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Nonexistence of bifurcation from Crapper's pure capillary waves*

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§1. Introduction.

We consider a free boundary problem of progressive waves of two dimensional irrotational flow of inviscid incompressible fluid. In [1] Crapper showed that this free boundary problem has a family of solutions expressed by elementary functions in the case where the gravity is neglected and only the capillary force is taken into account. The purpose of the present paper is to prove that there is no bifurcation from this family of solutions, in other words, this family is isolated from any other solutions. The reason that we give a complete but rather elementary proof is that the result contributes to clarifying the global bifurcation diagram of the solutions. In fact, when we take account of both the gravity and the capillary force, numerical simulations show that there is a rather complicated bifurcation structure including secondary bifurcations in the set of the solutions ([6,7]). Yet, in the limit of zero gravity, our theorem implies that no secondary bifurcation is possible.

The present paper consists of five sections. The precise mathematical formulation is given in §2. Basic facts on Crapper's waves are briefly reviewed in §3. Theorem and its proof is given in §4. We present some numerical results in §5, which visually show the meaning of our theorem and seems, at the same time, to indicate the following conjecture: only Crapper's waves are solutions when the gravity is zero.

§2. Formulation of the problem.

We take a coordinate system (x, y) moving with the wave with the same speed. The x-coordinate is taken horizontally to the right and y-coordinate is taken vertically upward. By definition, the wave profile of a progressive wave is at rest in this moving frame and there is an underlying flow travelling in the opposite direction. We consider two dimensional irrotational flow of incompressible inviscid fluid. As usual in this problem, we assume that the wave profile is periodic in x with a period L, and is symmetric with respect to a vertical line passing through a crest. We take the line as y-axis. We consider a flow of infinite depth with a smooth free boundary expressed by $y = h(x)$. Consequently it is sufficient to consider the flow in $\Omega_h \equiv \{z = x + iy \in \mathbb{C}; -L/2 < x < L/2, -\infty < y < h(x)\}$.

The problem is to find an even function $y = h(x)$ $(|x| < L/2)$, and an analytic function $f(z)$ $(z \in \Omega_h)$ called a complex potential satisfying the following (2.1-4):

(2.1) $$V = 0 \qquad \text{on} \quad y = h(x).$$

(2.2) $$U = \pm cL/2 \qquad \text{on} \quad x = \pm L/2, \quad \text{respectively.}$$

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$$(2.3)\qquad \frac{1}{2}\left|\frac{df}{dz}\right|^2 + gy - \frac{T}{m}\left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x = \text{constant} \qquad \text{on} \quad y = h(x),$$

$$(2.4)\qquad \frac{df}{dz} \to c \qquad \text{as} \quad y \to -\infty.$$

where the subscript implies the differentiation, g, m and T are positive constants called the gravity constant, the mass density, and the surface tension coefficient, respectively. We require that $f(z)$ is analytic in Ω_h and continuously differentiable (in the real variables) in $\overline{\Omega_h}$. The complex potential $f(z)$ determines the velocity vector (u, v) by $u - iv = df/dz$. For the derivation of (2.1-4), see [2] or [4].

We can give an alternative formulation to this free boundary problem. The original problem (2.1-4) is formulated by the relation between f and z. The new formulation is written in terms of $\zeta \equiv \exp(-2\pi i f/(cL))$ and $\omega \equiv i\log(c^{-1}df/dz)$. This formulation is due to Levi-Civita and expressed as follows. *Find a function* $\omega = \omega(\zeta)$ *which is continuous on* $|\zeta| \le 1$, *is analytic in* $|\zeta| < 1$ *and satisfies* $\omega(0) = 0$ *and*

$$(2.5)\qquad e^{2\tau}\frac{\partial \tau}{\partial \sigma} - pe^{-\tau}\sin\theta + q\frac{\partial}{\partial \sigma}\left(e^{\tau}\frac{\partial \theta}{\partial \sigma}\right) = 0 \qquad on \quad \rho = 1,$$

where (ρ, σ) *is the polar coordinate of* ζ, $\theta = \theta(\rho, \sigma)$ *is the real part of* ω, *and* $\tau = \tau(\rho, \sigma)$ *the imaginary part.* p *and* q *are nondimensional parameters defined by* $p = gL/(2\pi c^2), q = 2\pi T/(mc^2 L)$. The derivation of (2.5) is given in [5]. We remark that θ is related with the free boundary by the following relation:

$$(2.6)\qquad \tan\theta = \frac{d}{dx}h(x).$$

Since an analytic function is uniquely determined by its boundary value, a further reduction of the equation (2.5) is possible. In fact we can write (2.5) only by $\theta(1, \sigma)$ $(0 \le \sigma < 2\pi)$. To this end, we define a Hilbert transform on S^1:

$$H\left(\sum_{n=1}^{\infty}\left(a_n \sin n\sigma + b_n \cos n\sigma\right)\right) = \sum_{n=1}^{\infty}\left(-a_n \cos n\sigma + b_n \sin n\sigma\right).$$

This is a linear isomorphism from $L^2(S^1)/\mathbb{R}$ onto itself. $H(\theta(1, \sigma))$ is well-defined, since $\int_0^{2\pi}\theta(1, \sigma)d\sigma = 0$ is assured by $\omega(0) = 0$ (see [5]). Then we have $\tau(1, \sigma) = H(\theta^*)$, where $\theta^*(\sigma) = \theta(1, \sigma)$. The equation (2.5) is now written as

$$(2.7)\qquad e^{2H\theta^*}\frac{dH\theta^*}{d\sigma} - pe^{-H\theta^*}\sin\theta^* + q\frac{d}{d\sigma}\left(e^{H\theta^*}\frac{d\theta^*}{d\sigma}\right) = 0 \quad (0 \le \sigma < 2\pi).$$

Accordingly the problem is to find a 2π-periodic function θ^* satisfying (2.7). Equivalence of (2.7) and (2.5) with $\omega(0) = 0$ is also shown in [5]. Obviously, $\theta^* \equiv 0$ satisfies (2.7) for all p and q. This is a trivial solution. In fact, (2.6) implies $h \equiv$ *constant.* Thus free boundary is completely flat. By the definition of ω, $\theta(1, \sigma) \equiv 0$ implies $df/dz \equiv c$, which shows that the velocity field is uniform: $(u, v) \equiv (c, 0)$. The problem of periodic progressive water waves is to find θ^* which is not identically zero and satisfies (2.7).

§3. Crapper's waves.

In what follows we consider (2.5) or (2.7) in the case of $p = 0$. Namely we neglect the gravity force and only the capillary force is assumed. In this extreme case, Crapper [1] found a family of solutions which are expressed by elementary functions. We put

$$F(q,u) = \frac{d}{d\sigma}\left(\frac{1}{2}e^{2Hu} + qe^{Hu}\frac{du}{d\sigma}\right).$$

Thus we wish to find zeros of F. F is defined in a certain Banach space, which will be defined in the next section. Since we consider only Crapper's special solutions in this secton, the functional analytic formalism in the next section is not necessary here. Note that (q,u) is a zero of F, if $q du/d\sigma = -\sinh(Hu)$.

THEOREM 3.1 (CRAPPER [1]). *Define an analytic function* ω *by*

$$\omega = \theta + i\tau = 2i \log \frac{1 + A\zeta}{1 - A\zeta}, \tag{3.1}$$

where A is a real parameter satisfying $-1 < A < 1$ and the principal branch of log is taken. We also define

$$q = \frac{1 + A^2}{1 - A^2}. \tag{3.2}$$

Then $u = \theta(1,\sigma)$ satisfies $F(q,u) = 0$ for $-1 < A < 1$.

PROOF: For $A \in (-1,1)$, the relation (3.1) defines an analytic function in the unit disk. We have

$$\begin{aligned} \tau(1,\sigma) &= 2\log\left|\frac{1 + Ae^{i\sigma}}{1 - Ae^{i\sigma}}\right| = \log \frac{1 + A^2 + 2A\cos\sigma}{1 + A^2 - 2A\cos\sigma} \\ &= 4\left(A\cos\sigma + \frac{A^3}{3}\cos 3\sigma + \frac{A^5}{5}\cos 5\sigma + \cdots\right) \end{aligned} \tag{3.3}$$

and

(3.4)

$$\theta(1,\sigma) = -2\arctan\left(\frac{2A\sin\sigma}{1 - A^2}\right) = -4\left(A\sin\sigma + \frac{A^3}{3}\sin 3\sigma + \frac{A^5}{5}\sin 5\sigma + \cdots\right).$$

It also holds that $\tau(1,\sigma) = H(\theta(1,\sigma))$ and

$$e^{\tau(1,\sigma)} = \frac{1 + A^2 + 2A\cos\sigma}{1 + A^2 - 2A\cos\sigma} = \frac{1 + 3A^2}{1 - A^2} + \frac{4(1 + A^2)}{1 - A^2}\sum_{n=1}^{\infty} A^n \cos n\sigma, \tag{3.5}$$

and

$$e^{-\tau(1,\sigma)} = \frac{1 + A^2 - 2A\cos\sigma}{1 + A^2 + 2A\cos\sigma} = \frac{1 + 3A^2}{1 - A^2} + \frac{4(1 + A^2)}{1 - A^2}\sum_{n=1}^{\infty}(-A)^n \cos n\sigma. \tag{3.6}$$

These two equations give

$$\sinh \tau = \frac{4(1+A^2)}{1-A^2}\left(A\cos\sigma + A^3\cos 3\sigma + \cdots\right).$$

This equality and (3.4) prove that

$$q\frac{d}{d\sigma}\theta(1,\sigma) = -\sinh\tau.$$

It is now easy to check $F(q,u)=0$. ∎

We call the solution in Theorem 3.1 Crapper's wave. By the definition, Crapper's wave satisfies $\omega \equiv 0$ at $q=1$. This means that $df/dz \equiv c$. Thus, at $q=1$, Crapper's wave becomes a trivial solution. In other words, Crapper's wave bifurcates at $q=1$ from the trivial solution. (3.2) also shows that the bifurcation takes place supercritically in q. The bifurcation diagram is illustrated by

$$q = \cosh\left(\frac{1}{2}\tau(1,0)\right),$$

which follows from (3.2, 3). This curve and the trivial solutions $\{(q,0); 0<q<\infty\}$ form a pitchfork.

If we linearize F at $u=0$, then we obtain the following Fréchet derivative:

$$D_uF(q,0)w = \frac{d}{d\sigma}\left(Hw + q\frac{dw}{d\sigma}\right).$$

This formula yields $D_uF(q,0)(\cos n\sigma) = n(1-nq)\cos n\sigma$ and $D_uF(q,0)(\sin n\sigma) = n(1-nq)\sin n\sigma$ for $n=1,2,\cdots$. Therefore, the Fréchet derivative at the trivial solution has nontrivial kernel if and only if $q=1/n$ for some $n=1,2,\cdots$. The solutions (3.4) corresponds to the one emanating from $(q,u)=(1,0)$. The solutions bifurcating from $(q,u)=(1/N,0)$ $(N=1,2,\cdots)$ are given by

$$q = \frac{1}{N}\frac{1+A^2}{1-A^2}, \qquad u(\sigma) = \theta(1,N\sigma),$$

where θ is defined by (3.4). We call this family of solutions Crapper's solutions of mode N. We now have a diagram as is illustrated schematically in Figure 1.

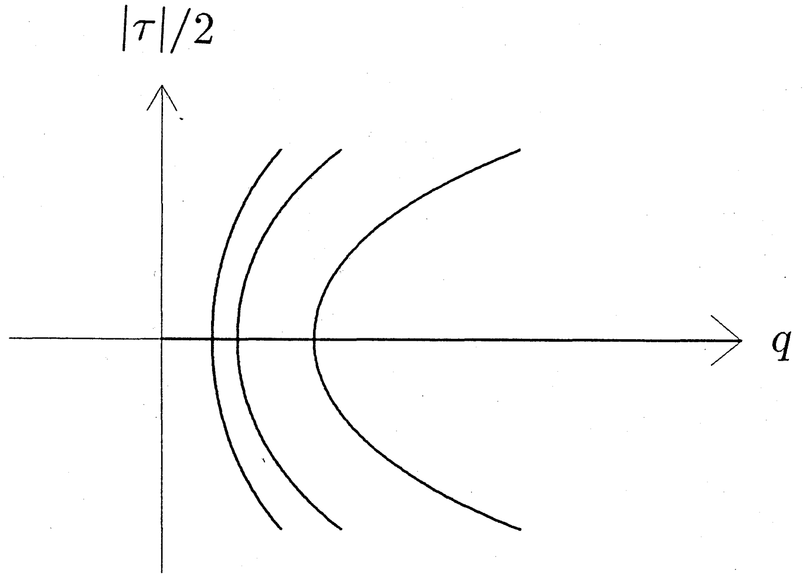

Figure 1. Bifurcation diagram of Crapper's waves

Since (3.1) yields that

$$\frac{dz}{d\zeta} = c^{-1}\left(\frac{1-A\zeta}{1+A\zeta}\right)^2 \frac{df}{d\zeta} = -\frac{L}{2\pi i\zeta}\left(\frac{1-A\zeta}{1+A\zeta}\right)^2,$$

we have the following parametrization of the free boundary:

(3.7) $$\frac{x}{L} = \alpha - \frac{2}{\pi}\frac{A\sin 2\pi\alpha}{1+A^2+2A\cos 2\pi\alpha} \qquad (0<\alpha<1)$$

(3.8) $$\frac{y}{L} = -\frac{2}{\pi} + \frac{2}{\pi}\frac{1+A\cos 2\pi\alpha}{1+A^2+2A\cos 2\pi\alpha} \qquad (0<\alpha<1).$$

By this formula we can draw the figures of the free boundaries. In Figure 2, free boundaries with negative A are given. Curves with positive A are obtained by shifting those figures with $-A$ by half a wave length. While $|A|$ is small, the wave profiles look sinusoidal. They, however, form particular shapes for large $|A|$ and, eventually, y is not a single valued function of x when $|A| > 0.414215\cdots$. If $|A| > 0.454670\cdots$, then the free boundary has a self-intersection and becomes physically meaningless. In this sense we redraw Figure 1 as Figure 1', where solid curves represent physical solutions and broken curves represent unphysical ones.

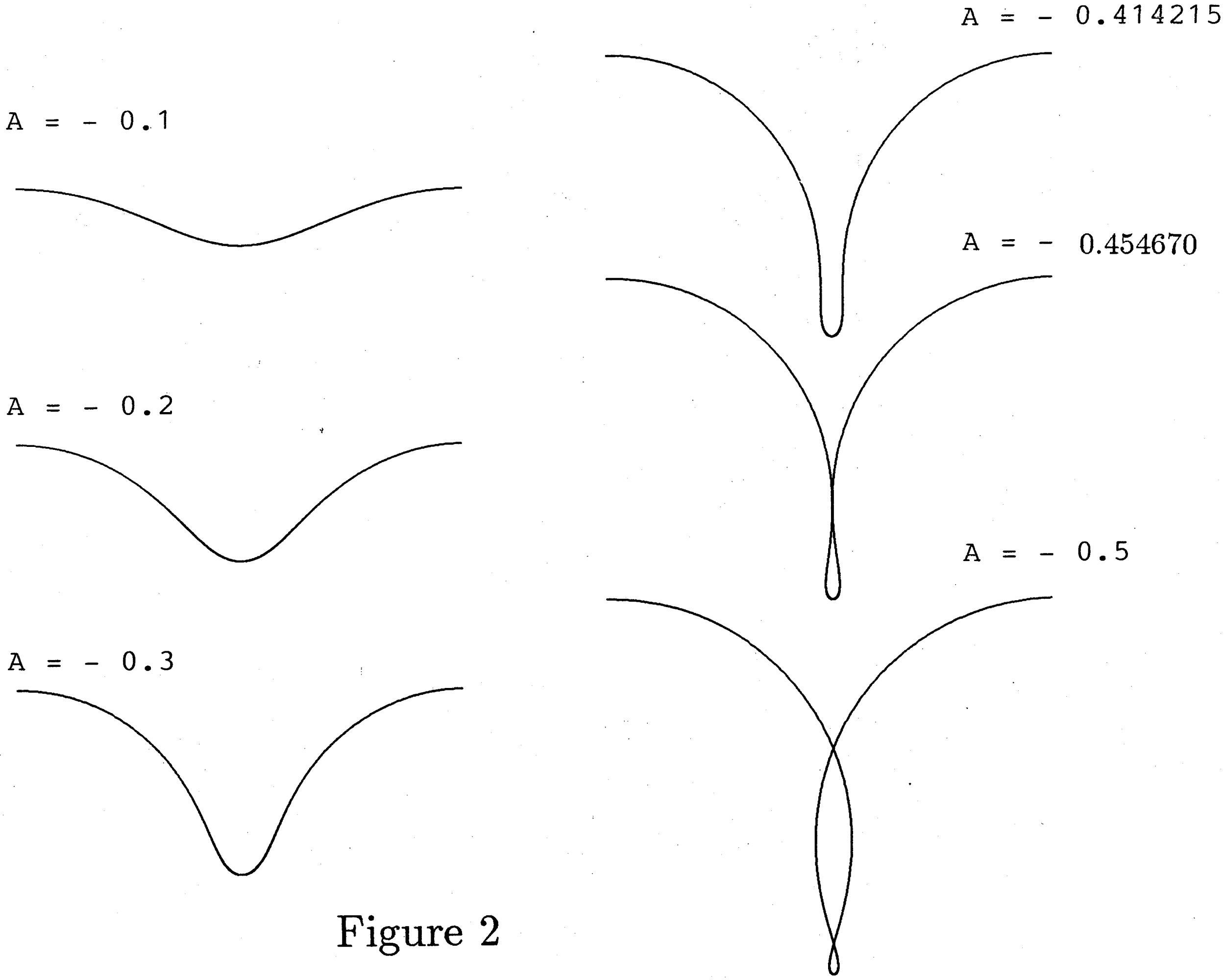

Figure 2

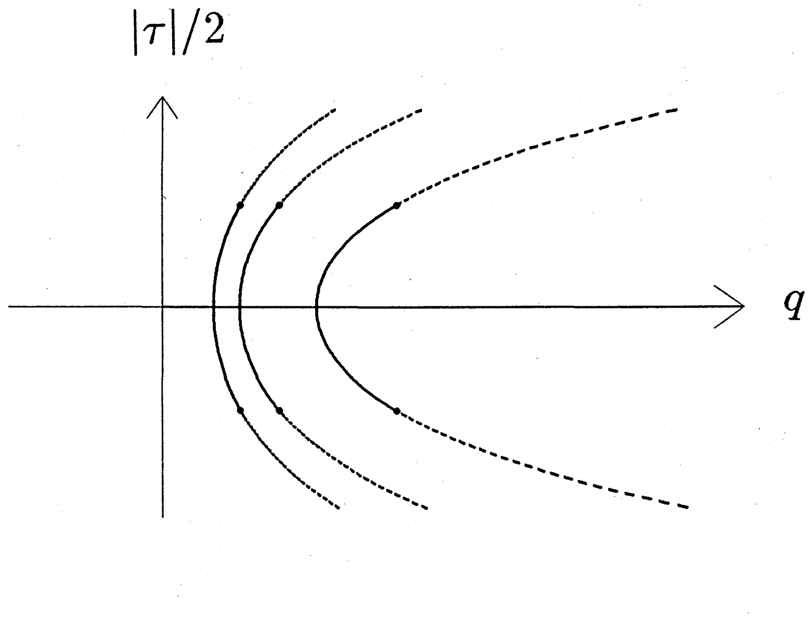

Figure 1'

Note, however, that (3.1) is well-defined for all $A \in (-1,1)$ and is a mathematical solution to $F=0$. Computations in [7] pointed out that: acquisition of not only physical solutions with no self-intersection but also self-intersecting, unphysical solutions is necessary for the complete understanding of global structure of the solutions. In fact, consider a bifurcation point at which the wave profile is self-intersecting. The new bifurcating solutions near the bifurcation point are self-intersecting. The branch, however, may go back to the region of physical solutions and there may be physically meaningful solutions other than Crapper's waves. A possible example is illustrated by Figure 3, where solid curves represents physical solutions and the broken curves do unphysical ones. Suppose we are tracing solutions along the curve AB and find that the solutions begin forming self-intersections. If we neglect solutions after B then we clearly overlook the physical solutions C. Bifurcation of this nature do occur when both surface tension and gravity are taken into account ([7]). This point is explained in more detail later in §4. Accordingly we consider (3.4) for all $A \in (-1,1)$.

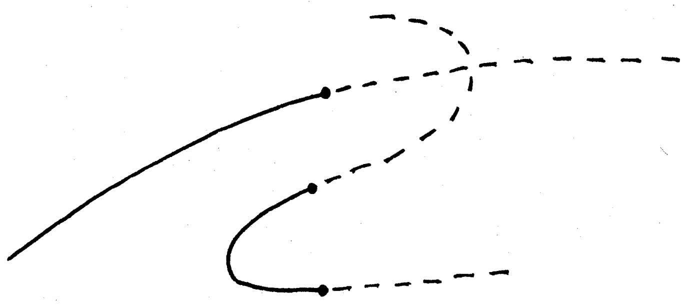

Figure 3. Possible recovery from self-intersection

§4. No bifurcation from Crapper's waves.

In this section, we prove nonexistence of a bifurcation from the branch of Crapper's waves. In order to state a theorem precisely, we define the following function spaces for a nonnegative integer m:

$$\begin{aligned} X^m = &\ H^m(S^1)/\mathbb{R} \\ = &\left\{ f = \sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma); \quad \sum_{n=1}^{\infty} n^{2m}\left(|a_n|^2 + |b_n|^2\right) < \infty \right\}, \end{aligned}$$

where $H^m(S^1)$ is the Sobolev space. The mapping F is a smooth mapping from $\mathbb{R}\times X^2$ into X^0 (see [5]). We define $Y^m = \{f \in X^m; \int_0^{2\pi} f(\sigma)\cos n\sigma d\sigma = 0 \quad (n = 1, 2, \cdots)\}$.

Then F sends every element of Y^2 into Y^0 ([5]). For the sake of convenience, we write for a positive integer N,

$$(4.1)\qquad Q_N(A)=\frac{1+A^2}{N(1-A^2)},\qquad \Theta_N(A,\sigma)=-2\arctan\left(\frac{2A\sin(N\sigma)}{1-A^2}\right).$$

We then have $F(Q_N(A),\Theta_N(A,\cdot))=0$ for all $-1<A<1$. We note that Crapper's solutions (4.1) lie in Y^2.

THEOREM 4.1. *For any $N=1,2,\cdots$ and any $A\in(-1,1)$ with $A\neq 0$, Crapper's solutions (4.1) are isolated. Namely, there is no bifurcation from Crapper's waves except for $(q,u)=(1/N,0)$.*

PROOF: We must be careful about the meaning of isolatedness. In fact, if $\theta(\sigma)$ is a solution, then $\theta(\sigma+\beta)$ is a solution, too, where β is a constant in $[0,2\pi)$. In $\mathbb{R}\times Y^2$, Crapper's waves form a pitchfork given by (2.2, 4). In $\mathbb{R}\times X^2$, we have a set of solutions which is obtained by revolving the pitchfork around the q-axis. Thus we have a bifurcation diagram consisting of a parabolic surface and a line intersecting with it. What we have to show is that any point of the parabola except for $(q,u)=(1,0)$ has a neighborhood in $\mathbb{R}\times X^2$ such that the zeros of F in the neighborhood are only those points on the parabola.

We first calculate the Fréchet derivative:

$$(4.2)\qquad F_u(q,u)w=\frac{d}{d\sigma}\left[e^{2Hu}Hw+qe^{Hu}\frac{du}{d\sigma}Hw+qe^{Hu}\frac{dw}{d\sigma}\right]\qquad(w\in X^2\ ).$$

In what follows, we fix a positive integer N. We write $F_u(Q_N(A),\Theta_N(A,\cdot))$ as L_A. We note that

$$(4.3)\qquad F(Q_N(A),\Theta_N(A,\cdot+\beta))=0$$

for all $A\in(-1,1)$ and $\beta\in[0,2\pi)$. By differentiating with respect to β and putting $\beta=0$, we obtain $L_A(\partial\Theta_N/\partial\sigma)=0$. Differentiating (4.3) in A and putting $\beta=0$, we have

$$L_A\left(\frac{\partial\Theta_N}{\partial A}\right)+F_q(Q_N,\Theta_N)\frac{\partial Q_N}{\partial A}=0.$$

We consider the following equation in $\mathbb{R}\times X^2$:

$$(4.4)\qquad L_Aw+\Phi_A\lambda=0,$$

where $\Phi_A=F_q(Q_N,\Theta_N)\frac{\partial Q_N}{\partial A}$. By the above consideration, we already have two independent solutions to this equation, i.e.,

(4.5)
$$(\lambda,w)=(0,\frac{\partial\Theta_N}{\partial\sigma})=\left(0,-4AN(\cos N\sigma+A^2\cos 3N\sigma+A^4\cos 5N\sigma+\cdots)\right)$$

(4.6)
$$(\lambda,w)=(1,\frac{\partial\Theta_N}{\partial A})=\left(1,-4(\sin N\sigma+A^2\sin 3N\sigma+A^4\sin 5N\sigma+\cdots)\right),$$

We prove the following lemma at the end of this section.

LEMMA. *Suppose that $-1 < A < 1$ and $A \neq 0$. Then the set of all $(\lambda, w) \in \mathbb{R} \times X^2$ satisfying (3.4) is exactly equal to the two dimensional subspace spanned by (4.5,6).*

For the moment, we admit the validity of this lemma and complete the proof of Theorem 4.1. To this end, let us consider a mapping G defined by

$$G(A, \beta, u) = F(Q_N(A), \Theta_N(A, \cdot + \beta) + u).$$

Let Z be the set of all the elements of X^2 which is L^2-orthogonal to $\partial\Theta_N/\partial\sigma$. We regard G as a mapping from $(-1,1) \times [0, 2\pi) \times Z$ to X^0. Suppose $(\bar{A}, \overline{\beta}) \in (-1,1) \times [0, 2\pi)$ is fixed. We notice that the Fréchet derivative $U := DG(\bar{A}, \overline{\beta}, 0)$ is represented as follows:

$$\begin{aligned} U(a, b, w) &= F_q(Q_N(\bar{A}), \Theta_N(\bar{A}, \cdot + \overline{\beta}))\frac{\partial Q_N}{\partial A}(\bar{A})a \\ &\quad + D_u F(Q_N(\bar{A}), \Theta_N(\bar{A}, \cdot + \overline{\beta})\Big(\frac{\partial \Theta_N}{\partial A}(\bar{A}, \cdot + \overline{\beta})a + \frac{\partial \Theta_N}{\partial \sigma}(\bar{A}, \cdot + \overline{\beta})b + w\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\quad a, b \in \mathbb{R} \quad w \in Z\ ). \end{aligned}$$

The equation $U(a, b, w) = 0$ is, by (4.2), equivalent to

$$\Phi_A a + L_A\Big(\frac{\partial \Theta_N}{\partial q}a + \frac{\partial \Theta_N}{\partial \sigma}b + w_\beta\Big) = 0,$$

where $w_\beta(\sigma) = w(\sigma - \beta)$ and we have dropped the bars of $\bar{A}$ and $\overline{\beta}$. Therefore the lemma above shows that the kernel of $DG(\bar{A}, \overline{\beta}, 0)$ is equal to $\mathbb{R}^2$. It is not difficult by (4.1) to see that the range of U is closed and U has a left inverse from X^0 to Z. Namely there is a bounded linear operator $V : X^0 \to Z$ such that $VUw = w$ for all $w \in Z$. We now recall the implicit function theorem. A proof of the uniqueness of the implicit function works in this situation and we see that $u \equiv 0$ is the only solution to $G(A, \beta, u(q, \beta)) = 0$ in some neighborhood of $(\bar{A}, \overline{\beta})$. Thus we are done.

PROOF OF LEMMA: Suppose now that (λ, w) satisfies (4.4) at $A = \bar{A}$. For simplicity, $Q_N(\bar{A})$ is denoted by q. Then there exists a constant c such that

$$\begin{aligned} c &= \Big(e^{2\tau} + q\frac{\partial \Theta_N}{\partial \sigma}e^{\tau}\Big)Hw + qe^{\tau}\frac{dw}{d\sigma} + \lambda e^{\tau}\frac{\partial \Theta_N}{\partial \sigma}\frac{\partial Q_N}{\partial A}(\bar{A}) \\ &= e^{\tau}\cosh\tau Hw + qe^{\tau}\frac{dw}{d\sigma} - \frac{\lambda}{q}e^{\tau}\sinh\tau\frac{\partial Q_N}{\partial A}(\bar{A}), \end{aligned}$$

where, $\tau = H(\Theta_N(\bar{A}, \cdot))$. Hereafter we write A instead of $\bar{A}$. By this equation we have

$$ce^{-\tau} = \cosh\tau Hw + q\frac{dw}{d\sigma} - \frac{\lambda}{q}\sinh\tau\frac{\partial Q_N}{\partial A} \tag{4.7}$$

We expand w as follows:

$$w = \sum_{n=1}^{\infty} a_n \sin n\sigma + \sum_{n=1}^{\infty} b_n \cos n\sigma.$$

It holds that

$$\cosh\tau = \frac{1 + A^4 + 4A^2 + 2A^2 \cos 2N\sigma}{1 + A^4 - 2A^2 \cos 2N\sigma}, \qquad \sinh\tau = \frac{4A(1+A^2)\cos N\sigma}{1 + A^4 - 2A^2 \cos 2N\sigma}.$$

Taking the even and the odd part of (4.7), we have

(4.8)
$$ce^{-\tau} + \frac{\lambda}{q}\sinh\tau \frac{\partial Q_N}{\partial A} = \cosh\tau \left(-\sum_{n=1}^{\infty} a_n \cos n\sigma \right) + q\sum_{n=1}^{\infty} n a_n \cos n\sigma$$

(4.9)
$$0 = \cosh\tau \left(\sum_{n=1}^{\infty} b_n \sin n\sigma \right) - q\sum_{n=1}^{\infty} n b_n \sin n\sigma$$

We first consider (4.9). It yields

$$(1 + A^4 + 4A^2 + 2A^2 \cos 2N\sigma) \sum_{n=1}^{\infty} b_n \sin n\sigma = (1 + A^4 - 2A^2 \cos 2N\sigma) q \sum_{n=1}^{\infty} n b_n \sin n\sigma.$$

For a positive integer n, we define $b_{-n} = -b_n$ and put $b_0 = 0$ Then we have

$$\left(1 + A^4 + 4A^2 + A^2 e^{2Ni\sigma} + A^2 e^{-2Ni\sigma}\right) \sum_{n\in\mathbb{Z}} b_n e^{in\sigma}$$
$$- q\left(1 + A^4 - A^2 e^{2Ni\sigma} - A^2 e^{-2Ni\sigma}\right) \sum_{n\in\mathbb{Z}} |n| b_n e^{in\sigma} = 0.$$

Consequently (4.9) is equivalently rewritten as the following recurrence relation:

(4.10)
$$\left(1 + A^4 + 4A^2 - \frac{|n|}{N}\frac{1+A^2}{1-A^2}(1+A^4)\right) b_n + A^2\left(1 + \frac{|n-2N|}{N}\frac{1+A^2}{1-A^2}\right) b_{n-2N}$$
$$+ A^2\left(1 + \frac{|n+2N|}{N}\frac{1+A^2}{1-A^2}\right) b_{n+2N} = 0, \qquad (\, n \in \mathbb{Z}\,).$$

Note that (4.10) with $n = 0$ holds trivially and that the relations with negative n are obtainable from those with positive n. Therefore it is sufficient to consider (4.10) for positive integers n. We now consider the case where $n \geq 2N$. In this case the relation (4.10) is written as

$$B_n = \mu_n B_{n-2N},$$

where $B_n = b_{n+2N} - A^2 b_n$, and

$$\mu_n = \frac{n(1+A^2) - N(1+3A^2)}{A^2[N(3+A^2) + n(1+A^2)]}.$$

Since $\mu_n > 0$ for $n \geq 2N$ and since

$$\lim_{n\to\infty} \mu_n = \frac{1}{A^2} > 1,$$

either of the followings holds for each $k = 0, 1, 2, \cdots, 2N-1$:

$$B_k = B_{2N+k} = \cdots = 0, \qquad \text{or } \lim_{m\to\infty} |B_{2mN+k}| = \infty.$$

On the other hand, B_n can not increase indefinitely, since $\{b_n\}$ is square summable hence is bounded. Accordingly we obtain

$$b_{2mN+k} = A^{2m} b_k \qquad (k = 0, 1, \cdots, 2N-1, \quad m = 1, 2, \cdots).$$

By virtue of $b_0 = 0$, b_{2mN} vanishes for any $m = 1, 2, \cdots$.

We now consider (4.9) in the case where $1 \leq n \leq 2N-1$. In this case we have

$$\left(1 + A^4 + 4A^2 - \frac{n}{N}\frac{1+A^2}{1-A^2}(1+A^4)\right)b_n + A^2\left(1 - \frac{n-2N}{N}\frac{1+A^2}{1-A^2}\right)b_{n-2N}$$
$$+ A^2\left(1 + \frac{n+2N}{N}\frac{1+A^2}{1-A^2}\right)b_{n+2N} = 0. \qquad (\ n = 1, 2, \cdots, 2N-1\ ).$$

Noting that $b_{2N+n} = A^2 b_n$ and $b_{n-2N} = -b_{2N-n}$, we obtain after some computation,

$$[N(1+3A^2) - n(1+A^2)]b_n - A^2[N(3+A^2) - n(1+A^2)]b_{2N-n} = 0.$$

Replacing n by $2N - n$ we have

$$[N(-1+A^2) + n(1+A^2)]b_{2N-n} - A^2[N(1-A^2) + n(1+A^2)]b_n = 0.$$

When $n = N$ these two equalities identically hold and imply nothing. Therefore we have $b_{2mN+N} = A^{2m} b_N$, $(m = 1, 2, \cdots)$ as a solution. When $n \neq N$, then these two equalities implies $b_n = b_{2N-n} = 0$ or

$$\frac{A^2[N(3+A^2) - n(1+A^2)]}{N(1+3A^2) - n(1+A^2)} = \frac{-N(1-A^2) + n(1+A^2)}{A^2[N(1-A^2) + n(1+A^2)]}.$$

By an elementary computation, we see that this equation does not hold for $A \in (-1, 1)$ and $n \neq N$. Consequently the only nontrivial solution is:

$$b_{2mN+N} = A^{2m} b_N \qquad (m = 1, 2, \cdots), \qquad b_k = 0 \qquad \text{for other } k.$$

Thus the even part of w is a constant multiple of $\partial\theta/\partial\sigma$.

We now consider (4.8). Writing

$$\eta = \frac{\lambda}{q}\frac{\partial Q_N}{\partial A}$$

we have

$$c\Big[(1+4A^2+A^4-4A(1+A^2)\cos N\sigma + 2A^2\cos 2N\sigma\Big] + 4A(1+A^2)\eta\cos N\sigma$$
$$= \big(1+4A^2+A^4+2A^2\cos 2N\sigma\big)\sum_{n=1}^{\infty}(-a_n\cos n\sigma)$$
$$+ q\left(1+A^4-2A^2\cos 2N\sigma\right)\sum_{n=1}^{\infty} na_n\cos n\sigma.$$

For those n's which are not integer multiples of N, a_n satisfies the same recurrence relation as b_n, hence it must vanish. Consequently we have only to consider a_{mN}, $(m = 1, 2, \cdots)$. For $m \in \mathbb{N}$, we define $d_{-m} = d_m = a_{mN}$. We then have

$$c\Big[(1+4A^2+A^4-2A(1+A^2)(e^{i\xi}+e^{-i\xi})+A^2(e^{2i\xi}+e^{-2i\xi})\Big] + 2A(1+A^2)\eta(e^{i\xi}+e^{-i\xi})$$
$$= -\left(1+4A^2+A^4+A^2(e^{2i\xi}+e^{-2i\xi})\right)\sum_{m\in\mathbb{Z}}\frac{d_m}{2}e^{im\xi}$$
$$+ q\left(1+A^4-A^2(e^{2i\xi}+e^{-2i\xi})\right)\sum_{m\in\mathbb{Z}}|m|N\frac{d_m}{2}e^{im\xi},$$

where we have put $\xi = N\sigma$. This equality gives the following relations:

$$c(1+A^4+4A^2) = -A^2\frac{d_{-2}+d_2}{2} - 2qNA^2\frac{d_{-2}+d_2}{2} \tag{4.11}$$

$$-2cA(1+A^2) - 2A(1+A^2)\eta = -\left(1+A^4+4A^2-(1+A^4)qN\right)\frac{d_1}{2} - A^2\frac{d_{-1}+d_3}{2} - A^2qN\frac{d_{-1}+3d_3}{2}, \tag{4.12}$$

$$cA^2 = -\left(1+A^4+4A^2-2(1+A^4)qN\right)\frac{d_2}{2} - A^2\left(4qN+1\right)\frac{d_4}{2}, \tag{4.13}$$

and for $m \geq 3$,

$$\left(1+A^4+4A^2-m(1+A^4)qN\right)d_m = -A^2\left((m+2)q+1\right)d_{m+2} - A^2\left((m-2)q+1\right)d_{m-2}. \tag{4.14}$$

This equation (4.14) is of the same form as (4.10). Therefore, it holds that

$$(4.15) \qquad d_{2k+1} = A^{2k} d_1 \qquad d_{2k+2} = A^{2k} d_2 \quad (k = 0, 1, 2, \cdots)$$

Substituting $d_4 = A^2 d_2$ into (4.13), we obtain $d_2 = 2cA^2$. Substituting this into (4.11), we obtain

$$c\left(1 + A^4 + 4A^2 + 4A^4 \frac{1 + A^2}{1 - A^2} + 2A^4\right) = 0.$$

Hence we have $c = 0$ and $d_2 = 0$. Then (3.16) implies that $d_{2k} = 0$ for all k. By $c = 0$, (4.12) and $d_3 = A^2 d_1$, we obtain $\eta = -d_1 A/(1 - A^4)$, which implies that $\lambda = -d_1/4$. We have thus shown that the solutions to (4.4) consists of two dimensional space spanned by (4.5,6). ∎

Remark. Kinnersley [3] obtained a formula for the free boundary in the case of finite depth. His formula is written in terms of Jacobi's elliptic functions. We do not know whether we can generalize Theorem 3.1 so as to include the case of finite depth.

§5. Numerical solutions.

In this section, we present a numerical result, which seems to indicate a conjecture. By theorem 4.1 we know that, as we gradually increase the parameter q, Crapper's waves bifurcate from the trivial solution and there is no secondary bifurcation from the branch of Crapper's waves. It is, however, possible that a solution may exists which is not connected with neither the trivial solution nor Crapper's waves. In fact, for relatively large positive p, it is known that such isolated solutions exist ([**6**,**7**]). We conjecture that

if $p = 0$, then any solution to (2.7) is either a trivial solution or one of Crapper's waves.

The following numerical results seems to support this conjecture. We computed bifurcation diagrams (Figure 4 and 5) in the (q, A_1, A_2) space with different p, where A_1 and A_2 are the Fourier coefficients of $\sin\sigma$ and $\sin 2\sigma$ of $dx/d\sigma$. Do not confuse these with the parameter A in §3. Figure 4 shows the bifurcation diagrams for $p = 0.7, 0.6666667, 0.64$. The left hand sides show views in an obligue direction and the right hand sides show the views in the direction of A_1-axis. All the figures are symmetric with respect to $q - A_2$ plane. At $p = 0.7$, we observe a loop which is connected to q-axis and the upper pitchfork branch. The loop has two turning points. As p decreases, the joint of the loop and q-axis approaches the pitchfork bifurcation point and at $p = 1/3$ the loop emanates from the bifurcation point. As p decreases further, the joint climb up the pitchfork. At approximately $p = 0.54$, the turning points disappear (see Figure 5). As p decrease further, the size of the loop decrease. At $p = 0.465$ we can observe only mode 1 branch and mode 2 branch. Thus numerical results comply with our conjecture.

We presents another numerical results in Figure 6 and 7, where we computed solutions of mode 3. They are drawn in (q, A_1, A_3) space. Diagrams in Figure 4 and 5 are symmetric with respect to $q - A_2$ plane. Those in Figure 6 and 7 are, however, symmetric with respect to the q axis. For big p, we have interecting loops. Note that some on the loops are physical and others are unphysical. As p decreases, the loops shrink to points and we can see only two pitchfork branches when $p < 0.58$.

THEOREM 4.1 and our numerical computations seem to suggest our conjecture above. The proof, however, is not known to the authors.

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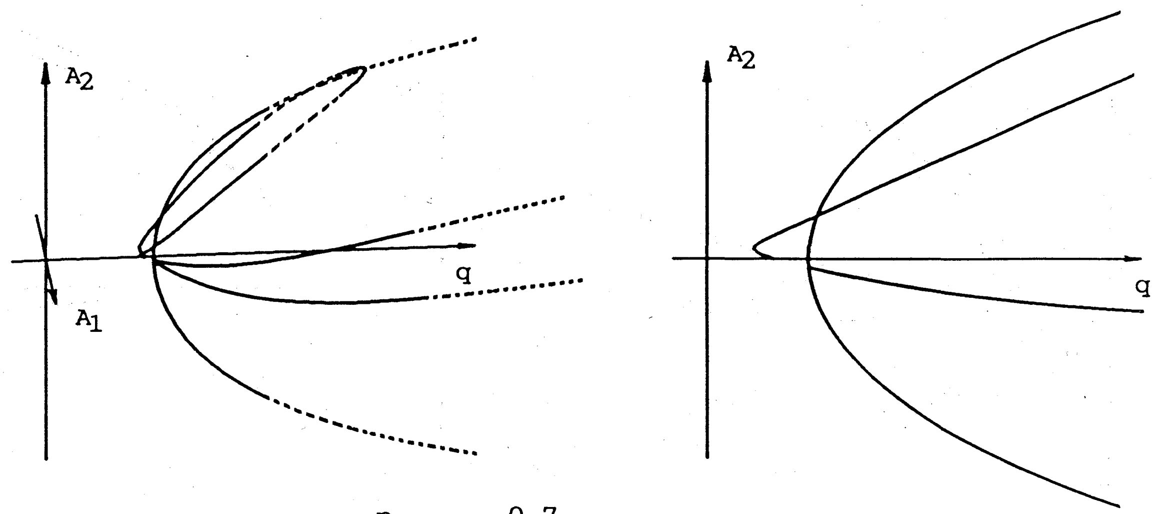
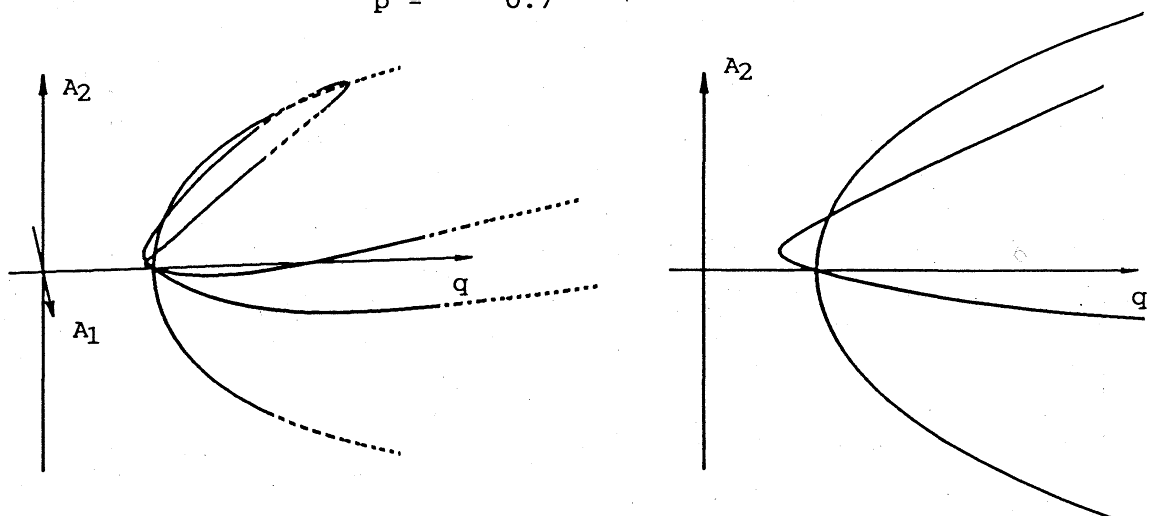


p = 0.7



p = 0.6666667

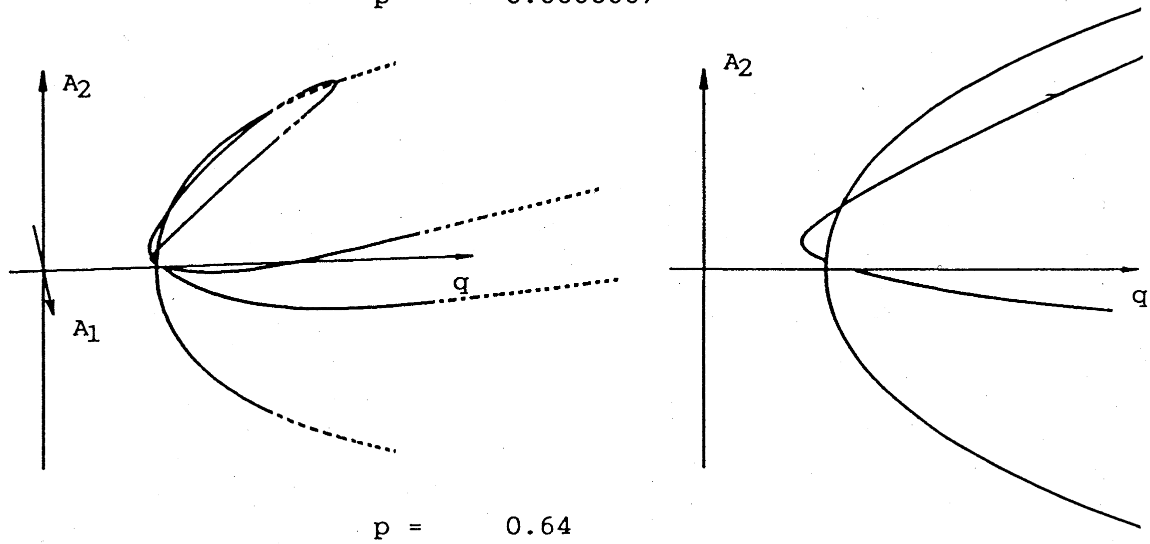


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Figure 4

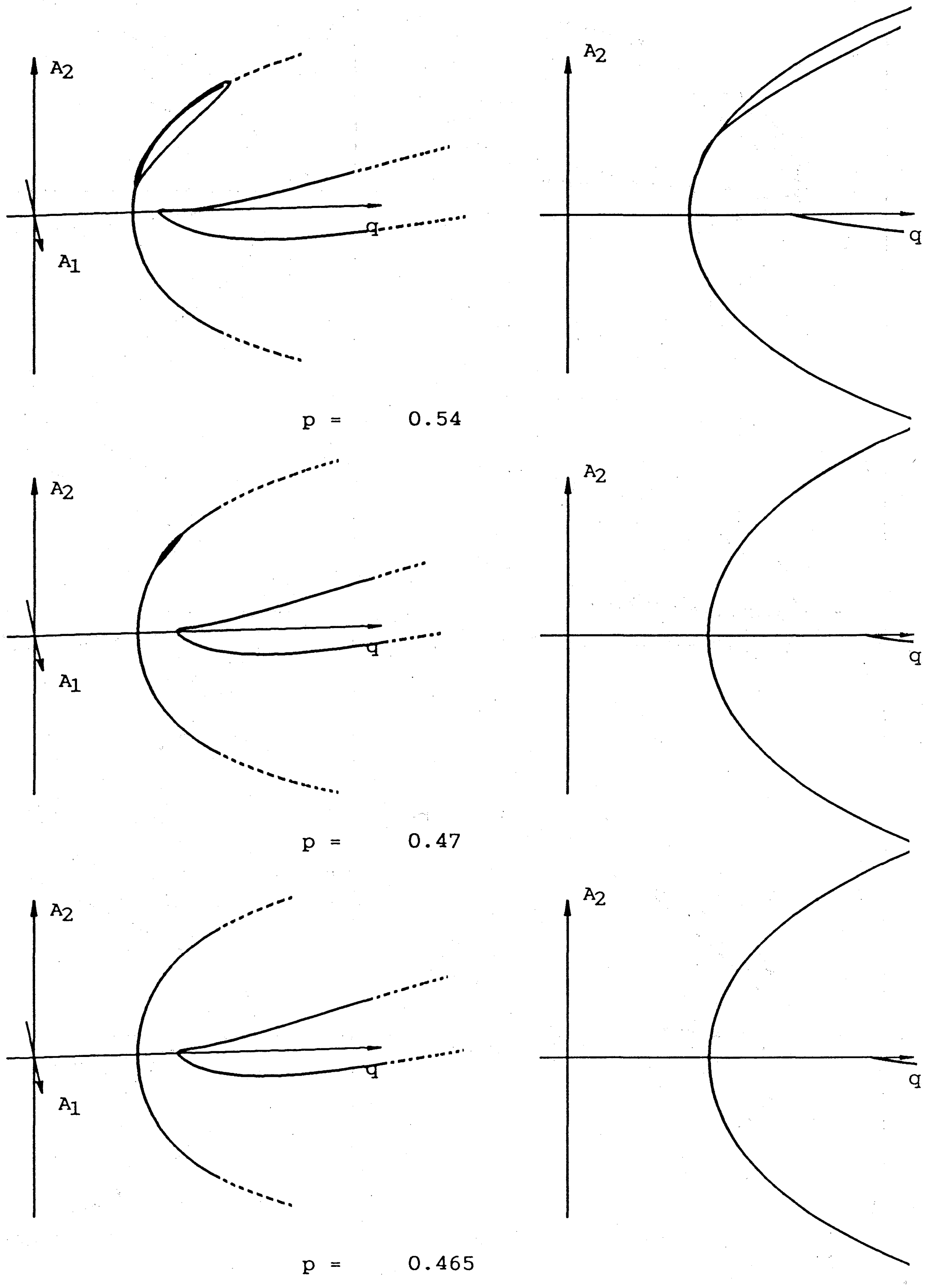


Figure 5

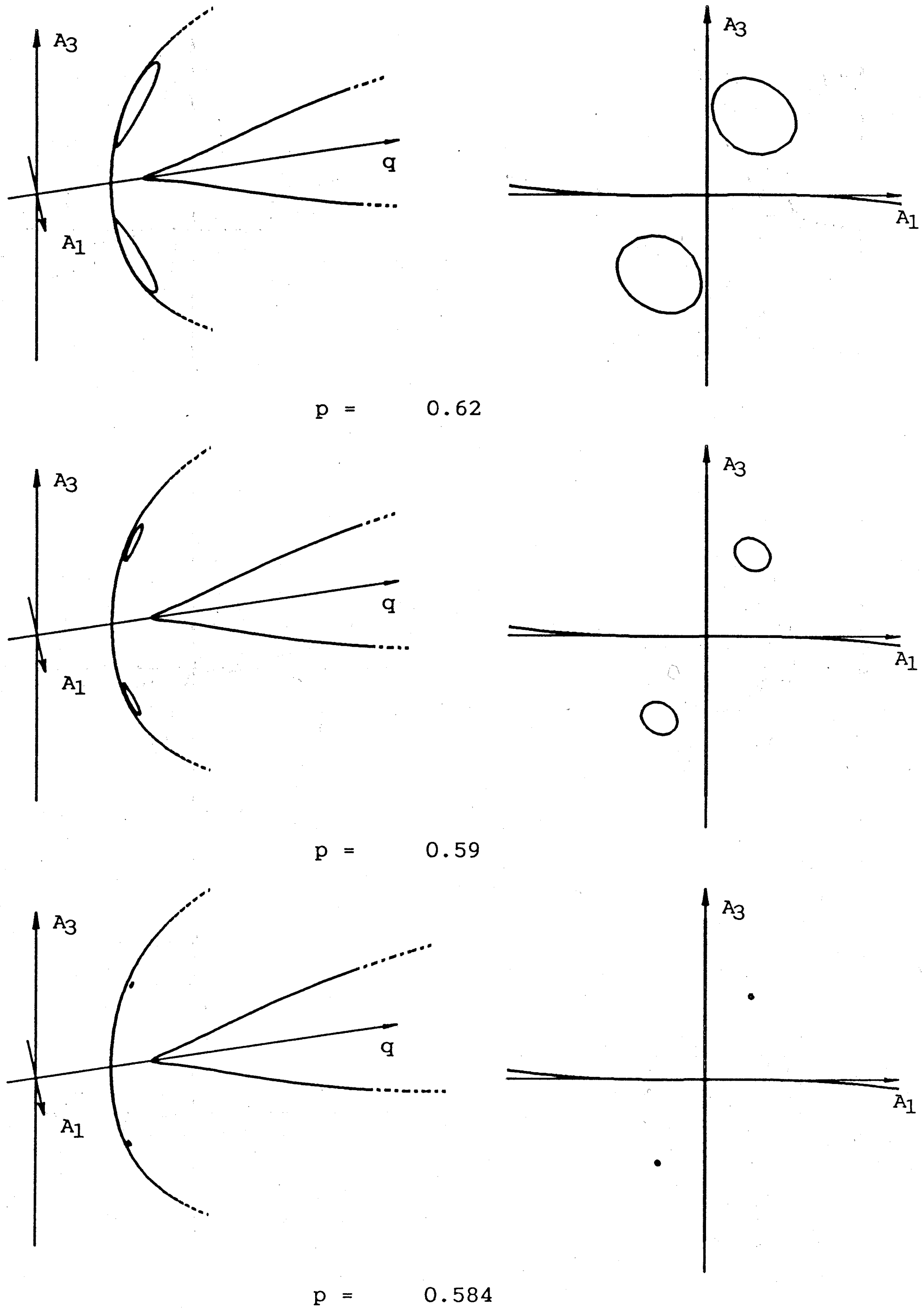
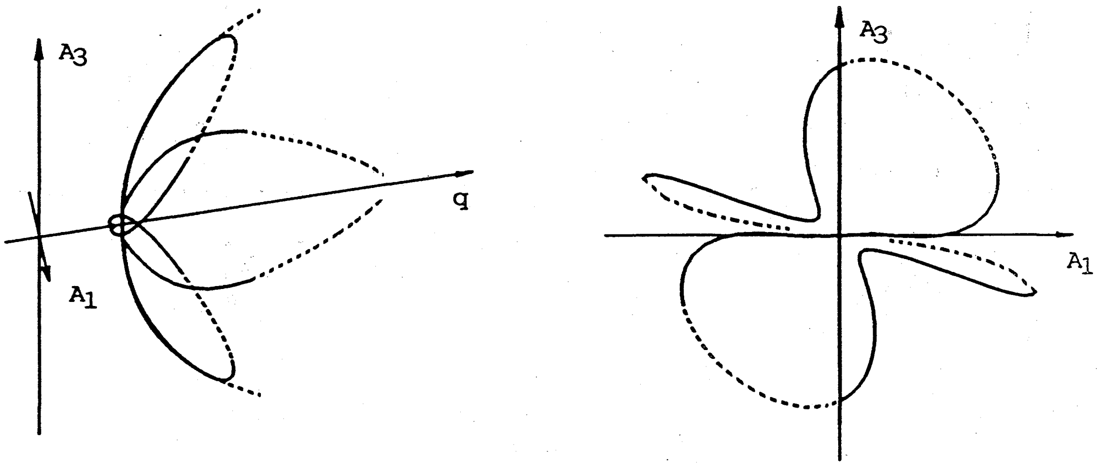
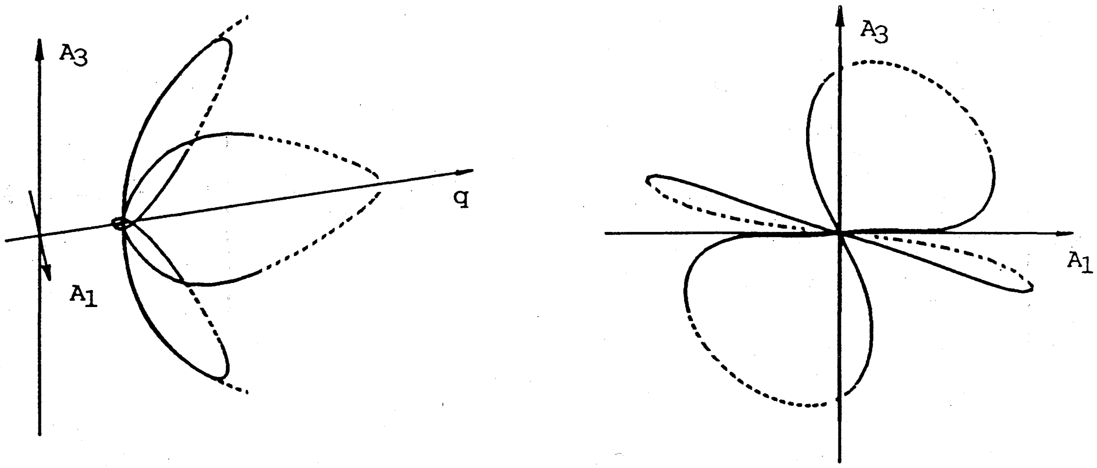


Figure 7



p = 0.77



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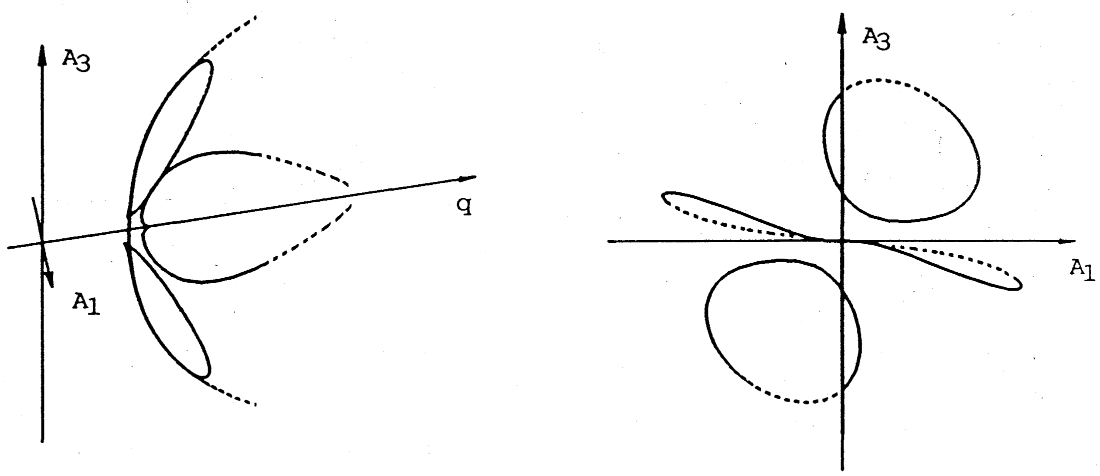

p = 0.68

Figure 6