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Restrictions of reproducing kernel Hilbert spaces to subsets (Preliminary Reports)

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1. Introduction.

Let E be an arbitrary set and K(p,q) be a complex valued positive matrix on E in the sense of Aronszajn-Moore [3]; that is, for any finite points $\{p_j\}_{j=1}^n$ of E and for any complex numbers $\{C_j\}_{j=1}^n$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} C_i \overline{C_j} K(p_j,p_i) \geq 0.$$

Then, by the fundamental theorem of Aronszajn-Moore [3], there exists a uniquely determined functional Hilbert (possibly finite dimensional) space H_K (reproducing kernel Hilbert space admitting the reproducing kernel K(p,q)) composed of functions f(p) on E such that

$$K(p,q) \in H_K \quad \text{for any fixed } q \in E$$

and, for any $f \in H_K$ and for any $q \in E$,

$$(f(p),\ K(p,q))_{H_K} = f(q).$$

For the general properties of reproducing kernel Hilbert spaces, see Aronszajn [3] and Saitoh [10].

The general properties of the subspaces of reproducing kernel

Hilbert spaces were examined by Chalmers [5]. Some special subspaces of the Bergman spaces were examined by Davis [6] from the viewpoint of doubly orthogonal functions, approximation and analytic extension problems.

Meanwhile, Okubo [7] examined a special and new subspace of the Szegö space on the unit disc in the investigations of the high energy physics. See also Aikawa-Hayashi-Saitoh [1] for some subspaces of this type for the Hilbert spaces of Szegö type on strip domains. For the Bergman and the Szegö spaces, see Bergman [4] and Saitoh [10].

We will examine the subspaces of Okubo type for reproducing kernel Hilbert spaces from the viewpoint of the general theory of reproducing kernels by Aronszajn [3]. Here, the subspace of Okubo type will be stated in a general situation as follows:

For a subset $X \subset E$, we consider a Hilbert (possibly finite dimensional) space $H(X)$ composed of complex-valued functions on X. We assume that

(a) for the restrictions $f|_X$ of the members f of H_K to the set X, $f|_X$ belong to the Hilbert space $H(X)$,

and

(b) the linear operator $Tf = f|_X$ is continuous from H_K into $H(X)$.

The subscript in T_p indicates that T is applied on a function of p.

We introduce the inner product $(f,g)_{H_K[H(X)]}$ for the members f and g of H_K by the sum

$$(f,g)_{H_K[H(X)]} = (f,g)_{H_K} + (Tf,Tg)_{H(X)}. \tag{1.1}$$

We will examine this Hilbert space $H_K[H(X)]$ of Okubo type equipped with the inner product (1.1) in connection with the Hilbert spaces H_K and $H(X)$.

When the two Hilbert spaces H_K and $H(X)$ are reproducing kernel Hilbert spaces on the same set E (i.e. E = X) without assumptions (a) and (b), the Hilbert space $H_K[H(X)]$ for $H_K \cap H(X)$ equipped with the inner product (1.1) was examined by Ando [2]. The construction of the reproducing kernel of this Hilbert space seems to be abstract. Ando's theorem ([2, p. 34, Theorem 2.3]) and our argument in the special case will be understood as a "dual" of those for the Hilbert space ([3, pp. 352-354]) admitting the reproducing kernel which is the sum of reproducing kernels.

Meanwhile, recall that the norms of the type (1.1) are similar to those of the Sobolev spaces in the framework of Hilbert spaces.

In this paper, we will furthermore examine some related extremal problems and approximations in connection with the linear operator T. The contents are as follows:

§2. Properties of the restriction operator T.

§3. Hilbert spaces derived from the space H(X).

§4. Extremal function of $\sup_{\|f\|_{H_K} \leq 1} \|Tf\|_{H(X)}$.

§5. Best approximation of $g \in H_K$ by H_K functions taking assigned values.

§6. Best approximation of $F \in H(X)$ by H_K functions.

§7. Examples.

For the sake of the nice properties of T and its adjoint operator T^* in our situation, we will be able to give "algorithmics" to get constructively the reproducing kernel for the space $H_K[H(X)]$, the extremal functions in §4, and best approximations in §5 and §6. Indeed, we shall give intrinsic relations and representations of these functions in terms of the given data. This point of view will be important in this paper.

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2. Properties of the restriction operator T.

Since the restriction operator Tf is continuous from H_K into H(X), we can define the continuous adjoint operator T^* from H(X) to H_K by the rule

$$(F, Tg)_{H(X)} = (T^*F, g)_{H_K} \text{ for all } F \in H(X) \text{ and for all } g \in H_K.$$

Then, we obtain the following fairly simple expression of the adjoint operator T^*:

Lemma 2.1. We define the linear mapping from $F \in H(X)$ into the functions on E by

$$f(p) = (F(\cdot), TK(\cdot,p))_{H(X)}, \quad p \in E. \tag{2.1}$$

Then, we have

$$f = T^*F \tag{2.2}$$

and so, in particular, $f \in H_K$. Furthermore, we obtain the identity

$$\|f\|^2_{H_K} = \Big(F(p), T_p(F(\cdot), TK(\cdot,p))_{H(X)}\Big)_{H(X)}. \tag{2.3}$$

Proof. We set $\mathbb{K}_p(\cdot) = K(\cdot,p)$. Then, we have directly

$$f(p) = (F(\cdot), TK(\cdot,p))_{H(X)}$$

$$= (F, T\mathbb{K}_p)_{H(X)}$$

$$= (T^*F, \mathbb{K}_p)_{H_K}$$

$$= (T^*F)(p),$$

which proves (2.2).

Furthermore, we have

$$\|f\|^2_{H_K} = \|T^*F\|^2_{H_K}$$

$$= (T^*F, T^*F)_{H_K}$$

$$= (F, TT^*F)_{H(X)}$$

$$= (F(p), T_p(F(\cdot), TK(\cdot,p))_{H(X)})_{H(X)},$$

which is the desired identity.

Lemma 2.2. The following items are equivalent:

(i) $K(p,q) >> (TK(\cdot,q), TK(\cdot,p))_{H(X)}$;

that is,

$$K(p,q) - (TK(\cdot,q), TK(\cdot,p))_{H(X)}$$

is a positive matrix.

(ii) $\|T\| \le 1$.

(iii) For any $F \in H(X)$ and for $f = T^*F$, $\|f\|_{H_K} \le \|F\|_{H(X)}$.

Proof. Since $K(p,q) = (\mathbb{K}_q, \mathbb{K}_p)_{H_K}$, from the identity

$$(TK(\cdot,q),\ TK(\cdot,p))_{H(X)} = (T\mathbb{K}_q,\ T\mathbb{K}_p)_{H(X)} = (T^*T\ \mathbb{K}_q,\ \mathbb{K}_p)_{H_K}$$

we have

$$K(p,q) - (TK(\cdot,q),\ TK(\cdot,p))_{H(X)}$$

$$= ((I - T^*T)\mathbb{K}_q,\ \mathbb{K}_p)_{H_K}.$$

We thus have the desired results:

$$K(p,q) - (TK(\cdot,q),\ TK(\cdot,p))_{H(X)} \gg 0$$

$$\iff \quad I - T^*T \gg 0$$
$$\iff \quad \|T\| \le 1$$
$$\iff \quad \|T^*\| \le 1.$$

The images $f(p) = (T^*F)(p)$ in (2.1) for $F \in H(X)$ belong intrinsically to the Hilbert space $H_{\mathbb{K}}$ admitting the reproducing kernel

$$\mathbb{K}(p,q) = (T\mathbb{K}_q, T\mathbb{K}_p)_{H(X)}$$

and we have the inequality

$$\|f\|_{H_{\mathbb{K}}} \le \|F\|_{H(X)}$$

([10, p. 83, Theorem 3.2). Hence, if an item in Lemma 2.2 is valid, then we have the sharp inequalities in (iii)

$$\|f\|_{H_K} \le \|f\|_{H_{\mathbb{K}}} \le \|F\|_{H(X)}.$$

Furthermore, for the inverse of the mapping $f(p) = (T^*F)(p)$, see [9] and [10, chapter VI].

Lemma 2.3. The mapping $f(p) = (T^*F)(p)$ from $H(X)$ into H_K is onto H_K if and only if there exists a positive constant L such that

$$K(p,q) \ll L(T\mathbb{K}_q, T\mathbb{K}_p)_{H(X)}.$$

Proof. That the mapping T^* from $H(X)$ is onto H_K is equivalent to the relation $H_K \subset H_{\mathbb{K}}$. The relation $H_K \subset H_{\mathbb{K}}$ is equivalent to the condition in Lemma 2.3 ([3, p. 383, Corollary IV_2]), and so, we have the desired result.

When the norm in the space $H(X)$ is realized in terms of a σ finite positive measure $d\mu$ on X in the form

$$\|F\|^2_{H(X)} = \int_X |F(p)|^2 \, d\mu(p),$$

there exists a general method to check our basic assumption (b).

Indeed, from the reproducing property of $K(p,q)$ for H_K, we have the inequality

$$|f(p)|^2 \leq K(p,p) \, \|f\|^2_{H_K},$$

and so, it is sufficient to see that

$$\int_X K(p,p) \, d\mu(p) < \infty,$$

for the validness of (b).

3. Hilbert spaces derived from the space $H(X)$.

By definition, we note that for any $f \in H_K[H(X)]$

$$\|f\|_{H_K[H(X)]} \geq \|f\|_{H_K} \tag{3.1}$$

and so, there exists a reproducing kernel $K_{H(X)}(p,q)$ for the space $H_K[H(X)]$ such that

$$K_{H(X)}(p,q) << K(p,q) \ ; \tag{3.2}$$

([3, p. 355, Theorem II]).

Meanwhile, since $f \in H_K[H(X)]$ for any $f \in H_K$, from the closed graph theorem ([3, p. 382, Theorem IV]), we see that there exists a positive constant M satisfying

$$K(p,q) << M\ K_{H(X)}(p,q)$$

and so, for any $f \in H_K$

$$\|f\|_{H_K} \geq \left(\sqrt{M}\right)^{-1} \|f\|_{H_K[H(X)]}.$$

Of course, as the sets of functions we have

$$H_K = H_K[H(X)].$$

For a characterization of the reproducing kernel $K_{H(X)}(p,q)$ we obtain

Theorem 3.1. The reproducing kernel $K_{H(X)}(p,q)$ is characterized as the solution $\tilde{K}(p,q)$ of the functional equation

$$K(p,q) = \tilde{K}(p,q) + (T\tilde{K}(\cdot,q),\ TK(\cdot,p))_{H(X)} \tag{3.3}$$

satisfying the condition

$$\tilde{K}(\cdot,q) \in H_K \quad \text{for all } q \in E. \tag{3.4}$$

Proof. We set $\tilde{\mathbb{K}}_p(\cdot) = \tilde{K}(\cdot,p)$. Then, we have directly: for every $q \in E$,

$$(\mathbb{K}_q, \mathbb{K}_p)_{H_K} = (\tilde{\mathbb{K}}_q, \mathbb{K}_p)_{H_K} + (T\tilde{\mathbb{K}}_q, T\mathbb{K}_p)_{H(X)} \text{ for all } p \in E$$

$<=>$

$$\mathbb{K}_q = \tilde{\mathbb{K}}_q + T^*T\tilde{\mathbb{K}}_q$$

$<=>$

$$(f, \mathbb{K}_q)_{H_K} = (f, \tilde{\mathbb{K}}_q)_{H_K} + (f, T^*T\tilde{\mathbb{K}}_q)_{H_K} \quad \text{for all } f \in H_K$$

$<=>$

$$f(q) = (f(\cdot), \tilde{\mathbb{K}}(\cdot,q))_{H_K[H(X)]} \text{ for all } f \in H_K.$$

which implies the desired result.

If, for any $f \in H_K$

$$\|f\|_{H_K} \geq \|Tf\|_{H(X)}$$

and, equality holds if and only if $f = 0$, then we can introduce the pre-Hilbert space H' equipped with the inner product

$$(f,g)_{H'} = (f,g)_{H_K} - (Tf,Tg)_{H(X)} .$$

In order to consider a functional completion of H' admitting a reproducing kernel, recall Theorem of N. Aronszajn [3, p. 347]:

In order that there exists a functional completion of H' it is necessary and sufficient that (1) for every fixed $p \in E$ the linear functional $f(p)$ defined on H' is bounded and (2) for a Cauchy sequence $\{f_n\} \subset H'$, the condition $f_n(p) \to 0$ for every $p \in E$ implies $\|f_n\|_{H'} \to 0$. If the functional completion is possible, it is unique.

When the conditions (1) and (2) are satisfied, we denote the functional completion of H' by $H_K^-[H(X)]$. Then, for any $f \in H_K$

$$\|f\|_{H_K^-[H(X)]} \leq \|f\|_{H_K}$$

and so, H_K is a subspace of $H_K^-[H(X)]$. We denote a reproducing kernel for the Hilbert space $H_K^-[H(X)]$ by $K_{H(X)}^-(p,q)$. For this reproducing kernel $K_{H(X)}^-(p,q)$, we obtain the similar characterization as in Theorem 3.1 by exchanging + by - in (3.3). This argument will be understood as a "dual" of that for the Hilbert space admitting the reproducing kernel which is a positive matrix expressed by the difference of two reproducing kernels ([3, pp. 354-357]).

4. Extremal function of $\sup_{\|f\|_{H_K} \leq 1} \|Tf\|_{H(X)}$.

Under assumptions (a) and (b) for the linear operator T from H_K into H(X), we shall consider the extremal problem

$$\sup_{\|f\|_{H_K} \leq 1} \|Tf\|_{H(X)}.$$

For this problem, we have

Theorem 4.1. We obtain the identities

$$\sup_{f \in H_K} \frac{\left(T_q f(q),\ T_q(Tf(\cdot),\ TK(\cdot,q))_{H(X)}\right)_{H(X)}}{\|Tf\|^2_{H(X)}}$$

$$= \text{the maximum eigenvalue } \mu \text{ of } TT^*$$

$$= \|T^*\|^2 = \|T\|^2. \tag{4.1}$$

When there exists, we set the extremal function (eigenfunction) as $\hat{f}$ for this extremal problem. Then, we obtain

$$\max_{\|f\|_{H_K}^2 \le 1} \|Tf\|_{H(X)}^2 \le \mu \tag{4.2}$$

and the extremal function h(p) attaining the equality is given by

$$h(p) = \frac{(T\hat{f}(\cdot),\ TK(\cdot,\ p))_{H(X)}}{(T_q\hat{f}(q), T_q(T\hat{f}(\cdot), TK(\cdot,q))_{H(X)})_{H(X)}^{1/2}} . \tag{4.3}$$

Proof. The identity (4.1) follows directly from the identity

$$\left(T_q f(q),\ T_q(Tf(\cdot),\ TK(\cdot,q))_{H(X)}\right)_{H(X)}$$

$$= (Tf,\ TT^*\cdot Tf)_{H(X)} .$$

Furthermore, when there exists the extremal function $\hat{f}$, we have

$$T\,T^*\cdot T\,\hat{f} = \mu\, T\,\hat{f}$$

and so,

$$T^*\,T\cdot (T^*T\,\hat{f}) = \mu\, T^*\,T\,\hat{f},$$

and we thus obtain (4.3).

5. Best approximation of $g \in H_K$ by H_K functions taking assigned values.

For any fixed $F \in H(X)$, we consider the subset $H_K(X,F)$ of H_K such that

$$H_K(X,F) = \{f \in H_K\ ;\ Tf(p) = F(p) \text{ on } X\}. \tag{5.1}$$

Then, we shall determine the best approximation of any $g \in H_K$ by $H_K(X,F)$ functions in the sense that

$$\min_{f \in H_K(X,F)} \|f - g\|^2_{H_K} . \tag{5.2}$$

For this purpose, recall the construction of the Hilbert space $H_{K|X}$ admitting the reproducing kernel $K(p,q)|X$ restricted to the subset X ([3, pp. 350-352]). The reproducing kernel Hilbert space $H_{K|X}$ is composed of functions on X which are obtained by the restrictions $f|_X$ of H_K functions f to X. The norm in $H_{K|X}$ is given by

$$\|f|_X\|_{H_{K|X}} = \min \|h\|_{H_K}$$

where, the minimum is taken over all functions $h \in H_K$ whose restrictions are $f|_X$ on X.

For any $f \in H_K$, we have the expression, by the reproducing property of $K(p,q)|X$

$$f(p) = (f|_X(\cdot),\ K(\cdot,p)|X)_{H_{K|X}} \quad \text{on } X. \tag{5.3}$$

Then, for $f|_X = F$, we define the function f^* on E by

$$f^*(p) = (F(\cdot),\ TK(\cdot,p))_{H_{K|X}} . \tag{5.4}$$

Then, for $H(X) = H_{K|X}$ in the general situation, assumptions (a) and (b) are satisfied, and furthermore, the items in Lemma 2.2 are valid. From (5.3) and (5.4), we first see that

$$Tf^*(p) = F(p) \quad \text{on } X. \tag{5.5}$$

In particular, note that

$$f^* \in H_K \text{ and } \|f^*\|_{H_K} = \|F\|_{H_{K|X}}.$$

We thus see that f^* is the extremal function in the sense that

$$\|f^*\|_{H_K} = \min_{f \in H_K(X,F)} \|f\|_{H_K} . \tag{5.6}$$

By the translation f - g, we obtain

Theorem 5.1. For any given $g \in H_K$, the extremal function f^* $\in H_K$ satisfying

$$\min_{f \in H_K(X,F)} \|f - g\|^2_{H_K} = \|f^* - g\|^2_{H_K}$$

is given by

$$f^*(p) = g(p) + (F(\cdot) - Tg(\cdot),\ TK(\cdot,p))_{H_{K|X}} .$$

6. Best approximation of $F \in H(X)$ by H_K functions.

We will consider a fundamental approximation problem in the two Hilbert spaces $H(X)$ and H_K such that for any $F \in H(X)$

$$\inf_{f \in H_K} \|Tf - F\|_{H(X)} .$$

When there exist the extremal functions $f^* \in H_K$ in the sense that

$$\min_{f \in H_K} \|Tf - F\|_{H(X)} = \|Tf^* - F\|_{H(X)}, \tag{6.1}$$

note that f^* are characterized by the orthogonality

$$(Tf^* - F,\ Tf)_{H(X)} = 0 \quad \text{for all } f \in H_K;$$

which is equivalent to

$$(Tf^*(\cdot) - F(\cdot),\ TK(\cdot,p))_{H(X)} = 0 \quad \text{for all } p \in E.$$

Furthermore, it is equivalent to

$$T^*Tf^* = T^*F$$

$$\Longleftrightarrow$$

$$T^*F \in \text{range}\ (T^*T).$$

When there exist the extremal functions f^*, from the relation

$$\ker(T^*T) \oplus \overline{\text{range}(T^*T)} = H_K,$$

we see that the mapping

$$T^*T \mid_{\overline{\text{range}(T^*T)}}$$

is one to one and onto range(T^*T). So, we denote its inverse by $(T^*T)^{-1}$. Then, we note the identity

$$\min \|f^*\|_{H_K} = \|f^{**}\|_{H_K} = \|(T^*T)^{-1} \cdot T^*F\|_{H_K}.$$

We shall give a realization of the abstract operator $(T^*T)^{-1}$. For this purpose, we shall determine the functions f^* by the functional equation

$$(Tf^*(\cdot),\ TK(\cdot,p))_{H(X)} = (T^*F)(p) \in \text{range}\ (T^*T).$$

We have the expression

$$\begin{aligned}(T^*F)(p) &= (Tf^*(\cdot),\ TK(\cdot,p))_{H(X)} \\ &= (f^*(\cdot),\ T^*TK(\cdot,p))_{H_K} \\ &= \Big(f^*(r),\ (TK(\cdot,p),\ TK(\cdot,r))_{H(X)}\Big)_{H_K}. \qquad (6.2)\end{aligned}$$

In order to determine the natural Hilbert space formed by the images of the transform (6.2) by, in general, H_K functions, we shall compute the kernel form on E

$$k(p,q) = (T^*TK(\cdot,q),\ T^*TK(\cdot,p))_{H_K}$$

$$= (TK(\cdot,q),\ TT^*TK(\cdot,p))_{H(X)}$$

$$= \left(T_r K(r,q),\ T_r(TK(\cdot,p),\ TK(\cdot,r))_{H(X)}\right)_{H(X)}. \qquad (6.3)$$

Then, the images belong to the Hilbert space H_k admitting the reproducing kernel k(p,q) on E ([10, p. 83, Theorem 3.2]).

In the transform (6.2), we shall determine the natural inverse $f^{**} \varepsilon H_K$ for $T^*F \varepsilon H_K$ in the sense that

$$\|f^{**}\|_{H_K} = \min \|f^*\|_{H_K} \qquad (6.4)$$

among all the functions $f^* \varepsilon H_K$ satisfying the functional equation

$$(T^*F)(p) = (f^*(\cdot),\ T^*\ TK(\cdot,p))_{H_K} \qquad (6.5)$$

In order to express f^{**} in terms of T^*F, we can use Theorem 4.1 & 4.2 in [9] and Theorems 4.1 & 4.2 in [10, chapter VI]. Here, we shall write them formally

$$f^{**}(p) = (T^*F(\cdot),\ T^*TK(\cdot,p))_{H_k}, \qquad (6.6)$$

in the sense of their Theorems, respectively. Then, we obtain

Theorem 6.1. We assume that $T^*F \varepsilon$ range (T^*T). Then, there exist the extremal functions f^* in the sense of (6.1). Then, by using the formal expression (6.6), the function

$$f^{**}(p) = ((F(s), T_s K(s,\cdot))_{H(X)}, T^* TK(\cdot,p))_{H_k}$$

gives the extremal function in the sense that

$$\min \|f^*\|_{H_K} = \|f^{**}\|_{H_K}$$

among the functions $f^* \varepsilon H_K$ satisfying the equation

$$\min_{f \varepsilon H_K} \|Tf - F\|_{H(X)} = \|Tf^* - F\|_{H(X)}.$$

In order to assure a unique correspondence in the mapping

$$T^* F \dashrightarrow f^* \text{ in } H_K,$$

we shall need the natural assumptions that

(c) $\{Tf;\ f \varepsilon H_K\}$ is complete in $H(X)$,

and

(d) $Tf = 0$ in $H(X)$ for $f \varepsilon H_K$ implies $f = 0$ on E.

Indeed, when these assumptions (c) and (d) are valid, for any $f \varepsilon H_K$ satisfying

$$(f(\cdot), T^* TK(\cdot,p))_{H_K} = 0 \text{ on } E$$

we have

$$(Tf(\cdot), TK(\cdot,p))_{H(X)} = 0$$

and, by (c)

$$Tf = 0 \text{ in } H(X),$$

and so, by (d) we have the desired result that $f = 0$ on E.

7. Examples.

(I). It is in general difficult to solve explicitly the equation (3.3) satisfying (3.4) for a given reproducing kernel $K(p,q)$. For a general method, see also [7] in connection with an integral equation. Here, we will examine two typical cases where the solutions $K_{H(X)}(p,q)$ are explicitly determined.

First, in our general situation, we assume that

$$X = \{\xi_1, \xi_2, \ldots, \xi_n\} \tag{7.1}$$

and, for $f \in H_K$

$$\|f\|^2_{H(X)} = \sum_{\nu=1}^{n} |f(\xi_\nu)|^2 \ ; \tag{7.2}$$

that is, $H(X)$ is the usual Euclidean n-dimensional space. Then, from (3.3) we have

$$K(p,q) = K_{H(X)}(p,q) + \sum_{\nu=1}^{n} K_{H(X)}(\xi_\nu,q)\ K(p,\xi_\nu). \tag{7.3}$$

By setting $p = \xi_\mu$, $\mu = 1,2,\ldots,n$, we have the linear equations

$$\sum_{\nu=1}^{n} \left(\delta^\mu_\nu + K(\xi_\mu,\xi_\nu)\right) K_{H(X)}(\xi_\nu,q) = K(\xi_\mu,q), \tag{7.4}$$

$$\mu = 1,2,\ldots, n.$$

Since the matrix

$$\left\| \delta^\mu_\nu + K(\xi_\mu,\xi_\nu) \right\|$$

is positive definite, we denote its inverse by $\|A_{\mu\nu}\|$. Then, we have

$$K_{H(X)}(\xi_\nu, q) = \sum_{\mu=1}^{n} A_{\nu\mu} K(\xi_\mu, q), \quad \nu = 1,2,\ldots,n.$$

Hence, from (7.3) we obtain the explicit solution

$$K_{H(X)}(p,q) = K(p,q) - \sum_{\nu,\mu=1}^{n} K(p,\xi_\nu) A_{\nu\mu} K(\xi_\mu, q). \tag{7.5}$$

In particular, for one point $X = \{\xi\}$, we have

$$K_{H(X)}(p,q) = K(p,q) - \frac{K(p,\xi)\, K(\xi,q)}{1 + K(\xi,\xi)} .$$

(II). We consider the positive matrix, for $q > 0$

$$K_q(z,\bar{u}) = \frac{1}{2\pi} \int_{-q}^{q} e^{izt} e^{-i\bar{u}t}\, dt = \frac{\sin q(z - \bar{u})}{\pi(z - \bar{u})} , \tag{7.6}$$

on the entire complex plane $\mathbb{C}$. The positive matrix $K_q(z,\bar{u})$ is a reproducing kernel of Szegö type in our sense [1] and the Hilbert space H_{K_q} admitting the reproducing kernel $K_q(z,\bar{u})$ is composed of entire functions $f(z)$ of exponential type $\le q$ with finite norms

$$\|f\|^2_{H_{K_q}} = \int_{-\infty}^{\infty} |f(x)|^2\, dx. \tag{7.7}$$

Note that for $f \in H_{K_q}$

$$\int_{-\infty}^{\infty} |f(x + \xi i)|^2\, dx < \infty$$

for any real number ξ. For this fact, see also (7.10) - (7.12). So, we will consider

$$X = \{\xi + p_\nu i \ ; \ \nu = 1,2,\ldots,n, \ \xi \in (-\infty, \infty)\}$$

and

$$\|f\|^2_{H(X)} = \sum_{\nu=1}^{n} \int_{-\infty}^{\infty} |f(\xi + p_\nu i)|^2 \, d\xi. \tag{7.8}$$

Then, $K_{H(X)}(z,\bar{u})$ satisfies the functional equation following (3.3)

$$\frac{\sin q(z - \bar{u})}{\pi(z - \bar{u})} = K_{H(X)}(z,\bar{u})$$

$$+ \sum_{\nu=1}^{n} \frac{1}{\pi} \int_{-\infty}^{\infty} K_{H(X)}(\xi + p_\nu i,\bar{u}) \, \overline{\frac{\sin q(\xi + p_\nu i - \bar{z})}{\xi + p_\nu i - \bar{z}}} \, d\xi. \tag{7.9}$$

In order to find an explicit solution $K_{H(X)}(z,\bar{u})$, we use a general method for integral transforms ([10, chapter 6]).

From the identity (7.6), we see that $f(z) \in H_{K_q}$ is expressible in the form

$$f(z) = \frac{1}{2\pi} \int_{-q}^{q} F(t) \, e^{izt} \, dt \tag{7.10}$$

for a function $F(t)$ satisfying

$$\int_{-q}^{q} |F(t)|^2 \, dt < \infty$$

and, the identity

$$\|f\|^2_{H_{K_q}} = \frac{1}{2\pi} \int_{-q}^{q} |F(t)|^2 \, dt \tag{7.11}$$

is valid. Then, note that from Parseval's identity

$$\|f\|^2_{H_{K_q}[H(X)]} = \frac{1}{2\pi} \int_{-q}^{q} |F(t)|^2 \left(1 + \sum_{\nu=1}^{n} e^{-2p_\nu t}\right) dt. \tag{7.12}$$

This identity (7.12) and (7.10) conversely imply that the reproducing kernel $K_{H(X)}(z,\bar{u})$ is expressible in the form

$$K_{H(X)}(z,\bar{u}) = \frac{1}{2\pi}\int_{-q}^{q} \frac{e^{izt}\, e^{-i\bar{u}t}}{1 + \sum_{\nu=1}^{n} e^{-2p_\nu t}}\, dt, \qquad (7.13)$$

which gives the desired solution.

(III). Let D be a bounded domain in $\mathbb{C}$ and we consider the Bergman space $AL_2(D)$ on D composed of analytic functions f(z) on D with finite norms

$$\left\{\iint_D |f(z)|^2\, dxdy\right\}^{\frac{1}{2}} < \infty.$$

We denote the Bergman kernel for the Hilbert space $AL_2(D)$ by $K(z,\bar{u})$. Let $\{D_j\}_{j=1}^{n}$ be any finite number of open discs $\{|z - p_j| < r_j\}$ on D which are disjoint. Then, by the submean property of $|f(z)|^2$, we obtain the inequality

$$|f(p_j)|^2 \leq \frac{1}{\pi r_j^2}\iint_{|z-p_j|<r_j} |f(z)|^2\, dxdy,$$

and so we have

$$\pi\sum_{j=1}^{n} r_j^2\, |f(p_j)|^2 \leq \iint_D |f(z)|^2\, dxdy.$$

We assume that for nonzero constants, the equality in the above inequality does not hold. Then, we can introduce the Hilbert space $H_K^-\{D_j\}$ equipped with the inner product

$$(f,g)_{H_K^-\{D_j\}} = (f,g)_{AL_2(D)} - \pi\sum_{\nu=1}^{n} r_\nu^2\, f(p_\nu)\,\overline{g(p_\nu)}$$

for the members f and $g \in AL_2(D)$. The reproducing kernel $K(z,\bar{u};\{D_j\})$ for the space $H_K^-\{D_j\}$ can be determined by the functional equation

$$K(z,\bar{u}) = K(z,\bar{u};\{D_j\}) - \pi \sum_{\nu=1}^{n} r_\nu^2\, K(p_\nu,\bar{u};\{D_j\})\, \overline{K(p_\nu,\bar{z})}.$$

By setting $z = p_\mu$, $\mu = 1,2,\ldots,n$, we have

$$\sum_{\nu=1}^{n} \{\delta_\nu^\mu - \pi r_\nu^2\, \overline{K(p_\nu,\bar{p}_\mu)}\}\, K(p_\nu,\bar{u};\{D_j\}) = K(p_\mu,\bar{u}),\ \mu = 1,2,\ldots,n.$$

Since the matrix $\| \delta_\nu^\mu - \pi r_\nu^2\, \overline{K(p_\nu,\bar{p}_\mu)} \|$ is non-singular, we denote its inverse by $\|B_{\mu\nu}\|$. Then, we have

$$K(p_\nu,\bar{u};\{D_j\}) = \sum_{\mu=1}^{n} B_{\nu\mu}\, \overline{K(u,\bar{p}_\mu)},\ \nu = 1,2,\ldots,n.$$

Hence, we have

$$K(z,\bar{u};\{D_j\}) = K(z,\bar{u}) + \pi \sum_{\nu,\mu=1}^{n} K(z,\bar{p}_\nu)\, r_\nu^2\, B_{\nu\mu}\, \overline{K(u,\bar{p}_\mu)},$$

In particular, for one disc $D(p,r) = \{|z - p| < r\}$, we obtain

$$K(z,\bar{u};D(p,r)) = K(z,\bar{u}) + \frac{\pi r^2\, K(z,\bar{p})\, \overline{K(u,\bar{p})}}{1 - \pi r^2\, K(p,\bar{p})}.$$

(IV). In the case of a finite point set X as in (7.1) with (7.2), Theorem 4.1 was given by [8, Theorem 3] explicitly.

(V). In the case of a finite point set X as in (7.1), Theorem 5.1 was given by [8, Theorem 2] explicitly.

(VI). We can refer to the approximation problem of Davis [6] as a typical example satisfying the assumptions (c) and (d) :

Let G and B be two domains which are bounded and such that $\bar{G} \subset B$. Let $AL_2(G)$ and $AL_2(B)$ be the Bergman spaces on G and B, respectively. For a function $F \in AL_2(G)$, we are interested in approximating F on G by a function $f \in AL_2(B)$ such that

$$\min \|F - f\|_{AL_2(G)}.$$

From the point of view of the doubly orthogonal system which is closed on both domains G and B, Davis [6, p. 117] assumed that B and G are each bounded by a finite number of disjoint Jordan curves and the inner region G is such that it separates no point of B - G from the boundary of B. This assumption coincides with our assumption (d). Of course, other assumptions (a), (b) and (c) are trivially satisfied.

(VII). In the case of a finite point set X as in (7.1) with (7.2), the extremal function $f^{**}(p)$ in Theorem 6.1 is given explicitly by

$$f^{**}(p) = \left\| \begin{matrix} \sum_{\nu} K(\xi_\nu, p)\, \overline{K(\xi_\nu, \xi_1)} \\ \sum_{\nu} K(\xi_\nu, p)\, \overline{K(\xi_\nu, \xi_2)} \\ \vdots \\ \sum_{\nu} K(\xi_\nu, p)\, \overline{K(\xi_\nu, \xi_n)} \end{matrix} \right\|^{*}$$

$$\times \ \|k(\xi_\nu,\xi_\mu)\|^{\sim} \left\| \begin{matrix} \sum_\nu F(\xi_\nu)\ \overline{K(\xi_\nu,\xi_1)} \\ \sum_\nu F(\xi_\nu)\ \overline{K(\xi_\nu,\xi_2)} \\ \vdots \\ \sum_\nu F(\xi_\nu)\ \overline{K(\xi_\nu,\xi_n)} \end{matrix} \right\|$$

where

$$k(p,q) = \sum_{\nu,\mu=1}^{n} K(p,\xi_\nu)\ \overline{K(q,\xi_\mu)}\ K(\xi_\nu,\xi_\mu).$$

Here, for a positive definite Hermitian matrix A, $\tilde{A} = \overline{A^{-1}}$ and, * denotes the transpose of the complex conjugate vector. For a realization of the finite dimensional space H_k admitting the reproducing kernel k(p,q) on X, see [10,pp. 12-13]. In the above expression of f^{**}, we assumed that $\|k(\xi_\nu,\xi_\mu)\|_{n\times n}$ is positive definite.

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