq \phi$. Since X_j is decomposed into Z_1 and Z_2, this contradicts to (b).

Here we call $\{X_1, ..., X_m\}$ the minimal decomposition of X, and X is called connected if the minimal decomposition of X is X itself.

Theorem 2. Notation as in above. Then the equality in (1) holds if and only if x is a direct sum of $z_1, ..., z_m$, where each z_s is of form $\sum a_i x_i$ such that $x_i \in X_s$,

(5) $\qquad (a_i x_i, a_j x_j) \geq 0$, and $|a_i| = |a_j|$ for all i, j.

In particular, if X is connected, then $x = \sum a_i x_i$ and (5) is enjoyed.

Proof. We may assume that X is connected, that is, for each $i, (x_i, x_j) \neq 0$ for some $j \neq i$. Following Furuta, if we put $a_i = (x, x_i)/c_i$, where $c_i = \sum_{j=1}^n |(x_i, x_j)|$, then

$$
\begin{aligned}
&0 \leq \|x - \textstyle\sum a_i x_i\|^2 \\
&= \|x\|^2 - 2Re \textstyle\sum a_i^* (x, x_i) + \sum_{i,j} (a_i x_i, a_j x_j) \\
&\leq \|x\|^2 - 2Re \textstyle\sum a_i^* (x, x_i) + \sum_{i,j} (|a_i|^2 + |a_j|^2)|(x_i, x_j)|/2 \\
&= \|x\|^2 - 2Re \textstyle\sum a_i^* (x, x_i) + \sum |a_i|^2 c_i \\
&= \|x\|^2 - \textstyle\sum |(x, x_i)|^2 / c_i.
\end{aligned}
$$

Therefore the equality in (1) holds if and only if $x = \sum a_i x_i$ and

(6) $\qquad \sum_{i,j} 2Re(a_i x_i, a_j x_j) = \sum_{i,j} (|a_i|^2 + |a_j|^2)|(x_i, x_j)|$.

However, since

$$2Re(a_i x_i, a_j x_j) \leq (|a_i|^2 + |a_j|^2)|(x_i, x_j)|$$

is always valid for all i and j, (6) is equivalent to

(7) $\qquad 2Re(a_i x_*, a_j x_j) = (|a_i|^2 + |a_j|^2)|(x_i, x_j)|$

for all i and j. For each i, if we choose j adjacent to i, then (7) implies that

(8) $\qquad (a_i x_i, a_j x_j) = |(a_i x_i, a_j x_j)| \geq 0$ and $|a_i| = |a_j|$.

By the connectedness of X, (8) holds true for all i and j.

Conversely, if (8) is enjoyed for all i and j, then so is (7) and hence the equality in (1) holds.

Remark. As you know, the difference of the examples (2) and (3) is due to connectedness. Actually (2) is the connected case, but $\{y_1, y_2, y_3\}$ in (3) is decomposed into $\{y_1, y_2\}$ and $\{y_3\}$, which is the minimal decomposition. Furthermore (2) shows that a mixed type of $(x_1, x_2) = 0$ and (5) happens even if it is connected. To speak plainly, if X is connected and the equality in (1) holds, then (5) must be enjoyed nevertheless there are orthogonal vectors in X, cf. also [1 ; Remark 1].

§3. The Selberg operator.

In this section, we shall make an operator theoretic consideration for the Selberg inequality. First of all, we remark the following facts : The Schwarz inequality is represented by a rank one projection Q as follows;

(9) $$\|Qx\| \leq \|x\|$$

for all $x \in H$, see [3; Solution 2]. Along with this, if Q in (9) is a rank n projection, then it corresponds to the Bessel one. Furthermore (9) might be regarded as

(9)' $$(Qx, x) \leq \|x\|^2.$$

We thus define the Selberg operator S_X for $X = \{x_1, ..., x_n\}$ by (4). Noting that every Selberg operator is a finite rank positive contraction, we have

Theorem 3. Suppose $X = \{x_1, ..., x_n\}$ is connected. If 1 is an eigenvalueof the Selberg operator S_X, then it is simple. In other words, the eigenspace for 1 is at most one dimensional.

Proof. Since $S = S_X$ is a positive contraction, its eigenspace M for 1 coincides with the subspace $\{x \in H; \|Sx\| = \|x\|\}$. Then it follows from Theorem 2 that M is at most one dimensional. As a matter of fact, suppose that there exists a nonzero vector $x = \sum a_i x_i \in M$. For any adjacent j to 1, since $(a_1 x_1, a_j x_j) \geq 0$, we have

$$arg\ a_j = arg(x_1, x_j) + arg a_1.$$

Moreover, for any adjacent k to j,

$$arg\ a_k = arg\ (x_j, x_k) + arg\ a_j$$

Like this, since X is connected, the ratios of $\{arg\ a_i; i = 1, 2, ..., n\}$ is determined and does not depend on vectors in M. Therefore M is one dimensional.

From our graph theoretical viewpoint, the Bessel inequality is an extension of the Schwarz one on the number of connected components. The following corollary means that the Selberg inequality for connected $\{x_1, ..., x_n\}$ is another extension of the Schwarz one.

Ccrollary 4. Notation as in above. If the Selberg operator S_X is a nonzero projection, then the subspace spanned by X is one dimensional.

Remark. In the Selberg operator S_X, the attachment of normalizing constants $\sum_j |(x_i, x_j)|$ is very meaningful. That is actually clarified by the following example ;

$$x = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix},\ y = \begin{pmatrix} (-1+\sqrt{3})/6 \\ (-1-\sqrt{3})/6 \end{pmatrix} \text{ and } z = x + y.$$

Then $\{x, y, z\}$ is connected and moreover we have

$$x \otimes x + y \otimes y + z \otimes z = 1.$$

Selberg operators are not necessarily norm 1:

Corollary 5. If there is a circuit $\{x_1, ..., x_k\}$ in a connected X such that

$$arg(x_1, x_2)(x_2, x_3)...(x_{k-1}, x_k)(x_k, x_1) \neq 0,$$

then there is no nonzero x such that $\|S_X x\| = \|x\|$.

A simple example for the above is as follows ;

$$x_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

On the other hand, we have $\|S_X\| = 1$ in the following case.

Corollary 6. If X is a tree as a graph, then $\|S_X\| = 1$.

§4. Generalization.

K. and F. Kubo and Furuta gave a generalization of the Selberg inequality independently.

Theorem B. Let T be a bounded linear operator on a Hilbert space H. If $x_1, ..., x_n \notin$ kernel(T^*), then

$$\sum_{i=1}^{n} \frac{|(Tx, x_i)|^2}{\sum_{j=1}^{n} |(|T^*|^{2\beta} x_i, x_j)|} \leq \||T|^{\alpha} x\|^2$$

for all $x \in H$ and $0 \leq \alpha \leq 1$, where $\beta = 1 - \alpha$.

Furuta also discussed the equality case. So we review it in our situation. Let $T = U|T|$ be the polar decomposition of T. Following them, replacing $|T|^{\alpha} x$ to x and $|T|^{\beta} U^* x_i$ to x_i in Theorem 2, we have Theorem B. Now we define a graph K with vertices $1, 2, ..., n$ such that

$$(i, j) \text{ is an arc if } (|T^*|^{2\beta} x_i, x_j) \neq 0.$$

then we obtain the following equivalent condition to the equality case in (1) :

Theorem 7. Assume that K is connected. Then the equality in (1) holds if and only if $Tx = \sum a_i U |T^*|^{2\beta} x_i$, and

$$(a_i |T^*|^{2\beta} x_i, x_j) \geq 0 \quad \text{and} \quad |a_i| = |a_j|$$

for all i and j.

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