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Numerical Study of a Shape Design Problem in Elasticity by use of Auxiliary Domain Method

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Abstract. In this paper, we discuss numerical computation obtaining solution of a two-dimensional shape design problem in linear elasticity. Though this problem has been solved by using finite element method in [3], we present an alternative approach in order to use finite difference method. Auxiliary domain method is applied so that computation, especially at the boundary, can be more easier. One of the reasons why we use finite difference method is to avoid resetting of mesh in numerical computing while we change the domain which we consider. In that case, there are some ways with fixed mesh to solve the problem, for example, grid generation method. But these methods have some disadvantages such as losing linearity of the state equation and, obviously, become more complicated. Our technique provides to study numerical analysis of the problem as simple as possible.

1. Contact problem with friction

At first we will present steady state problem. Let a, b, c_o and c , where $a < b$ and $0 < c_o < c$, be given. We consider two-dimensional elastic body represented by

$$(1.1) \qquad \Omega = \Omega(\alpha) = \left\{(x_1, x_2) \in R^2 \mid \ a < x_1 < b \, , \ \alpha(x_1) < x_2 < c\right\}$$

as shown in the figure 1.1 .

$$(1.2) \qquad \Gamma_c = \Gamma_c(\alpha) = \left\{(x_1, x_2) \in R^2 \mid \ a < x_1 < b \, , \ x_2 = \alpha(x_1)\right\}$$

contacts with the rigid body which is lower-half of the plane.

Here $\alpha \in U_{ad}$ and the set U_{ad} of admissible controls is defined by

$$(1.3) \qquad U_{ad} = \left\{ \begin{aligned} &\alpha \in C^{1,1}([a,b]) \mid \ 0 \le \alpha(x_1) \le c_o, \\ &|\alpha'(x_1)| \le c_1 \quad \text{for } x_1 \in [a,b], \\ &|\alpha''(x_1)| \le c_2 \quad \text{a.e. in } (a,b), \\ &\text{measure of } \Omega(\alpha) = c_3 \end{aligned} \right\}.$$

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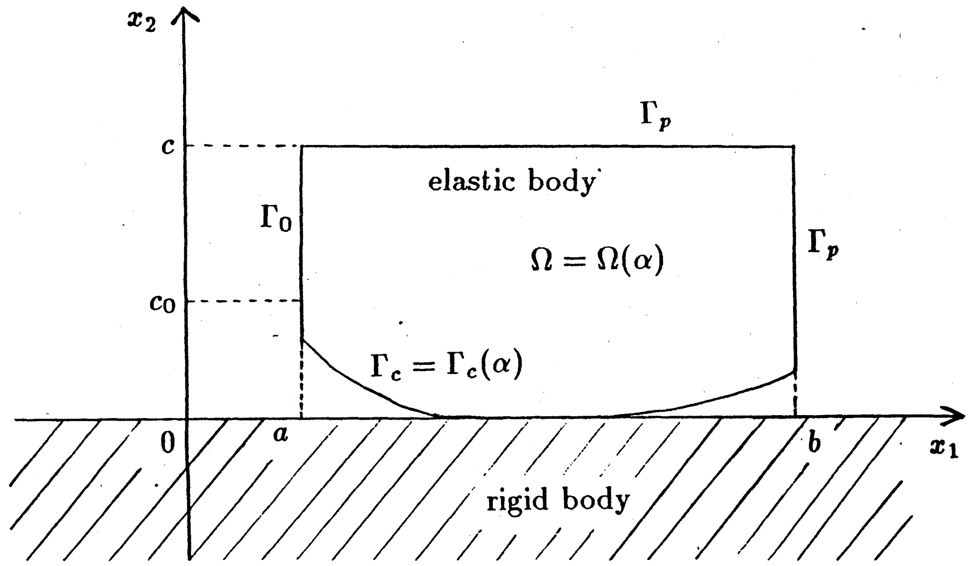


figure 1.1

The elastic body is subjected to forces and so deformation occurs. We denote the displacement vector by $u = (u_1(x_1, x_2), u_2(x_1, x_2))$. We assume that strain tensor

$$(1.4) \qquad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad , \quad i, j = 1, 2$$

and stress tensor $\tau_{ij}(u)$, $i, j = 1, 2$, satisfy the linearized Hooke's law

$$(1.5) \qquad \tau_{ij}(u) = \lambda\, \delta_{ij}\, (e_{11}(u) + e_{22}(u)) + 2\, \mu\, e_{ij}(u) \quad , \quad i, j = 1, 2\, .$$

$\lambda \geq 0$ and $\mu > 0$ are Lame's coefficients and constants. On the boundary of Ω , we define

$$(1.6) \qquad T_i = T_i(u) = \sum_{j=1}^{2} \tau_{ij}(u)\, n_j \quad , \quad i = 1, 2 \quad ,$$

where $n = (n_1, n_2)$ is outward unit normal vector.

Problem

Let force $f = (f_1, f_2)$ in Ω , stress $p = (p_1, p_2)$ on Γ_p and friction $g = constant$ on Γ_c be given. Then for some given $\alpha \in U_{ad}$, the problem is to find $u = u(\alpha)$ satisfying

$$(P) \quad \begin{cases} -\displaystyle\sum_{j=1}^{2} \frac{\partial \tau_{ij}(u)}{\partial x_j} = f_i \quad , \quad i = 1, 2 \quad , \quad in\ \Omega \\ u_i = 0 \quad , \quad i = 1, 2 \quad , \quad on\ \Gamma_0 \\ T_i(u) = p_i \quad , \quad i = 1, 2 \quad , \quad on\ \Gamma_p \\ u_2 + \alpha \geq 0 \quad , \quad T_2(u) \geq 0 \quad , \quad (u_2 + \alpha)\, T_2(u) = 0 \quad on\ \Gamma_c \\ |T_1(u)| \leq g \quad , \quad (g - |T_1(u)|)\, u_1 = 0 \quad , \quad u_1\, T_1(u) \leq 0 \quad on\ \Gamma_c \end{cases}$$

Differentiation which will be used throughout is taken in the sense of distribution.

In (P) , the condition $u_2 + \alpha \geq 0$ means that the elastic body can not penetrate into rigid body. On Γ_c , free boundaries will take place between touching parts where $u_2 + \alpha = 0$, $T_2(u) > 0$ and separating parts where $u_2 + \alpha > 0$, $T_2(u) = 0$. There may be another free boundaries between sliding parts where $|T_1(u)| = g$, $u_1 \neq 0$ and static parts where $|T_1(u)| < g$, $u_1 = 0$. See the figure 1.2 below. The condition $u_1 \; T_1(u) \leq 0$ states that, on sliding parts, u_1 and $T_1(u)$ are opposite.

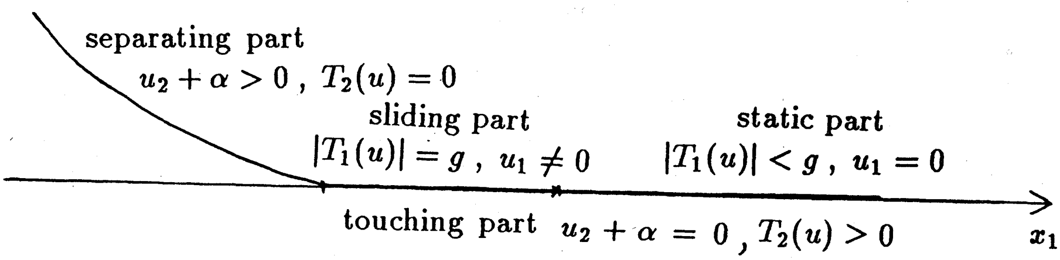


figure 1.2

2. Variational formulation

To express the problem (P) in weak form, we introduce the space

$$V = V(\alpha) = \left\{ v \in \left(H^1(\Omega(\alpha))\right)^2 \mid \quad v_i = 0, \quad i = 1,2, \quad on \; \Gamma_0 \right\}, \tag{2.1}$$

nonempty closed convex set

$$K = K(\alpha) = \{ v \in V \mid \quad v_2(x_1, \alpha(x_1)) + \alpha(x_1) \geq 0 \quad \text{a.e. in } (a,b) \}, \tag{2.2}$$

continuous bilinear form

$$\begin{aligned} a(u,v) &= \sum_{i,j=1}^{2} \int_\Omega \tau_{ij}(u) \, e_{ij}(v) \, dx , \qquad\qquad u, v \in V , \\ &= \int_\Omega \{ \, \lambda \, div(u) \, div(v) + 2 \, \mu \sum_{i,j=1}^{2} e_{ij}(u) e_{ij}(v) \, \} \, dx , \end{aligned} \tag{2.3}$$

continuous linear functional

$$\langle l, v \rangle = \sum_{i,j=1}^{2} \int_\Omega f_i v_i \, dx \; + \; \sum_{i,j=1}^{2} \int_{\Gamma_p} p_i v_i \, d\gamma, \quad v \in V \tag{2.4}$$

and continuous convex functional

$$(2.5) \qquad j_c(v) = \int_{\Gamma_c} g\,|v_1|\,d\gamma, \quad v \in V\,.$$

It can be shown that (P) is equivalent to variational inequality

$$(VI) \qquad \begin{cases} u = u(\alpha) \in K \ \text{ s.t.} \\ a(u, v-u) + j_c(v) - j_c(u) \geq \langle l, v-u \rangle \quad , \ \text{for all } \ v \in K\,. \end{cases}$$

For details, see [4]. Now the following classical result holds.

Theorem 2.1

Under the assumption of algebraic ellipticity condition for λ and μ , the solution of (VI) exists uniquely.

Since $a(.,.)$ is symmetric, we have equivalent minimization problem

$$(M) \qquad u = u(\alpha) \in K \ \text{ s.t. } \ J(u) = \min_{v \in K} J(v) \quad ,$$

where

$$(2.6) \qquad J(v) = \frac{1}{2}a(v,v) - \langle l, v \rangle + j_c(v)$$

which is, in physical meaning, total potential energy.

3. Setting of the problem

For $\alpha \in U_{ad}$, we define cost functional by

$$(3.1) \qquad E(\alpha) = J(u(\alpha)) \quad , \quad \text{where } u = u(\alpha) \text{ is the solution of } (VI)\,.$$

Then shape design problem is to find

$$(SD) \qquad \alpha^* \in U_{ad} \ \text{ s.t. } \ E(\alpha^*) = \min_{\alpha \in U_{ad}} E(\alpha)\,.$$

We can explain that it is to look for an optimal domain on which displacement field u causes minimum of total potential energy. Existence of such domain is proved in the following.

Theorem 3.1 (J. Haslinger and P. Neittaanmaki) [3]

There exists at least one solution of the problem (SD) .

4. Regularization and penalization

To approximate nondifferentiable functional j_c by a family of convex and differentiable functionals, we can regularize it as

$$(4.1) \qquad j_{c\varepsilon_r} = j_{c\varepsilon_r}(v) = \int_{\Gamma_c} g \sqrt{v_1^2 + \varepsilon_r^2}\; d\gamma\,, \quad (\varepsilon_r > 0).$$

And to replace the fact that $u \in K$, a penalization operator $\beta : v \longmapsto \beta(v)$ can be defined by

$$(4.2) \qquad (\beta(v)\,, w) = -\frac{1}{\varepsilon_p} \int_{\Gamma_c} (v_2 + \alpha)^- \; w_2 \, d\gamma\,, \quad w \in V\,, \quad (\varepsilon_p > 0) \quad .$$

Denote $\varepsilon = (\ \varepsilon_r\ ,\ \varepsilon_p\ )$. Now we consider the following regularized and penalized variational equation

$$(VE) \qquad \begin{cases} u_\varepsilon = u_\varepsilon(\alpha) \in V \ \text{ s.t.} \\ a(u_\varepsilon, v) + \ (j'_{c\varepsilon_r}(u_\varepsilon) + \beta(u_\varepsilon)\,, v) = \langle l, v\rangle \quad , \ \text{for all } \ v \in V\,. \end{cases}$$

Note that the operator $v \longmapsto j'_{c\varepsilon_r}(v)$ and β are monotone. Furthermore kernel of $\beta = K$ and β is lipschitz continuous. By using these facts, we get the following convergence result.[1][2]

Theorem 4.1

There exists unique solution u_ε of (VE) and $u_\varepsilon \rightarrow u$: solution of (VI) strongly in V as $\varepsilon = (\varepsilon_r, \varepsilon_p) \rightarrow (0,0)$.

As before, (VE) is the same as the problem

$$(M_\varepsilon) \qquad u_\varepsilon = u_\varepsilon(\alpha) \in V \ \text{ s.t. } \ J_\varepsilon(u_\varepsilon) = \min_{v \in V} \ J_\varepsilon(v) \quad ,$$

where

$$(4.3) \qquad J_\varepsilon(v) = \frac{1}{2} a(v,v) - \langle l, v\rangle + j_{c\varepsilon_r}(v) + \frac{1}{2\varepsilon_p}\ \Psi(v)$$

and

$$(4.4) \qquad \Psi(v) = \int_{\Gamma_c} [\ (v_2 + \alpha)^-]^2 \ d\gamma\,.$$

The problem becomes to find $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2})$ satisfying

$$(P_\varepsilon) \quad \begin{cases} -\displaystyle\sum_{j=1}^{2} \frac{\partial \tau_{ij}(u_\varepsilon)}{\partial x_j} = f_i \quad , \quad i = 1,2 \quad , \quad in\ \Omega \\ u_{\varepsilon i} = 0 \quad , \quad i = 1,2 \quad , \quad on\ \Gamma_0 \\ T_i(u_\varepsilon) = p_i \quad , \quad i = 1,2 \quad , \quad on\ \Gamma_p \\ T_2(u_\varepsilon) = \dfrac{1}{\varepsilon_p}\,(u_{\varepsilon 2} + \alpha)^- \qquad on\ \Gamma_c \\ T_1(u_\varepsilon) = -g\,\dfrac{u_{\varepsilon 1}}{\sqrt{u_{\varepsilon 1}^2 + \varepsilon_r^2}} \qquad on\ \Gamma_c\,. \end{cases}$$

One can notice that the boundary conditions on Γ_c become more simpler because of regularization and penalization. It will also be helpful in calculating the derivative of cost functional.

5. Auxiliary domain method

We construct an auxiliary domain $\widetilde{\Omega}$ as shown in the following figure 5.1 and denote $\widehat{\Omega} = (a,b) \times (\widetilde{c}, c)$.

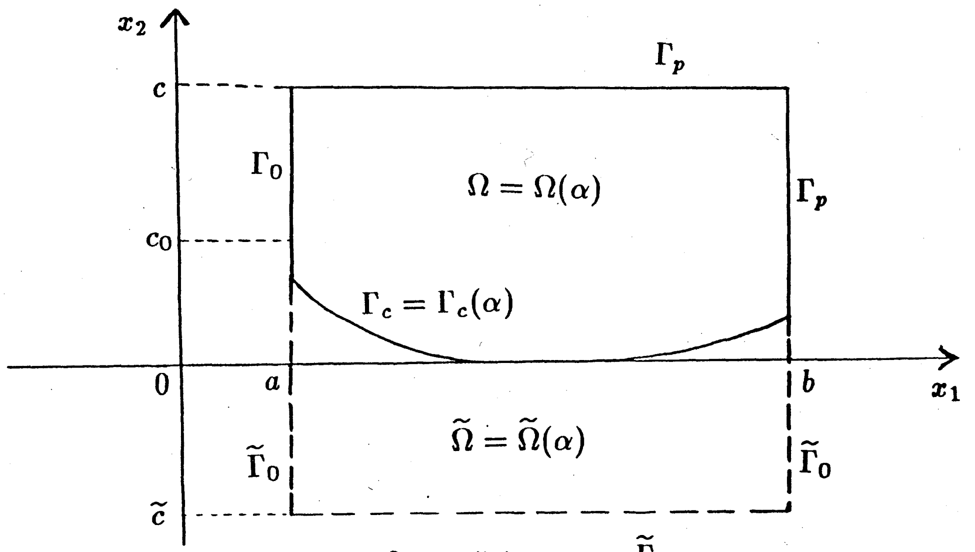


figure 5.1

Let us define again the space

$$(5.1) \qquad \widehat{V} = \left\{ v \in \left(H^1(\widehat{\Omega})\right)^2 \mid \ v_i = 0,\ i = 1,2\,, \quad on\ \Gamma_0 \cup \widetilde{\Gamma}_0 \right\}.$$

Let $\varepsilon_d > 0$. Our aim is to suppose the minimization problem , in place of (M_ε) ,

$$(\widehat{M}_\varepsilon) \qquad \widehat{u}_\varepsilon = \widehat{u}_\varepsilon(\alpha) \in \widehat{V} \ \text{ s.t. } \ \widehat{J}_\varepsilon(\widehat{u}_\varepsilon) = \min_{v \in \widehat{V}} \widehat{J}_\varepsilon(v) \quad ,$$

where

$$\widehat{J}_\varepsilon(v) = \frac{1}{2}a(v,v) + \frac{1}{2}\,\varepsilon_d\,\widetilde{a}(v,v) + \frac{1}{2}\int_{\widetilde{\Omega}} |v|^2\,dx \tag{5.2}$$
$$- \langle l, v\rangle + j_{c\varepsilon_r}(v) + \frac{1}{2\varepsilon_p}\,\Psi(v).$$

$\widetilde{a}(.,.)$ is as in (2.3) , but integration is based on the domain $\widetilde{\Omega}$. Here $\varepsilon = (\varepsilon_r\ ,\ \varepsilon_p\ ,\ \varepsilon_d\ )$. We can explain $(\widehat{M}_\varepsilon)$ as

$$(\widehat{P}_\varepsilon)\quad \begin{cases} -\displaystyle\sum_{j=1}^{2}\frac{\partial\tau_{ij}(\widehat{u}_\varepsilon)}{\partial x_j} = f_i \quad , \quad i=1,2 \quad , \quad in\ \Omega \\ \widehat{u}_{\varepsilon i} = 0 \quad , \quad i=1,2 \quad , \quad on\ \Gamma_0 \\ T_i(\widehat{u}_\varepsilon) = p_i \quad , \quad i=1,2 \quad , \quad on\ \Gamma_p \\ -\varepsilon_d\displaystyle\sum_{j=1}^{2}\frac{\partial\tau_{ij}(\widehat{u}_\varepsilon)}{\partial x_j} + \widehat{u}_{\varepsilon i} = 0 \quad , \quad i=1,2 \quad , \quad in\ \widetilde{\Omega} \\ \widehat{u}_{\varepsilon i} = 0 \quad , \quad i=1,2 \quad , \quad on\ \widetilde{\Gamma}_0 \\ T_2(\widehat{u}_\varepsilon) + \varepsilon_d\,\widetilde{T}_2(\widehat{u}_\varepsilon) = \dfrac{1}{\varepsilon_p}\,(\widehat{u}_{\varepsilon 2} + \alpha)^- \quad on\ \Gamma_c \\ T_1(\widehat{u}_\varepsilon) + \varepsilon_d\,\widetilde{T}_1(\widehat{u}_\varepsilon) = -g\,\dfrac{\widehat{u}_{\varepsilon 1}}{\sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}} \quad on\ \Gamma_c\ . \end{cases}$$

After all shape design problem to be considered is

$$(\widehat{SD}_\varepsilon)\qquad \alpha_\varepsilon^* \in U_{ad}\ \text{ s.t. }\ \widehat{E}_\varepsilon(\alpha_\varepsilon^*) = \min_{\alpha\,\in\,U_{ad}} \widehat{E}_\varepsilon(\alpha)\ ,$$

and the cost functional to be optimized is

$$\widehat{E}_\varepsilon(\alpha) = \widehat{J}_\varepsilon(\widehat{u}_\varepsilon(\alpha)) \quad , \quad \text{where } \widehat{u}_\varepsilon = \widehat{u}_\varepsilon(\alpha) \text{ is the solution of } (\widehat{M}_\varepsilon)\ . \tag{5.3}$$

6. Derivative of cost functional

We need to calculate Frechet derivative of $\widehat{E}_\varepsilon(\alpha)$ to be used in optimizer. Let us state two lemmas which are useful.

Lemma 6.1

Let $C(\alpha)$ be smooth function which is defined in $\Omega(\alpha)$. Then

$$\frac{d}{d\alpha}\left[\int_{\Omega(\alpha)} C(\alpha)\,dx\right]\delta\alpha = \int_{\Omega(\alpha)}\frac{\partial C(\alpha)}{\partial\alpha}\,\delta\alpha\,dx + \int_{\partial\Omega(\alpha)} C(\alpha)\,n.\nu\,\delta\alpha\,d\gamma.$$

Lemma 6.2

Under the same conditions as in Lemma 6.1 , we have

$$\frac{d}{d\alpha}\left[\int_{\partial\Omega(\alpha)} C(\alpha)\, d\gamma\right]\delta\alpha = \int_{\partial\Omega(\alpha)} \frac{\partial C(\alpha)}{\partial\alpha}\,\delta\alpha\, d\gamma + \int_{\partial\Omega(\alpha)} \{\, \nabla(C(\alpha))\,.\,n \,+\, H\, C(\alpha)\,\}\, n.\nu\,\delta\alpha\, d\gamma.$$

Here ∇ is gradient operator with respect to x, H is curvature of the curve $\partial\Omega(\alpha)$ and ν is to be taken as $(0,1)$ or $(0,-1)$.

Let us denote $w = (\, w_1 \,,\, w_2\,) = \left(\frac{\partial\widehat{u}_{\varepsilon 1}}{\partial\alpha}\,,\,\frac{\partial\widehat{u}_{\varepsilon 2}}{\partial\alpha}\right)$. With the help of Lemmas, we derive to obtain the followings.

(6.1)
$$\frac{d}{d\alpha}\left[\frac{1}{2}\int_{\Omega}\sum_{i,j=1}^{2} \tau_{ij}(\widehat{u}_\varepsilon)\, e_{ij}(\widehat{u}_\varepsilon)\, dx\right]\delta\alpha$$
$$= -\int_{\Omega}\sum_{i,j=1}^{2}\frac{\partial\tau_{ij}(\widehat{u}_\varepsilon)}{\partial x_j}\, w_i\, dx + \int_{\Gamma_c}\sum_{i=1}^{2} T_i(\widehat{u}_\varepsilon)\, w_i\, d\gamma$$
$$+ \int_{\Gamma_p}\sum_{i=1}^{2} T_i(\widehat{u}_\varepsilon)\, w_i\, d\gamma$$
$$+ \int_{\Gamma_c}\frac{1}{2}\sum_{i,j=1}^{2}\tau_{ij}(\widehat{u}_\varepsilon)\, e_{ij}(\widehat{u}_\varepsilon)\; n.\nu\,\delta\alpha\, d\gamma$$

(6.2)
$$\frac{d}{d\alpha}\left[\frac{\varepsilon_d}{2}\int_{\widetilde{\Omega}}\sum_{i,j=1}^{2} \tau_{ij}(\widehat{u}_\varepsilon)\, e_{ij}(\widehat{u}_\varepsilon)\, dx\right]\delta\alpha$$
$$= -\,\varepsilon_d\int_{\widetilde{\Omega}}\sum_{i,j=1}^{2}\frac{\partial\tau_{ij}(\widehat{u}_\varepsilon)}{\partial x_j}\, w_i\, dx + \varepsilon_d\int_{\Gamma_c}\sum_{i=1}^{2}\widetilde{T}_i(\widehat{u}_\varepsilon)\, w_i\, d\gamma$$
$$-\int_{\Gamma_c}\frac{\varepsilon_d}{2}\sum_{i,j=1}^{2}\widetilde{\tau}_{ij}(\widehat{u}_\varepsilon)\,\widetilde{e}_{ij}(\widehat{u}_\varepsilon)\; n.\nu\,\delta\alpha\, d\gamma$$

(6.3)
$$\frac{d}{d\alpha}\left[\frac{1}{2}\int_{\widetilde{\Omega}}\sum_{i=1}^{2}\widehat{u}_{\varepsilon i}^2\, dx\right]\delta\alpha$$
$$= \int_{\widetilde{\Omega}}\sum_{i=1}^{2}\widehat{u}_{\varepsilon i}\, w_i\, dx - \int_{\Gamma_c}\frac{1}{2}\sum_{i=1}^{2}\widehat{u}_{\varepsilon i}^2\; n.\nu\,\delta\alpha\, d\gamma\,.$$

$$\text{(6.4)} \qquad \frac{d}{d\alpha}\Big[-\int_{\Omega} \sum_{i=1}^{2} f_i\, \widehat{u}_{\varepsilon i}\, dx \Big] \delta\alpha$$

$$= -\int_{\Omega} \sum_{i=1}^{2} f_i\, w_i\, dx - \int_{\Gamma_c} \sum_{i=1}^{2} f_i\, \widehat{u}_{\varepsilon i}\;\; n.\nu\; \delta\alpha\; d\gamma$$

$$\text{(6.5)} \qquad \frac{d}{d\alpha}\Big[-\int_{\Gamma_p} \sum_{i=1}^{2} p_i\, \widehat{u}_{\varepsilon i}\, d\gamma \Big] \delta\alpha \;=\; -\int_{\Gamma_p} \sum_{i=1}^{2} p_i\, w_i\, d\gamma$$

$$\text{(6.6)} \qquad \frac{d}{d\alpha}\Big[\int_{\Gamma_c} g\, \sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}\;\; d\gamma \Big] \delta\alpha$$

$$= \int_{\Gamma_c} g\, \frac{\widehat{u}_{\varepsilon 1}}{\sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}}\, w_1\, d\gamma$$

$$+ \int_{\Gamma_c} g\, \Big\{ \nabla\, (\sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}\,)\; .\, n \;+\; H\, \sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2} \Big\}\; n.\nu\; \delta\alpha\; d\gamma$$

$$\text{(6.7)} \qquad \frac{d}{d\alpha}\Big[\frac{1}{2\varepsilon_p} \int_{\Gamma_c} [\; (\widehat{u}_{\varepsilon 2} + \alpha)^-]^2\; d\gamma \Big] \delta\alpha$$

$$= -\; \frac{1}{\varepsilon_p} \int_{\Gamma_c} (\widehat{u}_{\varepsilon 2} + \alpha)^-\; w_2\; d\gamma$$

$$+ \frac{1}{2\varepsilon_p} \int_{\Gamma_c} \Big\{ \nabla\, ([\; (\widehat{u}_{\varepsilon 2} + \alpha)^-]^2\,)\; .\, n$$

$$+\; H\; [\; (\widehat{u}_{\varepsilon 2} + \alpha)^-]^2 \Big\}\; n.\nu\; \delta\alpha\; d\gamma$$

(6.1) $\sim$ (6.7) and equations in $(\widehat{P}_\varepsilon)$ give the final result :

$$\text{(6.8)} \qquad \frac{d}{d\alpha}[\widehat{E}_\varepsilon(\alpha)]\; \delta\alpha$$

$$= \int_{\Gamma_c} \Big[\; \frac{1}{2} \sum_{i,j=1}^{2} \{\; \tau_{ij}(\widehat{u}_\varepsilon)\, e_{ij}(\widehat{u}_\varepsilon) - \varepsilon_d\, \widetilde{\tau}_{ij}(\widehat{u}_\varepsilon)\, \widetilde{e}_{ij}(\widehat{u}_\varepsilon)\; \}$$

$$- \sum_{i=1}^{2} \{\; \frac{1}{2}\, \widehat{u}_{\varepsilon i}^2 + f_i \widehat{u}_{\varepsilon i}\; \}$$

$$+ \nabla (g\, \sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}\; + \frac{1}{2\varepsilon_p}\, [(\widehat{u}_{\varepsilon 2} + \alpha)^-]^2\,).n$$

$$+ H \{\; g\, \sqrt{\widehat{u}_{\varepsilon 1}^2 + \varepsilon_r^2}\; + \frac{1}{2\varepsilon_p}\, [(\widehat{u}_{\varepsilon 2} + \alpha)^-]^2 \} \Big]\; n.\nu\; \delta\alpha\; d\gamma.$$

Since w is cancelled, adjoint equaion is not necessary to be considered. We will denote the right side of (6.8) by $\int_{\Gamma_c} \phi\, \delta\alpha\, d\gamma$.

In actual computation, we want to work out with the admissible set

$$W_{ad} = \Big\{ \alpha \in C^{1,1}([a,b]) \mid \quad 0 \le \alpha(x_1) \le c_o, \quad |\alpha'(x_1)| \le c_1 \quad \text{for } x_1 \in [a,b], \quad |\alpha''(x_1)| \le c_2 \quad \text{a.e. in } (a,b) \Big\} \tag{6.9}$$

from which restriction for measure of $\Omega(\alpha)$ is excluded. But it is unavoidable to retain that condition. Therefore we try to include it again in cost functional as

$$\widehat{F}_\varepsilon(\alpha) = \widehat{E}_\varepsilon(\alpha) + k\, |c_3 - m(\Omega)|^2 , \tag{6.10}$$

where k is a positive constant and $m(\Omega)$ is notation for measure of $(\Omega(\alpha))$.

At last it becomes to study numerically the shape design problem

$$(\widehat{\mathcal{SD}_\varepsilon}) \qquad \alpha_\varepsilon^* \in W_{ad} \quad \text{s.t.} \quad \widehat{F}_\varepsilon(\alpha_\varepsilon^*) = \min_{\alpha \in W_{ad}} \widehat{F}_\varepsilon(\alpha).$$

It can easibly be seen that

$$\frac{d}{d\alpha}[\widehat{F}_\varepsilon(\alpha)]\, \delta\alpha = \int_{\Gamma_c} \phi\, \delta\alpha\, d\gamma + 2\, k\, \big(c_3 - m(\Omega)\big) \int_{\Gamma_c} \frac{1}{\sqrt{1 + (\alpha'(x_1))^2}}\, \delta\alpha\, d\gamma. \tag{6.11}$$

For certain α , we can write the right side of (6.11) like as $\int_{\Gamma_c} \Phi\, \delta\alpha\, d\gamma$.

7. Optimizing algorithm

Numerical computations are carried out as stated below.

Step 0 : Initial $\alpha \in W_{ad}$ is given.

Step 1 : Solve $(\widehat{M_\varepsilon})$.

Step 2 : Compute Φ .

Step 3 : Take $\delta\alpha = -\rho\, \Phi \quad (\rho > 0)$.

Step 4 : Set $\alpha \leftarrow \alpha + \delta\alpha$.

Go to Step 1.

Relaxation method is used to solve $(\widehat{M}_\varepsilon)$. To optimize $\widehat{F}_\varepsilon$, according to Pironneau's method , we take $\delta\alpha$ as $\delta\alpha = -\rho\ \Phi$ $(\rho > 0)$ so that $\frac{d}{d\alpha}[\widehat{F}_\varepsilon(\alpha)]\ \delta\alpha \leq 0$. Suitable ρ had to be selected in order to that $\alpha + \delta\alpha$ lies in the admissible set W_{ad}.

8. Numerical examples

We take $f = (f_1, f_2) = (0,0)$ for simplicity and use the data :

$a = -1.0$, $b = 1.0$, $c = 1.5$, $\tilde{c} = -0.5$

$c_0 = 0.5$, $c_1 = 0.5$, $c_2 = 4.0$, $c_3 = 2.75$

$\lambda = 1.0$, $\mu = 1.0$, $g = 0.001$

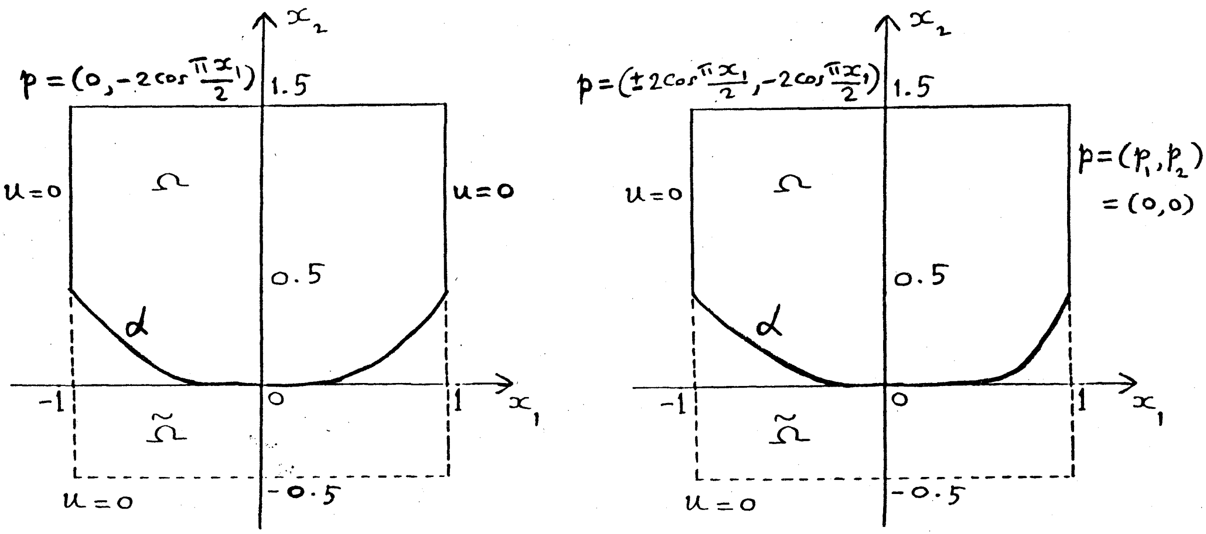
$k = 8.0$

mesh size $h = \frac{1}{8}, \frac{1}{16}$

$\varepsilon_r, \varepsilon_d = 2h^2$, h^2 , $\frac{2h^2}{2}$

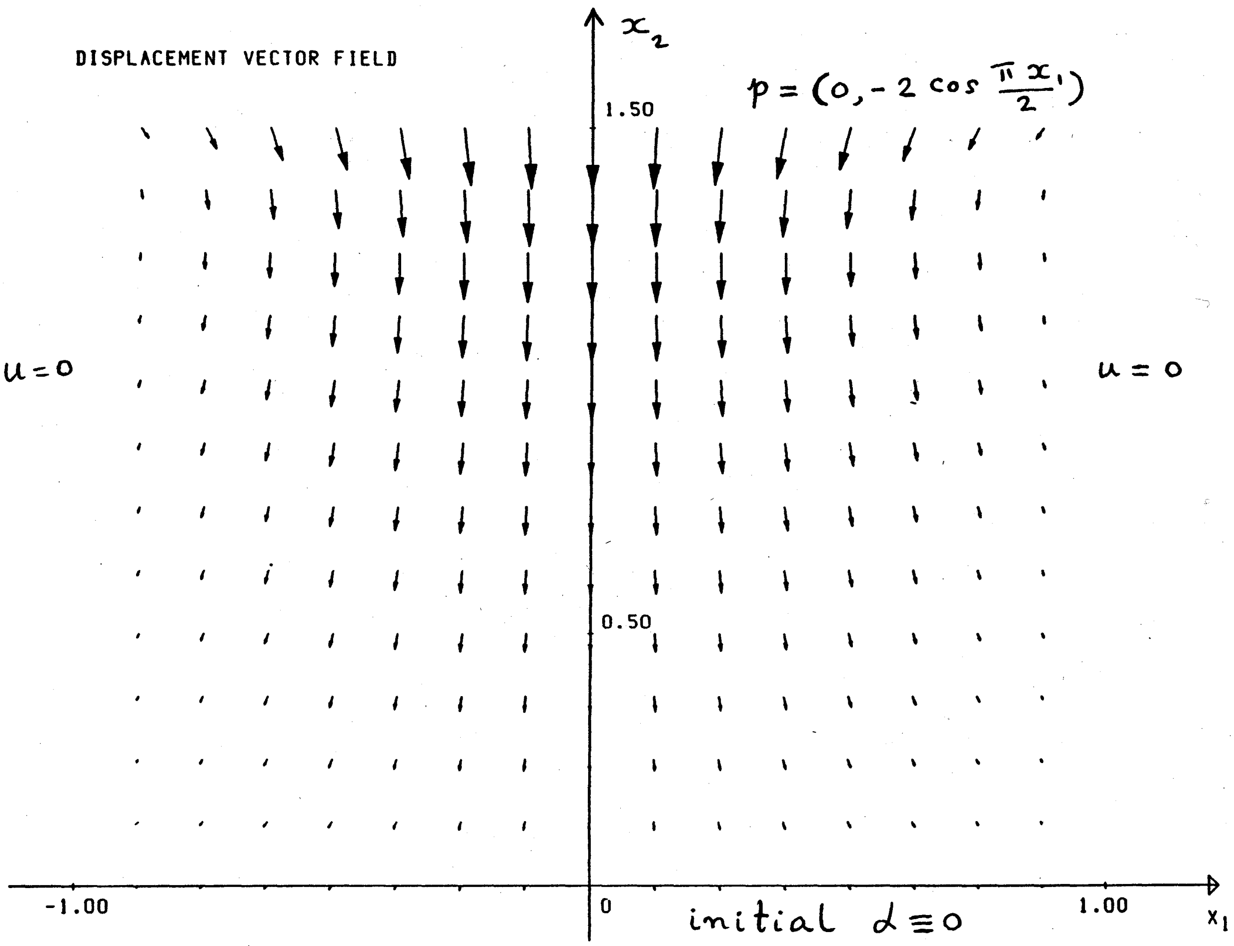
$\varepsilon_p = h$, $\frac{h}{2}$, $\frac{2h}{5}$

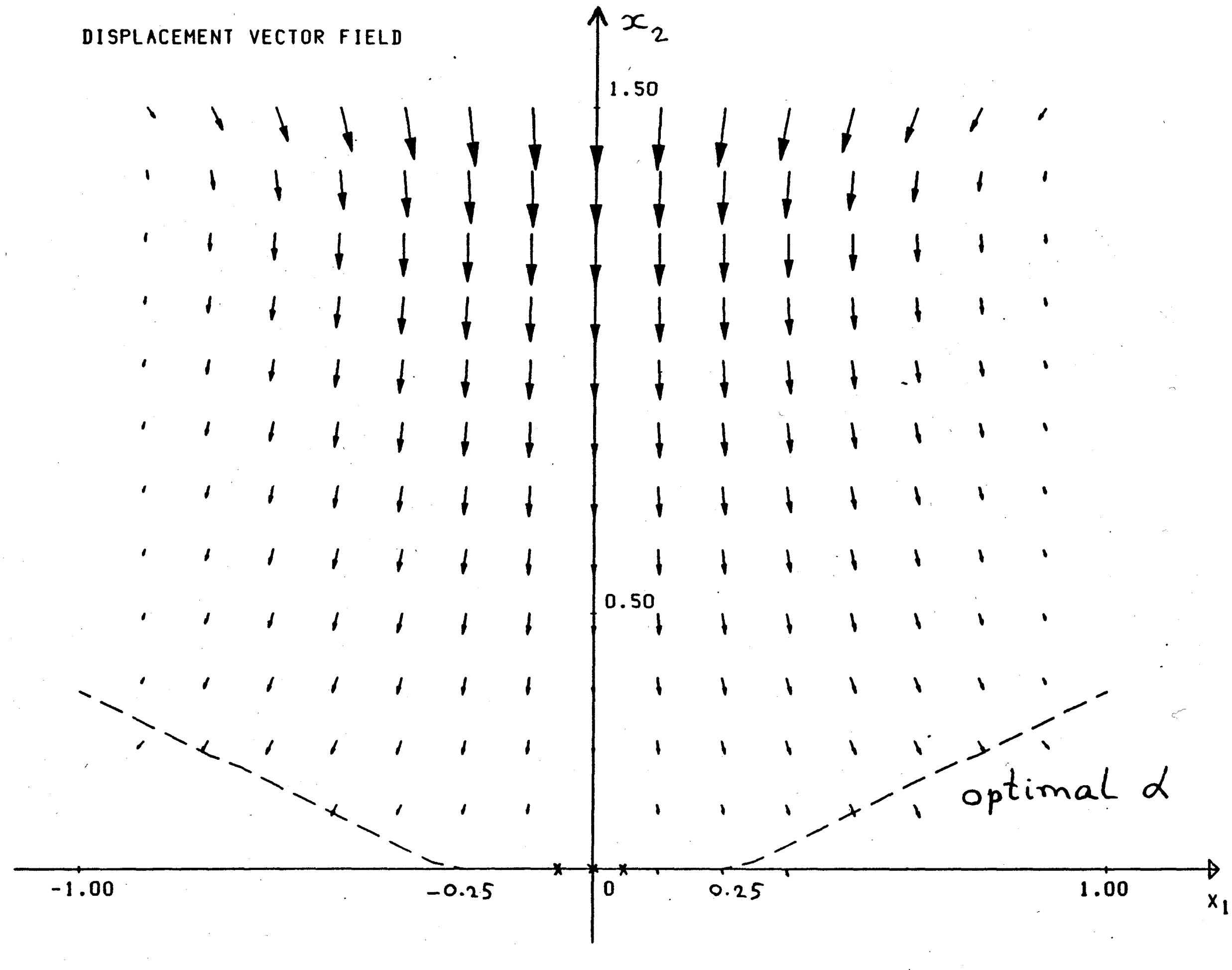
For $p = (p_1, p_2)$, see the following figures.

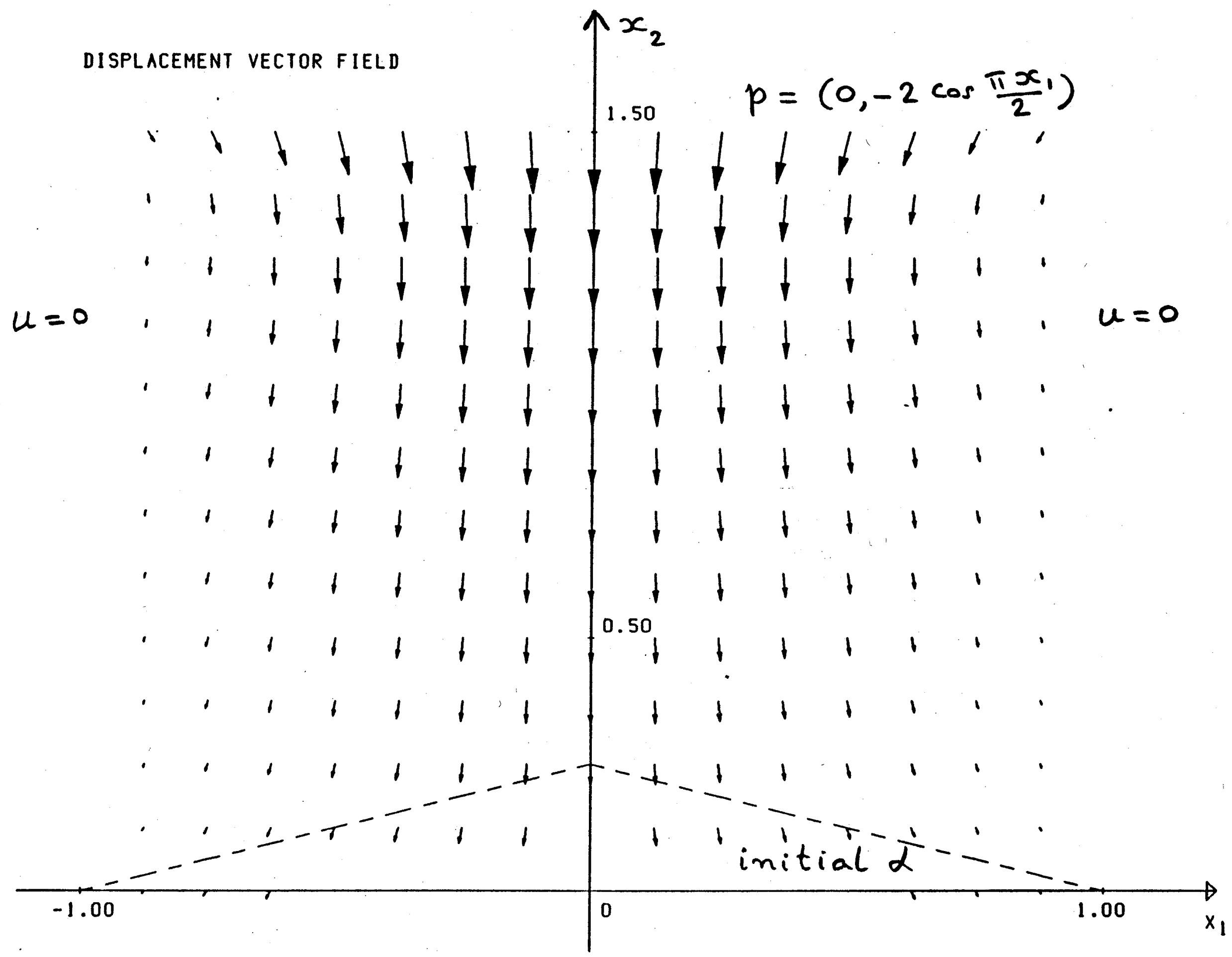


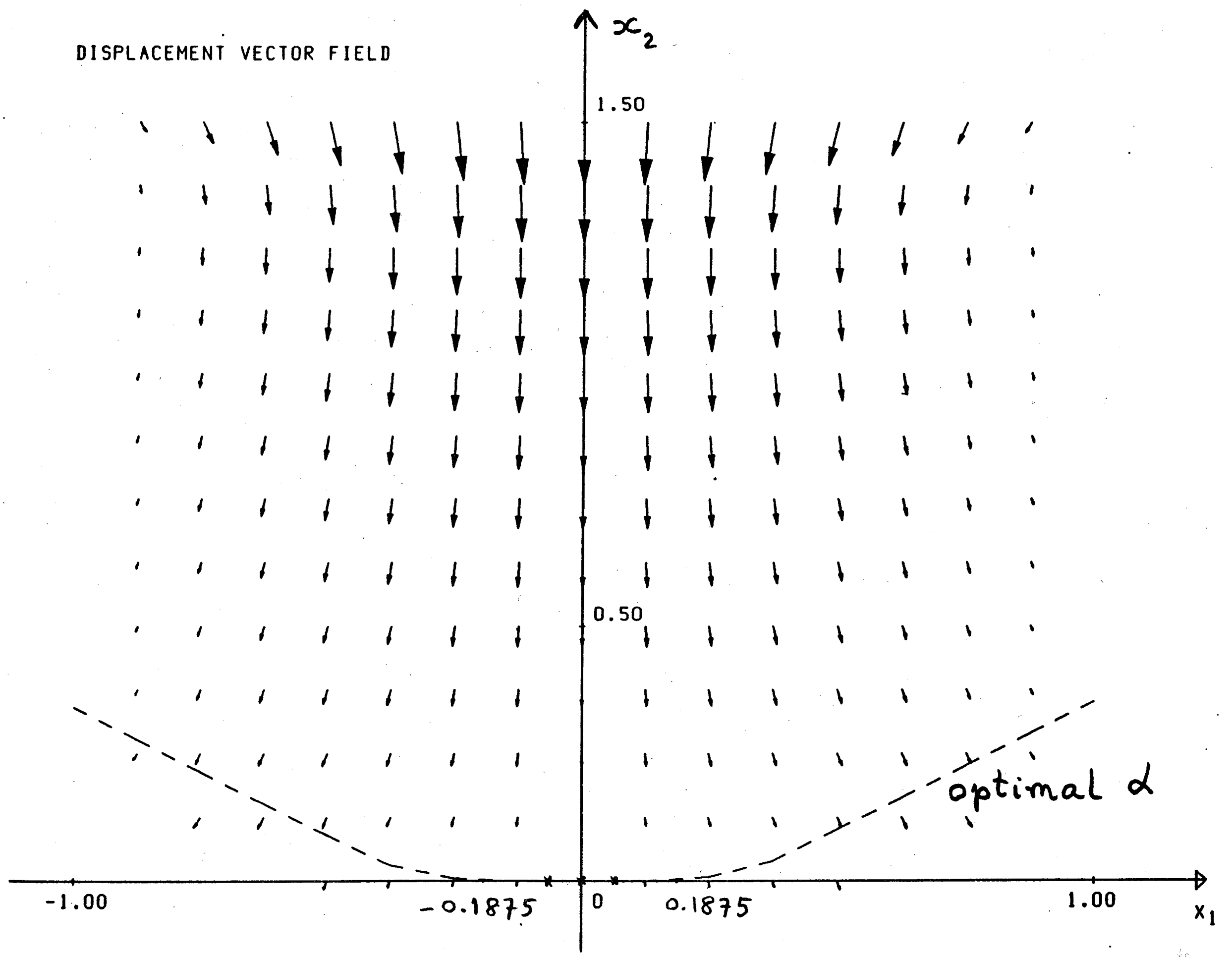
REFERENCES

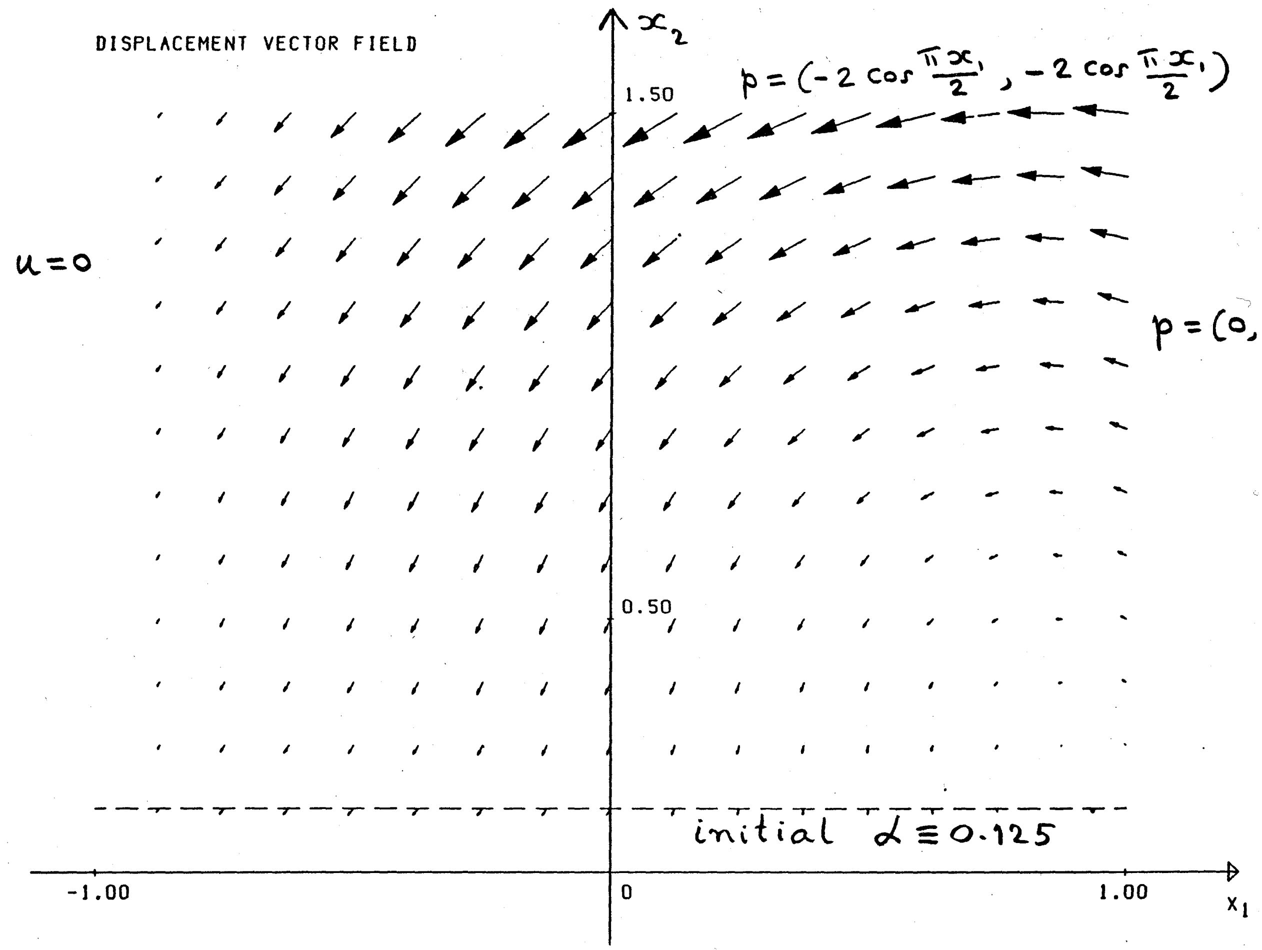
1. F.E. Browder, *The fixed-point theory of multi-valued problems*, Math. Annalen **1977** (1968), 283-301.
2. R. Glowinski, J.L. Lions and R. Tremolieres, "Numerical Analysis of Variational Inequalities," Studies in Mathematics and Its Applications, vol 8, North-Holland Publishing co., 1981.
3. J. Haslinger and P. Neittanmaki, "Finite Element Approximation for Optimal Shape Design," John Wiley and sons, 1988.
4. I. Hlavacek, J. Haslinger, J. Necas and J. Lovisek, "Solution of Variational Inequalities in Mechanics," Springer-Verlag, 1988.
5. H. Kawarada, "Free Boundary problems," Tokyo University, 1989. (in Japanese)
6. O. Pironneau, "Optimal Shape Design for Elliptic System," Springer-Verlag, 1983.

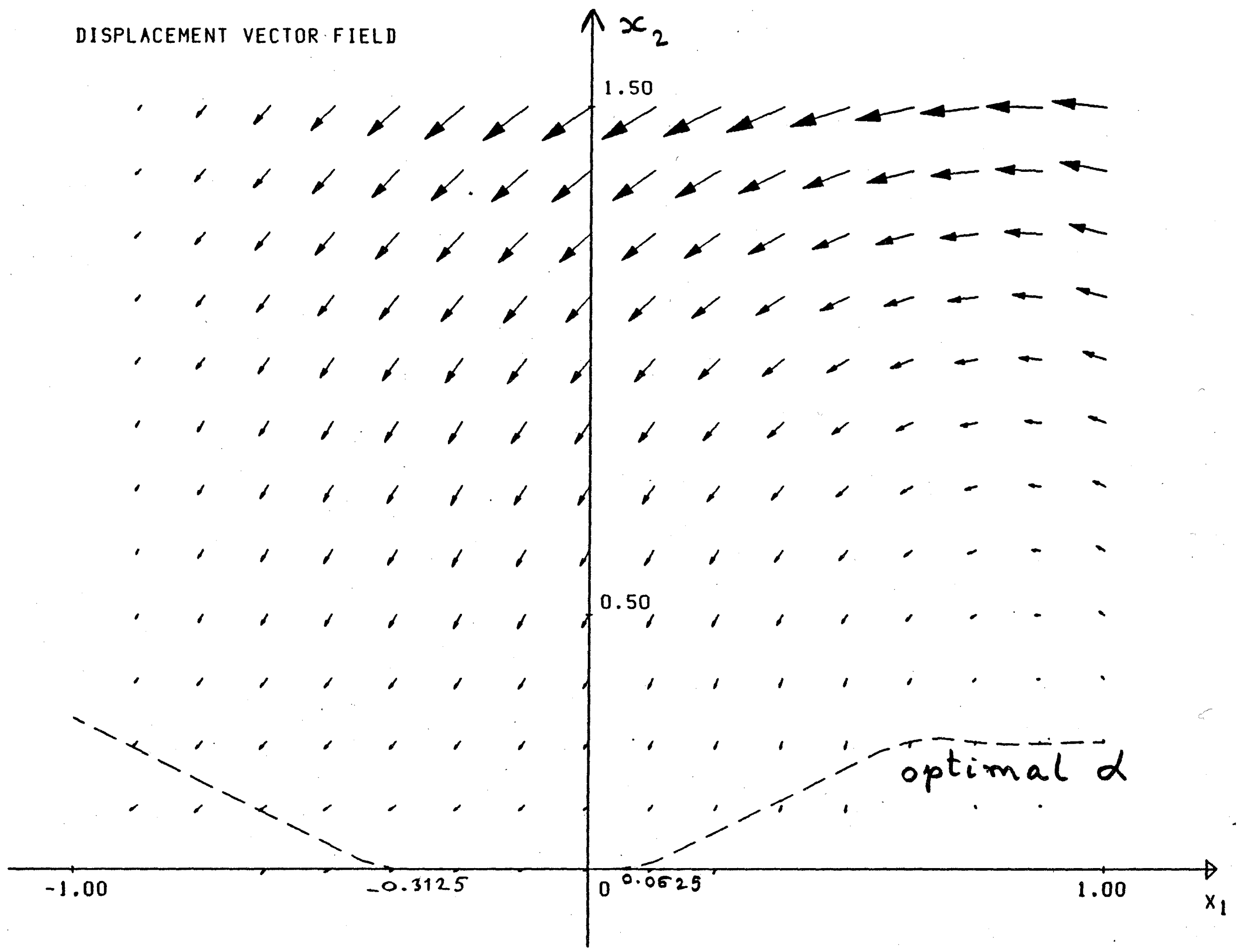

DISPLACEMENT VECTOR FIELD
x_2
$p = (0, -2\cos\frac{\pi x_1}{2})$
1.50
0.50
u = 0
u = 0
-1.00
0
1.00
x_1
initial $\alpha \equiv 0$

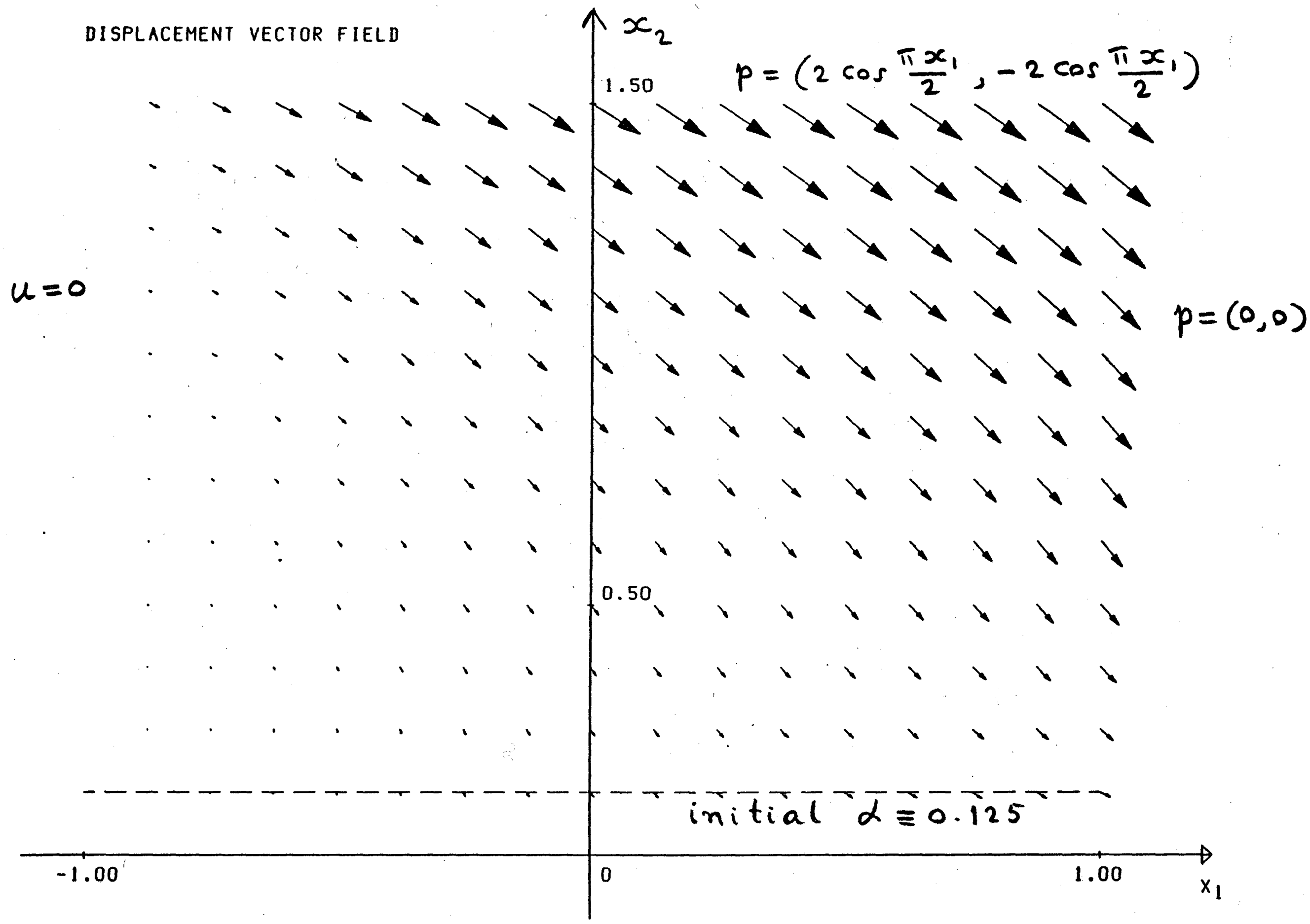

DISPLACEMENT VECTOR FIELD
x_2
1.50
0.50
optimal α
-1.00
-0.25
0
0.25
1.00
x_1

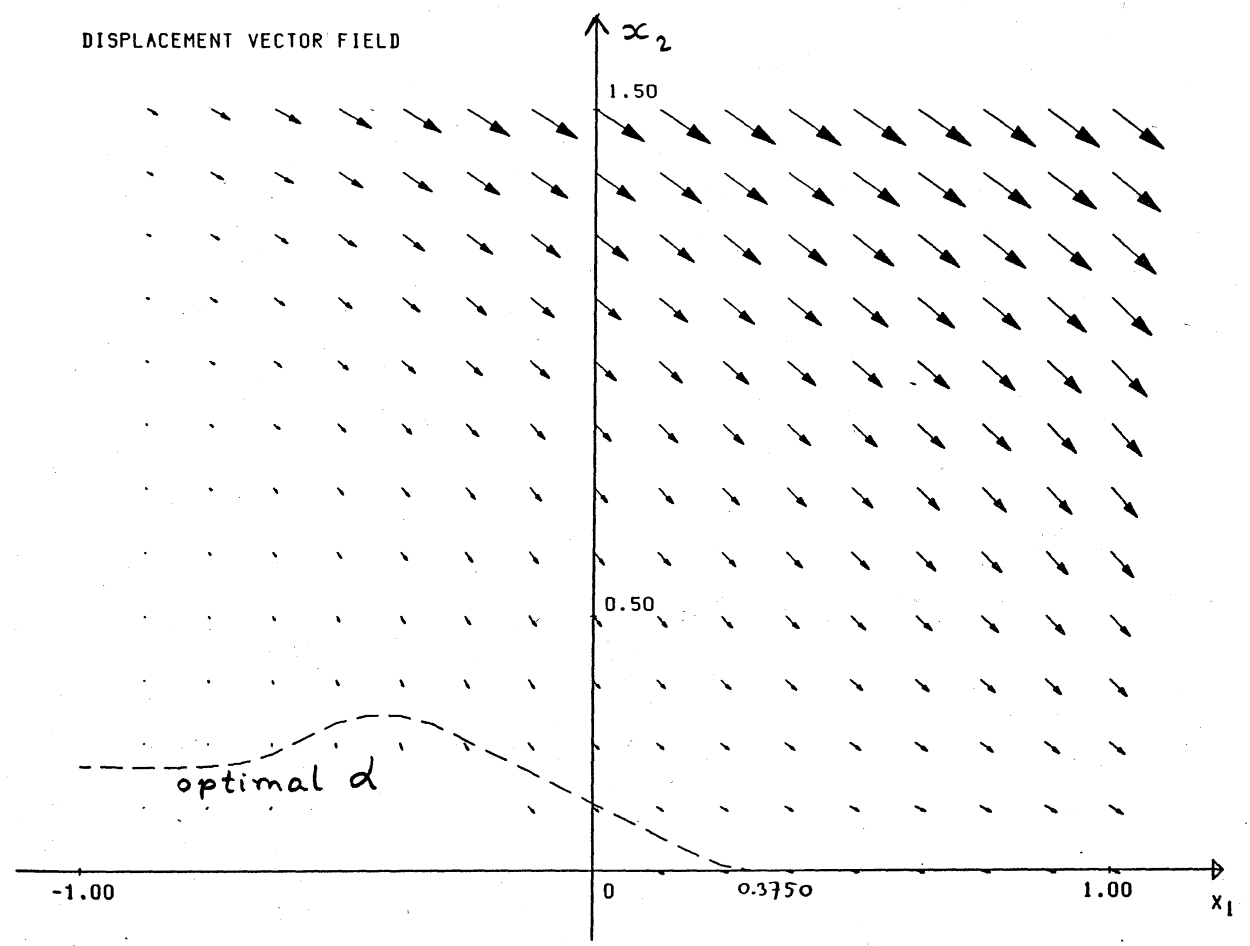

DISPLACEMENT VECTOR FIELD
x_2
$p = (0, -2\cos\frac{\pi x_1}{2})$
1.50
0.50
$u=0$
$u=0$
initial α
-1.00
0
1.00
x_1


DISPLACEMENT VECTOR FIELD
x_2
1.50
0.50
optimal α
-1.00
-0.1875
0
0.1875
1.00
x_1


DISPLACEMENT VECTOR FIELD
x_2
1.50
$p = (-2\cos\frac{\pi x_1}{2}, -2\cos\frac{\pi x_1}{2})$
$u=0$
$p=(0,0)$
0.50
initial $\alpha \equiv 0.125$
-1.00
0
1.00
x_1


DISPLACEMENT VECTOR FIELD
x_2
1.50
0.50
optimal α
-1.00
-0.3125
0
0.0625
1.00
x_1


DISPLACEMENT VECTOR FIELD
x_2
1.50
$p = (2\cos\frac{\pi x_1}{2}, -2\cos\frac{\pi x_1}{2})$
$u=0$
$p=(0,0)$
0.50
initial $\alpha \equiv 0.125$
-1.00
0
1.00
x_1


DISPLACEMENT VECTOR FIELD
x_2
1.50
0.50
optimal α
-1.00
0
0.3750
1.00
x_1