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L^2 *Decay for Navier-Stokes Flows in Unbounded Domains, with Application to Exterior Stationary Flows*

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1. Introduction

The motion of a viscous incompressible fluid filling a domain $D \subset R^n$ is governed by the Navier-Stokes initial value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\cdot\nabla u = \Delta u - \nabla p + f \quad & (x \in D, t > 0), \\ \nabla\cdot u = 0 \quad & (x \in D, t \geq 0), \\ u|_S = 0 \ ; \ u|_{t=0} = a & \end{aligned} \tag{NS}$$

for unknown velocity $u = (u_j)_{j=1}^n$ and pressure p. Here S is the boundary of D, $x = (x_1, \cdots, x_n)$ is a point of R^n, a and f denote, respectively, given initial velocity and external force ; and $u\cdot\nabla u = \sum_j u_j\partial_j u_j$, $\nabla\cdot u = \sum_j \partial_j u_j$, $\nabla p = (\partial_j p)_{j=1}^n$, $\partial_j = \partial/\partial x_j$. The fluid density and the kinematic viscosity are normalized to be one. It is known [16] that problem (NS) possesses at least one weak solution for an arbitrary initial velocity a in L^2. Uniqueness of weak solutions is proved by now only when $n = 2$.

This paper studies the existence problem of a weak solution, in an arbitrary unbounded domain, which goes to zero in L^2, as $t \to \infty$, with explicit rates. The L^2 decay problem for Navier-Stokes flows was first raised by LERAY [14] in case $D = R^3$. The first (affirmative) answer was given by KATO [13] in case $D = R^n$, $n = 3, 4$, through his study of strong solutions in general L^p spaces. A different approach was then given by SCHONBEK [20] which is based on the Fourier decomposition for the fluid velocity u ; see also [12,21,23]. The idea of SCHONBEK was then applied by the present authors [2,3] to the case where D is a halfspace of R^n, $n \geq 2$, or an exterior domain of R^n, $n \geq 3$. In this paper we first show, in Sections 2 and 3, that the method developed in [2,3,12] can be modified so that it applies to the case of an arbitrary unbounded domain in R^n, $n \leq 4$. The arguments developed in Sections 2 and 3 are then applied in Section 4 to the stability problem for exterior stationary flows in three-dimensions.

To state our main results, we use the standard notation : $C_{0,\sigma}^\infty(D)$ denotes the set of smooth and compactly supported solenoidal vector fields on D. We denote by H (resp. V) the L^2- (resp. H^1-) closure of $C_{0,\sigma}^\infty(D)$. The following orthogonal decomposition

$$(L^2(D))^n = H \oplus H^\perp, \quad H^\perp = \{\nabla p \in (L^2(D))^n \ ; \ p \in L^2_{loc}(D)\},$$

is well known [22, Chap.I]. We denote by P the associated orthogonal projector onto H. To the bilinear form $(\nabla u, \nabla v)$ defined on $V \times V$, we associate a (unique) positive and self-adjoint operator A in H such that $D(A^{1/2}) = V$ and $\|A^{1/2}u\|_2 = \|\nabla u\|_2$, where $\|\cdot\|_r$ $(1 \le r \le \infty)$ is the usual L^r-norm. By $\hat{V}$ we denote the completion of $C^\infty_{0,\sigma}(D)$ in the norm $\|\nabla \cdot\|_2$, and by $\hat{V}^*$ its dual space. For simplicity in notation we assume that $f = Pf$, using the above orthogonal decomposition. Then the function

$$v(t) = e^{-tA}a + \int_0^t e^{-(t-s)A} f(s)ds$$

with $a \in H$ and $f \in L^2_{loc}([0,\infty); H)$ solves the nonstationary Stokes system

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v = f - \nabla q \quad & (x \in D, t > 0) \\ \nabla \cdot v = 0 \qquad & (x \in D, t \ge 0) \\ v|_S = 0 \;\; ; \;\; v|_{t=0} = a & \end{aligned} \tag{S}$$

with an appropriate scalar distribution q ; so the problem (NS) is formally transformed into the integral equation :

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}\left[f(s) - P(u \cdot \nabla)u(s)\right] ds. \tag{I}$$

Given a and f as above, a weakly continuous function $u : [0,\infty) \to H$ is called a weak solution of (NS) (or equivalently of (I)) if it belongs to $L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, and satisfies $u(0) = a$ and

$$(u(t), \phi(t)) + \int_s^t [(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi)]\, d\tau = (u(s), \phi(s)) + \int_s^t [(u, \phi') + (f, \phi)]\, d\tau \tag{W}$$

for all $t \ge s \ge 0$, and all $\phi \in C([0,\infty); V) \cap C^1([0,\infty); H)$, where $(\cdot,\cdot)$ is the standard L^2-inner product and $\phi' = \partial\phi/\partial t$. The existence of a weak solution corresponding to arbitrary a and f is well known ; see, e.g., [16]. All the weak solutions obtained so far satisfy the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \|a\|_2^2 + 2\int_0^t (f, u)ds$$

for all $t \ge 0$, and the equality sign holds in case $n = 2$. In Section 4 we shall deal with a more stringent form of the above energy inequality. Our main results are the following

Theorem 1.1. *Let $n = 3, 4$, and let D be an arbitrary n-dimensional unbounded domain for which the Poincaré inequality for functions in $C_0^\infty(D)$ does not hold. If $a \in H$, if $f \in L^2_{loc}([0,\infty); H) \cap L^1(0,\infty; H) \cap L^1(0,\infty; \hat{V}^*)$, and if*

$$\int_0^\infty t\|f(t)\|_2 dt < +\infty,$$

then there is a weak solution u of (NS) *such that, as $t \to \infty$,*

(i) $\|u(t)\|_2 \to 0.$

(ii) *If* $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ *for some* $\alpha > 0$, *then,*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2 \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geq 1/2, \end{cases}$$

where $0 < \varepsilon < 1/2$ *is arbitrary in case* $n = 3$, *and* $\varepsilon = 0$ *in case* $n = 4$.

Theorem 1.2. *Let* $D \subset R^2$ *be an arbitrary unbounded domain for which the Poincaré inequality does not hold. Given* $a \in H$ *and* f *as in Theorem 1.1, there is a unique weak solution* u *such that, as* $t \to \infty$,

(i) $\|u(t)\|_2 \to 0.$

(ii) *If* $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ *for some* $\alpha > 0$, *then*

$$\|u(t)\|_2 = O((\log(t+e))^{-m/2})$$

for all integers $m \geq 1$.

(iii) *If in addition* $\|a\|_2$ *and* $\int_0^\infty \|f\|_2 ds$ *are small enough and* $f \in L^2(0,\infty;\hat{V}^*)$, *then*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < 1/2 \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geq 1/2, \end{cases}$$

where $\varepsilon > 0$ *is arbitrary.*

(iv) *If* $a \in R(A^\alpha)$ *for some* $0 < \alpha \leq 1/2$, *then assertion* (iii) *holds irrespective of the size of* $\|a\|_2$ *and* $\int_0^\infty \|f\|_2 ds$.

When D is the entire space R^n or the halfspace R^n_+, $n \geq 2$, and $f = 0$, it is known [2,12,21,23] that there exists a weak solution u satisfying

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{for } \alpha < (n+2)/4 \\ O(t^{-(n+2)/4}) & \text{for } \alpha \geq (n+2)/4, \end{cases}$$

provided that $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$. When $n \geq 3$, $f = 0$, and D is an exterior domain with smooth boundary, we have recently proved in [3] the existence of a weak solution u such that, under the assumption $\|e^{-tA}a\|_2 = O(t^{-\alpha})$,

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4 \\ O(t^{-n/4}) & \text{if } \alpha \geq n/4. \end{cases}$$

All these results were deduced by essential use of various properties of the operator A in general L^r spaces. In our present case, however, the class of domains D is so large that we cannot appeal to L^r-theories. So we restrict ourselves to the case of space dimensions $n \leq 4$ and deduce our results by applying only L^2-theory of the operator A. We note that

Theorem 1.2 partially extends our previous result in [3] to the case of two-dimensional exterior domains (with nonsmooth boundaries).

We prove Theorem 1.1 in Section 2, using a specific approximation scheme. Since the uniqueness of weak solutions remains open in case $n \geq 3$, we first consider in Section 2 the decay problem for general weak solutions satisfying the energy inequality and show that the time-average $t^{-1}\int_0^t \|u\|_2 ds$ of any such weak solution u decays in the same way as stated in Theorem 1.1 ; see Theorem 2.1. It turns out that Theorem 1.1 immediately follows from Theorem 2.1.

Our proofs of Theorems 1.1 and 2.1 systematically use the weak version of the Hölder and Young inequalities, thereby avoiding a direct application of the spectral measure associated to the self-adjoint operator A. However, this approach does not work in two-dimensional case. So, we give in Section 3 a detailed proof of Theorem 1.2 which uses the spectral decomposition for A. This approach was first suggested by SCHONBEK [20] and then systematically studied by [2,3,12]. It is also possible to prove Theorem 1.1 by using the spectral decomposition. But, we do not employ this approach, since our argument in Section 2 provides not only Theorem 2.1, which is difficult to obtain by applying the spectral decomposition, but also a stability result given in Section 4, which directly deals with a non-self-adjoint linearized operator instead of the self-adjoint operator A.

In both of Theorems 1.1 and 1.2, it is in general difficult to completely characterize the class of functions $a \in H$ satisfying the condition $\|e^{-tA}a\|_2 = O(t^{-\alpha})$. In Section 3 we show that this condition holds for a in some L^r spaces. This result is deduced from the fact that the range $R(A^\alpha)$ of the fractional power A^α remains invariant under the Navier-Stokes flow if $\alpha > 0$ satisfies an appropriate condition depending on the space dimension ; and this invariance property not only enables us to prove assertion (iv) of Theorem 1.2, but also implies the following

Corollary 1.3. (i) *If $n = 2$ and $a \in H \cap (L^r(D))^2$ for some $1 < r < 2$, then the corresponding weak solution u satisfies*

$$\|u(t)\|_2 = O(t^{-(1/r-1/2)})$$

provided that f satisfies the assumption in Theorem 1.2 (iii).

(ii) *If $n = 3$ and $a \in H \cap (L^r(D))^3$ for some $6/5 \leq r \leq 3/2$, then there is a weak solution u with $u(0) = a$ such that*

$$\|u(t)\|_2 = O(t^{-3(1/r-1/2)/2}).$$

(iii) *If $n = 4$ and $a \in H \cap (L^{4/3}(D))^4$, there is a weak solution u with $u(0) = a$ such that*

$$\|u(t)\|_2 = O(t^{-1/2}).$$

The problem of L^2 decay for Navier-Stokes flows is closely connected with the notion of *energy stability* in viscous fluid motions (see [6]). Indeed, Theorems 1.1 and 1.2 assert in particular that the trivial steady state : $u = 0$ is globally asymptotically stable in this sense in arbitrary unbounded domains. In Section 4 we apply the method of proof of Theorems 1.1 and 1.2 to the stability problem for exterior stationary flows in three-dimensions. We prove that an exterior stationary flow is globally asymptotically stable in energy sense provided that the associated Reynolds number is small enough. See Theorem 4.2. This result improves and supplements the known results as given for instance in [6,8,9,10,15]. A novel feature of our result is that we deal with global L^2-norms of disturbances and deduce their explicit decay rates. However, we believe that our result in this section is not the optimal one.

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2. Proof of Theorem 1.1

First we deal with general weak solutions satisfying the energy inequality :

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \|a\|_2^2 + 2\int_0^t (f, u) ds \tag{E}$$

for all $t \ge 0$. The following result can be regarded as a refinement of the decay result of MASUDA [16].

Theorem 2.1. *Let the assumptions in Theorem 1.1 be satisfied. If u is any weak solution satisfying the energy inequality* (E), *then, as $t \to \infty$,*

(i)

$$\int_t^{t+1} \|u\|_2 ds \to 0.$$

(ii) *If $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then,*

$$\frac{1}{t}\int_0^t \|u\|_2 ds = \begin{cases} O(t^{-\alpha}) & \text{if} \quad \alpha < 1/2; \\ O(t^{\varepsilon - 1/2}) & \text{if} \quad \alpha \ge 1/2. \end{cases}$$

where $0 < \varepsilon < 1/2$ is arbitrary in case $n = 3$, and $\varepsilon = 0$ in case $n = 4$.

For the proof we prepare three lemmas.

Lemma 2.2. *Let $L_w^p = L_w^p(R)$, $1 < p < \infty$, denote the Banach space of measurable functions f on the real line R with norm*

$$\|f\|_{p,w} \equiv \sup_E |E|^{-1+1/p} \int_E |f| ds < \infty$$

where $|E|$ is the Lebesgue measure of a measrable set E.

(i) *If $f \in L^p_w$, $g \in L^q_w$ and $1/p + 1/q = 1/r$, then $fg \in L^r_w$, and*

$$\|fg\|_{r,w} \le C\|f\|_{p,w}\|g\|_{q,w}$$

with $C > 0$ depending only on p and q.

(ii) *If $f \in L^p_w$, $g \in L^q_w$ and $1/p + 1/q = 1 + 1/r$, then the convolution $f * g$ is in L^r_w and there is a constant $C > 0$ depending only on p and q so that*

$$\|f * g\|_{r,w} \le C\|f\|_{p,w}\|g\|_{q,w}.$$

(iii) *If $f \in L^p_w$ and $g \in L^1$, then $f * g \in L^p_w$, and*

$$\|f * g\|_{p,w} \le \|f\|_{p,w}\|g\|_1.$$

It is easy to see that $L^p \subset L^p_w$ with continuous injection. A typical example of L^p_w-functions that we need is

$$f(t) = \begin{cases} t^{-1/p} & (t > 0) \\ 0 & (t \le 0) \end{cases}$$

Lemma 2.2 (i) is the weak version of Hölder's inequality, while (ii) and (iii) are the weak versions of Young's inequality. Although these inequalities seem to be well known (see [19, p.32] for (ii)), we give here an elementary proof for the reader's convenience.

Proof. (i) First observe that f is in L^p_w if and only if

$$\|f\|^*_{p,w} \equiv \sup_{t>0} t|E(|f| > t)|^{1/p} < +\infty,$$

where $E(|f| > t) = \{s \in R; |f(s)| > t\}$, and that

$$\|f\|^*_{p,w} \le \|f\|_{p,w} \le \frac{p}{p-1}\|f\|^*_{p,w}$$

as shown for instance in [7, p. 585]. Applying the classical Young's inequality :

$$|fg| \le \frac{r}{p}\varepsilon^{p/r}|f|^{p/r} + \frac{r}{q}\varepsilon^{-q/r}|g|^{q/r}$$

for any $\varepsilon > 0$, we get

$$E(|fg| > t) \subset E(|f| > c_1\varepsilon^{-1}t^{r/p}) \cup E(|g| > c_2\varepsilon t^{r/q})$$

with c_1 and c_2 depending only on p and q. Direct calculation thus gives

$$\left(\|fg\|^*_{r,w}\right)^r \le C_1\varepsilon^p\left(\|f\|^*_{p,w}\right)^p + C_2\varepsilon^{-q}\left(\|g\|^*_{q,w}\right)^q$$

for all $\varepsilon > 0$, where C_1 and C_2 depend only on p and q. The result now follows by taking the minimum with respect to $\varepsilon > 0$.

(ii) Fix an arbitrary $\alpha > 0$ and let

$$K_1(\tau) = \begin{cases} |f(\tau)| & (|f(\tau)| \le \alpha) \\ 0 & (|f(\tau)| > \alpha) \end{cases}$$

and $K_2(\tau) = |f(\tau)| - K_1(\tau)$. Then

$$|f * g|(s) \le K_1 * |g|(s) + K_2 * |g|(s) \equiv I_1(s) + I_2(s). \tag{2.1}$$

By the definition of the Lebesgue integral,

$$I_1(s) = \int_0^\infty dt \int_{E_t} |g(\tau)| d\tau \le \|g\|_{q,w} \int_0^\infty |E_t|^{1-1/q} dt,$$

where $E_t = \{\tau; K_1(s-\tau) > t\}$, so $|E_t| \le \left(\|f\|^*_{p,w}\right)^p t^{-p}$, and $|E_t| = 0$ if $t > \alpha$. Since $p(1-1/q) < 1$ by assumption, we obtain, with $1/q' = 1 - 1/q$,

$$I_1(s) \le \|g\|_{q,w} \left(\|f\|^*_{p,w}\right)^{p/q'} \int_0^\alpha t^{-p/q'} dt = C\alpha^{1-p/q'} \|g\|_{q,w} \left(\|f\|^*_{p,w}\right)^{p/q'},$$

with C depending only on p and q. We thus have

$$\int_E I_1(s) ds \le C\alpha^{1-p/q'} |E| \cdot \|g\|_{q,w} \left(\|f\|^*_{p,w}\right)^{p/q'}. \tag{2.2}$$

On the other hand, denoting by 1_E the indicator function of the set E, we have

$$\begin{aligned} \int_E I_2(s) ds &= \int\int 1_E(s) K_2(s-\tau) |g(\tau)| ds d\tau \\ &= \int |g(\tau)| d\tau \int 1_E(s+\tau) K_2(s) ds = \int K_2(s) ds \int 1_E(s+\tau) |g(\tau)| d\tau \\ &\le \|g\|_{q,w} |E|^{1-1/q} \int K_2(s) ds. \end{aligned}$$

Since $|E(|f| > t)| \le \left(\|f\|^*_{p,w}\right)^p t^{-p}$, the definition of the Lebesgue integral shows

$$\begin{aligned} \int K_2(s) ds &= \alpha |E(|f| > \alpha)| + \int_\alpha^\infty |E(|f| > t)| dt \\ &\le C \left(\|f\|^*_{p,w}\right)^p \alpha^{1-p} \end{aligned}$$

with C depending only on p and q. Hence

$$\int_E I_2(s) ds \le C\alpha^{1-p} |E|^{1-1/q} \|g\|_{q,w} \left(\|f\|^*_{p,w}\right)^p. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) gives

$$\int_E |f * g| ds \le C \|g\|_{q,w} \left[\alpha^{1-p/q'} \left(\|f\|^*_{p,w}\right)^{p/q'} |E| + \alpha^{1-p} \left(\|f\|^*_{p,w}\right)^p |E|^{1-1/q}\right].$$

Taking the minimum with respect to $\alpha > 0$ yields

$$\int_E |f * g| ds \le C\|f\|_{p,w}^* \|g\|_{q,w} |E|^{1-1/r}$$

and this proves (ii).

(iii) Direct calculation gives

$$\begin{aligned}\int_E |f * g| ds &\le \int\int 1_E(s)|f(s-\tau)| \cdot |g(\tau)| ds d\tau \\ &= \int |g(\tau)| d\tau \int 1_E(s+\tau)|f(s)| ds \le \|g\|_1 \|f\|_{p,w} |E|^{1-1/p}.\end{aligned}$$

This proves (iii).

Lemma 2.3. *Let $f \ge 0$ be a measurable function on R. Suppose there exist constants $M > 0$, $C > 0$ and $p > q > 1$ so that $0 \le f \le M$ and*

$$\int_E f ds \le C \left(|E|^{1-1/p} + |E|^{1-1/q}\right)$$

for all measurable subsets E. Then, there is another constant $C' > 0$ such that

$$\int_E f ds \le C' |E|^{1-1/p}$$

for all measurable E.

Proof. Since $1 - 1/p > 1 - 1/q$, the result is obvious for E with $|E| \ge 1$. So we may assume $|E| < 1$. Then, since

$$\int_E f ds \le 2C|E|^{1-1/q},$$

Hölder's inequality yields, with $\theta = 1 - q/p$,

$$\begin{aligned}\int_E f ds &\le M^\theta \int_E f^{1-\theta} ds \\ &\le M^\theta |E|^\theta \left[\int_E f ds\right]^{1-\theta} \le M^\theta (2C)^{1-\theta} |E|^{1-1/p},\end{aligned}$$

which completes the proof.

Lemma 2.4. *Let $n = 3, 4$. Then for all $v \in V$ and $w \in H^1(R^n)$ with $\nabla \cdot w = 0$,*

$$\|e^{-tA} P(w \cdot \nabla) v\|_2 \le C t^{-1/2} \left(\|w\|_2 \|v\|_2\right)^{1-n/4} \left(\|\nabla w\|_2 \|\nabla v\|_2\right)^{n/4}.$$

Proof. Let $\phi \in C_{0,\sigma}^\infty(D)$. Since

$$\|\nabla e^{-tA} \phi\|_2 = \|A^{1/2} e^{-tA} \phi\|_2 \le t^{-1/2} \|\phi\|_2$$

direct calculation gives

$$\begin{aligned} |(e^{-tA}P(w\cdot\nabla)v,\phi)| &= |(v,w\cdot\nabla e^{-tA}\phi)| \le \|v\|_4\|w\|_4\|\nabla e^{-tA}\phi\|_2 \\ &\le t^{-1/2}\|v\|_4\|w\|_4\|\phi\|_2. \end{aligned}$$

The result follows by applying the Sobolev inequality :

$$\|f\|_4 \le C\|f\|_2^{1-n/4}\|\nabla f\|_2^{n/4}.$$

Proof of Theorem 2.1. First observe that the energy inequality (E) gives

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \|a\|_2^2 + \int_0^t \|f\|_2\left(1+\|u\|_2^2\right) ds.$$

Applying Gronwall's lemma yields

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \left(\|a\|_2^2 + \int_0^\infty \|f\|_2 ds\right)\exp\left(\int_0^\infty \|f\|_2 ds\right).$$

Thus, $\|u\|_2 \in L^\infty$ and $\|\nabla u\|_2^2 \in L^1$. Now, substituting $\phi(\tau) = e^{-(t-\tau)A}\psi$, with $\psi \in C_{0,\sigma}^\infty(D)$, into (W) we obtain

$$(u(t),\psi) = (e^{-(t-s)A}u(s),\psi) - \int_s^t (u\cdot\nabla u(\tau), e^{-(t-\tau)A}\psi)d\tau + \int_s^t (f, e^{-(t-\tau)A}\psi)d\tau.$$

We apply Lemma 2.4 to estimate the nonlinear term, to get

$$\|u(t)\|_2 \le \|e^{-(t-s)A}u(s)\|_2 + C\int_s^t (t-\tau)^{-1/2}\left(\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} + \|f\|_{\hat{V}^*}\right)d\tau \tag{2.4}$$

when $n = 3$; and

$$\|u(t)\|_2 \le \|e^{-(t-s)A}u(s)\|_2 + C\int_s^t (t-\tau)^{-1/2}\left(\|\nabla u\|_2^2 + \|f\|_{\hat{V}^*}\right)d\tau \tag{2.5}$$

when $n = 4$, with C independent of $s \ge 0$. From now on we regard $\|u\|_2$ and $\|\nabla u\|_2$ and $\|f\|_{\hat{V}^*}$ as defined to be zero for $\tau \le s$. Suppose first that $n = 3$. Since $\|\nabla u\|_2^{3/2}\|u\|_2^{1/2} \in L^{4/3} \subset L_w^{4/3}$, it follows from Lemma 2.2 that the last term of right-hand side of (2.4) belongs to $L_w^4 + L_w^2$ and the norm is bounded by

$$C\left[\int_s^\infty \|\nabla u\|_2^2 d\tau\right]^{3/4} + C\int_s^\infty \|f\|_{\hat{V}^*}d\tau$$

with C independent of $s > 0$. For an arbitrary $\varepsilon > 0$, we choose $s > 0$ so that the above quantity is less than ε. Then, by the definition of L_w^p-norm we get for all $t > s$,

$$\int_t^{t+1} \|u\|_2 d\tau \le \int_t^{t+1} \|e^{-(\tau-s)A}u(s)\|_2 d\tau + \varepsilon.$$

Since $\|e^{-tA}a\|_2 \to 0$ because of the injectivity of A, application of the bounded convergence theorem yields

$$\limsup_{t\to\infty} \int_t^{t+1} \|u\|_2 d\tau \le \varepsilon$$

and this shows (i) in case $n = 3$. The case $n = 4$ is similarly treated by using (2.5) instead of (2.4). To prove (ii), we consider (2.4) and (2.5) with $s = 0$ and integrate both sides in t, to get

$$\frac{1}{t}\int_0^t \|u\|_2 ds \le \frac{1}{t}\int_0^t \|e^{-sA}a\|_2 ds + C\left(t^{-\beta} + t^{-1/2}\right) \tag{2.6}$$

where $\beta = 1/4$ when $n = 3$, and $\beta = 1/2$ when $n = 4$. This proves (ii) with $n = 4$. To prove (ii) with $n = 3$, we systematically apply Lemmas 2.2 and 2.3. First observe that (2.6) shows the result for $\alpha < 1/4$. When $\alpha \ge 1/4$, (2.4) shows that $\|u\|_2$ is bounded from above by a function belonging to $L_w^{1/\alpha} + L_w^4 + L_w^2$. Since $\|u\|_2 \in L^\infty$, Lemma 2.3 implies that $\|u\|_2 \in L_w^4$, and so $\|u\|_2^{1/2} \in L_w^8$. Thus, by Lemma 2.2 (i), $\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} \in L_w^p$ with $1/p = 1/8 + 3/4$. Since $1/2 + 1/p = 1 + 1/4 + 1/8$, Lemma 2.2 (ii) implies

$$\frac{1}{t}\int_0^t \|u\|_2 ds \le C\left(t^{-\alpha} + t^{-1/q} + t^{-1/2}\right)$$

with $1/q = 1/4 + 1/8$, and this shows the result for $\alpha < 1/q$. When $\alpha \ge 1/q$, (2.4) shows that $\|u\|_2$ is bounded from above by a function in $L_w^{1/\alpha} + L_w^q + L_w^2$; so $\|u\|_2 \in L_w^q$ by Lemma 2.3. Thus, the same argument as above gives

$$\int_0^t (t-s)^{-1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} ds \in L_w^r$$

with $1/r = 1/4 + 1/8 + 1/16$. Hence

$$\frac{1}{t}\int_0^t \|u\|_2 ds \le C\left(t^{-\alpha} + t^{-1/r} + t^{-1/2}\right)$$

and this proves the result for $\alpha < 1/r$. Repeating these processes eventually proves the desired result all the way. The proof is complete.

Proof of Theorem 1.1. We first construct approximate solutions of (NS), for $n = 3, 4$, by solving

$$u_k(t) = e^{-tA}a_k - \int_0^t e^{-(t-s)A}\left(P(\overline{u}_k\cdot\nabla)u_k - f_k\right)(s)ds, \quad k = 1, 2, \cdots, \tag{IE}$$

where $a_k = (I + k^{-1}A)^{-1}a$, $f_k = (I + k^{-1}A)^{-1}f$, and $\overline{u}_k = J_{1/k}\tilde{u}_k$ is the spatial mollification of the zero-extension $\tilde{u}_k$ of u_k. The unique solvability of (IE) in the space $C([0,T]; V)$ as well as the fact that u_k solves, in $L^2(0,T; D(A)) \cap W^{1,2}(0,T; H)$, the problem

$$\frac{du_k}{dt} + Au_k + P(\overline{u}_k\cdot\nabla)u_k = f_k, \quad \text{a.e. } t > 0\ ;\ u_k(0) = a_k \tag{2.7}$$

can be shown as in [2,18]. From (2.7) we get, for $t \geq s \geq 0$,

$$\|u_k(t)\|_2^2 + 2\int_s^t \|\nabla u_k\|_2^2 d\tau = \|u_k(s)\|_2^2 + 2\int_s^t (f_k, u_k) d\tau. \tag{2.8}$$

Upon taking $s = 0$ and using $\|a_k\| \leq \|a\|_2$, (2.8) implies

$$u_k \text{ is bounded in } L^\infty(0,T;H) \cap L^2(0,T;V).$$

Hence we may assume that a subsequence of u_k converges weakly-star in $L^\infty(0,T;H)$ and weakly in $L^2(0,T;V)$. Moreover, a standard argument ([22, Chap.III]) applies to show that if we define $v_k(t) = u_k(t)$ for $t \in (0,T)$ and $v_k(t) = 0$ otherwise, then the fractional derivatives $D_t^\gamma v_k$, defined via the Fourier transform of $v_k(t)$ in t, remain bounded in $L^2(R;H)$ provided $0 < \gamma < 1/4$. We thus conclude that a subsequence, denoted again u_k, converges in $L^2_{loc}([0,T] \times D)$ to a function u, and it is readily seen (cf. [16]) that the limit function u is a weak solution of (NS). In view of the convergence mode of u_k as mentioned above, we need only show that $u_k(t)$ decays in L^2 as indicated in Theorem 1.1 *uniformly in* k.

Now the energy equality (2.8) implies that

$$\|u_k(t)\|_2^2 + 2\int_s^t \|\nabla u_k\|_2^2 d\tau \leq \|u_k(s)\|_2^2 + \int_s^t \|f\|_2 \left(1 + \|u_k\|_2^2\right) d\tau$$

so that, by Gronwall's lemma,

$$\|u_k(t)\|_2^2 + 2\int_s^t \|\nabla u_k\|_2^2 d\tau \leq C\left(\|u_k(s)\|_2^2 + \int_s^t \|f\|_2 d\tau\right)$$

where $C = \exp\left(\int_0^\infty \|f\|_2 ds\right)$. Hence $\int_0^\infty \|\nabla u_k\|_2^2 ds$ is bounded uniformly in k and

$$\|u_k(t)\|_2 \leq C\left[\|u_k(s)\|_2 + \left(\int_s^t \|f\|_2 d\tau\right)^{1/2}\right].$$

Integrating this in s gives

$$\|u_k(t+1)\|_2 \leq C\left[\int_t^{t+1} \|u_k\|_2 ds + \left(\int_t^{t+1} \|f\|_2 ds\right)^{1/2}\right] \tag{2.9}$$

and

$$\begin{aligned} \|u_k(t)\|_2 &\leq Ct^{-1}\int_0^t \|u_k\|_2 ds + Ct^{-1/2}\left[\int_0^t \tau\|f(\tau)\|_2 d\tau\right]^{1/2} \\ &\leq Ct^{-1}\int_0^t \|u_k\|_2 ds + Ct^{-1/2}. \end{aligned} \tag{2.10}$$

On the other hand, applying Lemma 2.3 to (IE) gives

$$\|u_k(\tau)\|_2 \leq \|e^{-(\tau-s)A}u_k(s)\|_2 + C\int_s^\tau (\tau-\sigma)^{-1/2}\left(\|\nabla u_k\|_2^2 + \|f\|_{\hat{V}^*}\right) d\sigma \tag{2.11}$$

when $n = 4$; and

$$\|u_k(\tau)\|_2 \le \|e^{-(\tau-s)A}u_k(s)\|_2 + C\int_s^\tau (\tau-\sigma)^{-1/2}\left(\|u_k\|_2^{1/2}\|\nabla u_k\|_2^{3/2} + \|f\|_{\hat{V}^*}\right)d\sigma \quad (2.12)$$

when $n = 3$. Note that we have used $\|\overline{u}_k\|_2 \le \|u_k\|_2$ and $\|\nabla\overline{u}_k\|_2 \le \|\nabla u_k\|_2$. Since $\|e^{-tA}a_k\|_2 \le \|e^{-tA}a\|_2$, and since $\|u_k\|_2 \in L^\infty$ and $\|\nabla u_k\|_2^2 \in L^1$ uniformly in k, we see from (2.11) and (2.12) that the assertions of Theorem 2.1 hold for u_k uniformly in k. Hence (2.10) implies that u_k have the decay properties listed in Theorem 1.1 uniformly in k. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section we first prove Theorem 1.2, using the spectral decomposition of the positive self-adjoint operator A. To prove assertion (iv) of Theorem 1.2, we establish the invariance of some of the ranges $R(A^\alpha)$ of the fractional powers A^α. As a byproduct we obtain Corollary 1.3. The argument below originates from those of [12,21,23] and, as in Section 2, relies on the following

Lemma 3.1. (i) *Let* $A = \int_0^\infty \lambda dE_\lambda$ *be the spectral decomposition of* A. *If* $n = 2$, *we have the following estimate : for* $v \in V$,

$$\|E_\lambda P(v\cdot\nabla)v\|_2 \le C\lambda^{1/2}\|v\|_2\|\nabla v\|_2, \quad \lambda > 0.$$

(ii) *Under the same assumption as in* (i), *we have*

$$\|e^{-tA}P(v\cdot\nabla)v\|_2 \le Ct^{-1/2}\|v\|_2\|\nabla v\|_2, \quad t > 0.$$

Proof. By the definition of P and E_λ we easily see that, for $\phi \in C_{0,\sigma}^\infty(D)$,

$$\begin{aligned} |(E_\lambda P(v\cdot\nabla)v,\phi)| &= |(v, v\cdot\nabla E_\lambda\phi)| \le \|v\|_4^2\|\nabla E_\lambda\phi\|_2 \\ &= \|v\|_4^2\|A^{1/2}E_\lambda\phi\|_2 \le \lambda^{1/2}\|v\|_4^2\|\phi\|_2 \end{aligned}$$

Applying the Sobolev inequality :

$$\|f\|_4 \le C\|f\|_2^{1/2}\|\nabla f\|_2^{1/2},$$

to the last term yields the desired estimate. This proves (i). Assertion (ii) is proved in the same way as in Lemma 2.4. The proof is complete.

Proof of Theorem 1.2. The standard theory of the two-dimensional Navier-Stokes equations as given in [16,22] asserts the existence of a unique weak solution u satisfying the energy equality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds = \|a\|_2^2 + 2\int_0^t (f,u)ds$$

for all $t \geq 0$. Hence we have

$$\frac{d}{dt}\|u\|_2^2 + 2\|\nabla u\|_2^2 = 2(f,u) \leq 2\|f\|_2\|u\|_2. \tag{3.1}$$

Using the estimate $2\|f\|_2\|u\|_2 \leq \|f\|_2(1+\|u\|_2^2)$ and Gronwall's lemma, we easily see that

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \leq \left(\|a\|_2^2 + \int_0^\infty \|f\|_2 ds\right)\exp\left(\int_0^\infty \|f\|_2 ds\right) \tag{3.2}$$

and therefore $\|u\|_2 \in L^\infty$ and $\|\nabla u\|_2^2 \in L^1$. Now, since $\|\nabla u\|_2 = \|A^{1/2}u\|_2$, using the estimates

$$2\|A^{1/2}u\|_2^2 = 2\int_0^\infty \lambda d\|E_\lambda u\|_2^2 \geq 2\rho\int_\rho^\infty d\|E_\lambda u\|_2^2 = 2\rho(\|u\|_2^2 - \|E_\rho u\|_2^2)$$

and $\|E_\rho u\|_2 \leq \|u\|_2$, we see from (3.1) that

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \leq \rho\|E_\rho u\|_2 + \|f\|_2 \tag{3.3}$$

for all $\rho > 0$. To deal with the right-hand side we substitute $\phi(\tau) = e^{-(t-\tau)A}E_\rho\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, into (W) and get

$$(E_\rho u(t),\psi) = (E_\rho e^{-tA}a,\psi) + \int_0^t (E_\rho e^{-(t-s)A}f,\psi)ds - \int_0^t (E_\rho e^{-(t-s)A}P(u\cdot\nabla u),\psi)ds$$

so that, by duality and Lemma 3.1 (i),

$$\|E_\rho u(t)\|_2 \leq \|e^{-tA}a\|_2 + C\rho^{1/2}\int_0^t (\|u\|_2\|\nabla u\|_2 + \|f\|_{\hat{V}^*})\,ds.$$

Combining this with (3.3) and applying Hölder's inequality gives

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \leq C\rho\left[\|e^{-tA}a\|_2 + \rho^{1/2}\left(\int_0^t \|u\|_2^2 ds\right)^{1/2} + \rho^{1/2}\right] + \|f\|_2 \tag{3.4}$$

since $\|\nabla u\|_2^2 \in L^1$ by (3.2). We take in (3.4) $\rho = 2/(t+e)\log(t+e)$ and then multiply both sides by $(\log(t+e))^2$ to obtain

$$\frac{d}{dt}\left((\log(t+e))^2\|u\|_2\right) \leq 2C(t+e)^{-1}\log(t+e)\left[\|e^{-tA}a\|_2 + C\,(\log(t+e))^{-1/2}\right]$$
$$+C\,(\log(t+e))^2\,(t+e)^{-1}\|f\|_2(t+e).$$

Since $\|e^{-tA}a\|_2 \leq \|a\|_2$, since $\|e^{-tA}a\|_2 \to 0$ as $t\to\infty$ because A is injective, and since $\int_0^\infty \|f\|_2(t+e)dt$ is finite by assumption, we obtain

$$\|u(t)\|_2 \leq (\log(t+e))^{-2}\left[\|a\|_2 + C\int_0^t (s+e)^{-1}\log(s+e)\|e^{-sA}a\|_2 ds\right]$$
$$+C(\log(t+e))^{-1/2} + C(\log(t+e))^{-2} \to 0 \quad \text{as} \quad t\to\infty.$$

The proof of (i) is complete. We next prove (ii). Since $\|e^{-tA}a\|_2 \le C(t+e)^{-\alpha}$ by assumption, the proof of (i) shows

$$\|u(t)\|_2 \le C\left[(\log(t+e))^{-2} + (\log(t+e))^{-1/2}\right].$$

This shows (ii) for $m = 1$. From this we obtain (see [21,23])

$$\int_0^t \|u\|_2^2 ds \le C\int_0^t (\log(s+e))^{-1} ds \le C(t+e)(\log(t+e))^{-1}$$

so that, as in the proof of (i),

$$\frac{d}{dt}\left((\log(t+e))^2\|u\|_2\right) \le C(t+e)^{-1}\log(t+e)\left[(t+e)^{-\alpha} + (\log(t+e))^{-1}\right]$$
$$+ (\log(t+e))^2 (t+e)^{-1}\|f\|_2(t+e).$$

Integrating this gives

$$\|u(t)\|_2 \le C(\log(t+e))^{-1}.$$

This proves (ii) for $m = 2$. Now suppose (ii) is true for some $m \ge 2$. Taking in (3.4) $\rho = (m+1)/(t+e)\log(t+e)$ and then multiplying by $(\log(t+e))^{m+1}$ we obtain, since $\int_0^t (\log(s+e))^{-m} ds \le C_m(t+e)(\log(t+e))^{-m}$ ([21,23]),

$$\frac{d}{dt}\left((\log(t+e))^{m+1}\|u\|_2\right) \le C(t+e)^{-1}(\log(t+e))^{m}\left[(t+e)^{-\alpha} + (\log(t+e))^{-(m+1)/2}\right]$$
$$+C(\log(t+e))^{m+1}(t+e)^{-1}\|f\|_2(t+e).$$

Hence

$$\|u(t)\|_2 \le C(\log(t+e))^{-(m+1)/2}$$

and this completes the proof of (ii).

We next prove (iii), following [12, pp.142-143]. Inserting $\phi(\tau) = e^{-(t-\tau)A}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, into (W) and applying Lemma 3.1 (ii) gives

$$\|u(t)\|_2 \le \|e^{-tA}a\|_2 + C\int_0^t (t-s)^{-1/2}\left(\|u\|_2\|\nabla u\|_2 + \|f\|_{\hat{V}^*}\right) ds. \tag{3.5}$$

Assume first that $0 < \alpha < 1/2$ and choose $q > 2$ so that $q\alpha < 1$. Since $1 + 1/q = 1/2 + (q+2)/2q$, the Hardy-Littlewood-Sobolev inequality [19] applied to (3.5) gives

$$\left[\int_0^t \|u\|_2^q ds\right]^{1/q} \le C_1(t+1)^{1/q-\alpha} + C\left[\int_0^t \left(\|u\|_2\|\nabla u\|_2\right)^{2q/(q+2)} ds\right]^{(q+2)/2q}$$
$$+C\left[\int_0^t \|f\|_{\hat{V}^*}^{2q/(q+2)} ds\right]^{(q+2)/2q}.$$

Notice that the last term is bounded in t by the assumption $f \in L^p(0,\infty;\hat{V}^*)$ for $p=1,2$. Since $(q+2)/2q = 1/2+1/q$, Hölder's inequality and (3.2) together yield

$$\left[\int_0^t \|u\|_2^q ds\right]^{1/q} \le C_1(t+1)^{1/q-\alpha} + C_2\left[\int_0^t \|u\|_2^q ds\right]^{1/q} + C_3$$

where C_1 and C_3 are constants and $C_2 = C_2(a,f)$ is the square root of the right-hand side of (3.2). Here we assume $\|a\|_2$ and $\int_0^\infty \|f\|_2 dt$ are so small as $C_2 \le 1/2$, to obtain from the above

$$\left[\int_0^t \|u\|_2^q ds\right]^{1/q} \le C(t+1)^{1/q-\alpha} + C \le C(t+1)^{1/q-\alpha}$$

because $1/q-\alpha > 0$. Inserting this into (3.4) gives

$$\begin{aligned}\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 &\le C\rho\left[(t+1)^{-\alpha} + \rho^{1/2}(t+1)^{1/2-1/q}\left(\int_0^t \|u\|_2^q ds\right)^{1/q} + \rho^{1/2}\right] + \|f\|_2 \\ &\le C\rho\left((t+1)^{-\alpha} + \rho^{1/2}(t+1)^{1/2-\alpha}\right) + \|f\|_2.\end{aligned}$$

Taking $\rho = m/(t+1)$, multiplying by $(t+1)^m$, and then proceeding as in the proof of (ii) we obtain $\|u(t)\|_2 = O((t+1)^{-\alpha})$, and this proves (iii) in case $\alpha < 1/2$. If $\alpha \ge 1/2$, then $q\alpha > 1$ for all $q > 2$, and so $\int_0^\infty \|e^{-tA}a\|_2^q dt < +\infty$. The foregoing argument thus gives

$$\left[\int_0^t \|u\|_2^q ds\right]^{1/q} \le C_1 + C_2\left[\int_0^t \|u\|_2^q ds\right]^{1/q} + C_3$$

so that $[\int_0^t \|u\|_2^q ds]^{1/q} \le C$ if $C_2 \le 1/2$. Inserting this into (3.4) and repeating the same argument as in the case $\alpha < 1/2$, we obtain

$$\|u(t)\|_2 \le C\left((t+1)^{-\alpha} + (t+1)^{-1/q}\right).$$

Since $q > 2$ was arbitrary, the proof of (iii) is now complete.

To prove (iv), we need only show that the (unique) weak solution $u(t)$ belongs to $R(A^\alpha)$ for all $t > 0$ provided $a \in R(A^\alpha)$; because it then follows that

$$\|e^{-tA}u(s)\|_2 = O(t^{-\alpha})$$

for any fixed $s \ge 0$ and so the proof of (iii) applies if we choose $s \ge 0$ as the initial time so that $\|u(s)\|_2$ and $\int_s^\infty \|f\|_2 d\tau$ are small enough. To this end we use

Proposition 3.2. *Let $n=2$ and $0<\alpha\le 1/2$. If $v \in V$, then for all $\lambda > 0$,*

$$\|(\lambda+A)^{-\alpha}P(v\cdot\nabla)v\|_2 \le C\|v\|_2^{2\alpha}\|\nabla v\|_2^{2-2\alpha}$$

with C depending only on α. Moreover, we have

$$\|(\lambda+A)^{-\alpha}f\|_2 \le \|f\|_{\hat{V}^*}^{2\alpha}\|f\|_2^{1-2\alpha}.$$

We continue the proof of Theorem 1.2 (iv), admitting Proposition 3.2 for a moment. Let

$$u_\lambda(t) = (\lambda + A)^{-\alpha} u(t) \quad ; \quad a_\lambda = (\lambda + A)^{-\alpha} a.$$

Since $a \in R(A^\alpha)$ by assumption, $\|a_\lambda\|_2$ is uniformly bounded in $\lambda > 0$. On the other hand, inserting $\phi(\tau) = (\lambda + A)^{-\alpha} e^{-(t-\tau)A}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, into (W) and applying Proposition 3.2 (i) we obtain

$$\begin{aligned} \|u_\lambda(t)\|_2 &\le \|a_\lambda\|_2 + \int_0^t \|(\lambda + A)^{-\alpha} P(u\cdot\nabla)u(s)\|_2 ds + \int_0^t \|(\lambda + A)^{-\alpha} f\|_2 ds \\ &\le C_1 + C_2 \int_0^t \|u\|_2^{2\alpha} \|\nabla u\|_2^{2-2\alpha} ds + \int_0^t \|f\|_{V^*}^{2\alpha} \|f\|_2^{1-2\alpha} ds \\ &\le C_1 + C\left[\int_0^t \|u\|_2^2 ds\right]^\alpha + C_3. \end{aligned}$$

This shows that $u_\lambda(t)$ remains bounded in H for any fixed $t > 0$. Hence we may assume that $u_\lambda(t) \to w(t)$ as $\lambda \to 0$ weakly in H, and therefore, for any $\phi \in D(A^\alpha)$,

$$(u(t), \phi) = (u_\lambda(t), (\lambda + A)^\alpha \phi) \to (w(t), A^\alpha \phi).$$

Hence $u(t) \in R(A^\alpha)$ for all $t \ge 0$ and this proves (iv). The proof is complete.

Proof of Proposition 3.2. First we show the estimate

$$\|\phi\|_p \le C\|A^\alpha \phi\|_2 \tag{3.6}$$

for $1/p = 1/2 - \alpha$, and $0 \le \alpha < 1/2$. This is deduced from the following : (i) The family $\{D^\alpha; 0 \le \alpha \le 1/2\}$ of the completion D^α of $D(A^\alpha)$ in the norm $\|A^\alpha \cdot\|_2$ forms a complex interpolation family (see, e.g. [17]) ; (ii) $D^{1/2} \subset BMO$ with continuous injection (see, e.g. [7, Prop. 3.4]) ; and (iii) $[L^2, BMO]_\theta = L^{2/(1-\theta)}$, $0 \le \theta < 1$, where the bracket denotes the complex interpolation (see [11]). Now, estimate (3.6) implies that

$$\begin{aligned} |((\lambda + A)^{-\alpha} P(v\cdot\nabla)v, \phi)| &= |(v\cdot\nabla v, (\lambda + A)^{-\alpha}\phi)| \\ &\le \|\nabla v\|_2 \|v\|_{1/\alpha} \|(\lambda + A)^{-\alpha}\phi\|_p \\ &\le C\|\nabla v\|_2^{2-2\alpha} \|v\|_2^{2\alpha} \|A^\alpha(\lambda + A)^{-\alpha}\phi\|_2 \\ &\le C\|\nabla v\|_2^{2-2\alpha} \|v\|_2^{2\alpha} \|\phi\|_2 \end{aligned}$$

for $\alpha < 1/2$, and

$$\begin{aligned} |((\lambda + A)^{-1/2} P(v\cdot\nabla)v, \phi)| &= |(v, v\cdot\nabla(\lambda + A)^{-1/2}\phi)| \le \|v\|_4^2 \|\nabla(\lambda + A)^{-1/2}\phi\|_2 \\ &= C\|v\|_4^2 \|A^{1/2}(\lambda + A)^{-1/2}\phi\|_2 \le C\|v\|_2 \|\nabla v\|_2 \|\phi\|_2, \end{aligned}$$

for $\alpha = 1/2$. This proves the first assertion. The second assertion is obtained as

$$\|(\lambda + A)^{-\alpha} f\|_2 \le \|f\|_2^{1-2\alpha} \|(\lambda + A)^{-1/2} f\|_2^{2\alpha} \le \|f\|_2^{1-2\alpha} \|f\|_{V^*}^{2\alpha}.$$

by using the estimate

$$\|(\lambda + A)^{-1/2} f\|_2 \le \|f\|_{\hat{V}^*}.$$

This completes the proof of Proposition 3.2.

Using the estimates :

$$\|\phi\|_p \le C\|A^\alpha \phi\|_2$$

where $1/p = 1/2 - 2\alpha/3$, $1/4 \le \alpha \le 1/2$, if $n = 3$; and $p = 4$, $\alpha = 1/2$, if $n = 4$, we can also prove

Proposition 3.3. *Let $v \in V$ and $w \in H^1(R^n)$ with $\nabla \cdot w = 0$.*

(i) *If $n = 3$ and $1/4 \le \alpha \le 1/2$, then for all $\lambda > 0$,*

$$\|(\lambda + A)^{-\alpha} P(w \cdot \nabla)v\|_2 \le C\|w\|_2^{2\alpha - 1/2} \|\nabla w\|_2^{3/2 - 2\alpha} \|\nabla v\|_2$$

with C depending only on α.

(ii) *If $n = 4$, then for all $\lambda > 0$,*

$$\|(\lambda + A)^{-1/2} P(w \cdot \nabla)v\|_2 \le C\|\nabla w\|_2 \|\nabla v\|_2$$

with C independent of v and w.

(iii) *$R(A^\alpha)$ is invariant under the Navier-Stokes flow provided either $n = 3$, $1/4 \le \alpha \le 1/2$; or $n = 4$, $\alpha = 1/2$.*

Finally, Corollary 1.3 is immediately obtained from Theorems 1.1, 1.2 and the following

Corollary 3.4. *There holds the inclusion :*

$$H \cap (L^r(D))^n \subset R(A^\alpha), \quad \alpha = n(1/r - 1/2)/2$$

where $1 < r < 2$ if $n = 2$; $6/5 \le r \le 3/2$ if $n = 3$; and $r = 4/3$ if $n = 4$. More precisely, to each $a \in H \cap (L^r(D))^n$ there corresponds a unique $b \in D(A^\alpha)$ such that $a = A^\alpha b$ and $\|b\|_2 \le C\|a\|_r$ with $C > 0$ independent of a.

Proof. We consider only the case $n = 2$; the other cases are treated similarly. The function $a_\lambda = (\lambda + A)^{-\alpha} a$ satisfies

$$\begin{aligned} |(a_\lambda, \phi)| &= |(a, (\lambda + A)^{-\alpha}\phi)| \le \|a\|_r \|(\lambda + A)^{-\alpha}\phi\|_{r'} \\ &\le C\|a\|_r \|A^\alpha (\lambda + A)^{-\alpha}\phi\|_2 \le C\|a\|_r \|\phi\|_2 \end{aligned}$$

for all $\phi \in H$, where $1/r' = 1 - 1/r$. This shows that a_λ, $\lambda > 0$, is bounded in H ; so a subsequence converges weakly in H, as $\lambda \to 0$, to a function $b \in H$ with $\|b\|_2 \le C\|a\|_r$. But then,

$$(a, \phi) = (a_\lambda, (\lambda + A)^\alpha \phi) \to (b, A^\alpha \phi)$$

for all $\phi \in D(A^\alpha)$, so $a = A^\alpha b$ and b is thus determined uniquely. The proof is complete.

Remark. When $n \geq 3$ and D is an exterior domain of R^n with smooth boundary, the L^r-theory of the operator A as developed in [3] gives the estimate

$$\|\phi\|_p \leq C\|A^\alpha \phi\|_r$$

for $1 < r < n$, $1 < p < \infty$ and $1/p = 1/r - 2\alpha/n$. This implies that $R(A^\alpha)$ is invariant under the Navier-Stokes flow (with $f = 0$) provided $\alpha < n/4$, and that the inclusion relation of Corollary 3.4 holds for all $1 < r < 2$. More generally, the L^r-theory implies that

$$L^r_\sigma \cap L^q_\sigma \subset R(A^\alpha_q)$$

whenever $n' < q < \infty$, $1 < r < \infty$, and $1/r = 1/q + 2\alpha/n$. Here $1/n' = 1 - 1/n$; L^r_σ denotes the L^r-closure of $C^\infty_{0,\sigma}(D)$; and A_q is the operator A regarded as a closed linear operator in L^q_σ. We can further show that the space $L^r \cap H$ is invariant under the Navier-Stokes flow provided $1 < r \leq n'$. See [3, Sect.4–5] for the details.

4. Energy Stability of Exterior Stationary Flows

This section deals with the problem

$$\begin{aligned} \frac{\partial v}{\partial t} + v \cdot \nabla v &= \Delta v - \nabla p + f_0 \quad (x \in D,\ t > 0), \\ \nabla \cdot v &= 0 \qquad\qquad\qquad\quad (x \in D,\ t > 0), \\ v|_S = v^* \ ;\ v &\to v^\infty \quad (|x| \to \infty), \\ v|_{t=0} &= v_0, \end{aligned} \tag{4.1}$$

in an exterior domain D in R^3 with smooth boundary S. Here $v^* = v^*(x)$ is a given smooth vector field on S ; v^∞ is a given constant vector ; and $f_0 = f_0(x)$ is a given external force.

Under some assumptions on v^*, v^∞ and f_0, FINN [4,5] and BABENKO [1] proved the existence of a stationary solution w to problem (4.1) satisfying

$$w - v^\infty \in L^3(D) \ ;\ \nabla w \in L^3(D) \ ;\ C_0 \equiv \sup_D |x| \cdot |w(x) - v^\infty| < +\infty. \tag{4.2}$$

In this section we study the stability of these stationary solutions with respect to L^2 disturbances. Given a stationary solution w and disturbances f and $a = v_0 - w$, the time-evolution of $u = v - w$ is governed by

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u - w \cdot \nabla u - u \cdot \nabla w - \nabla q + f \quad (x \in D,\ t > 0), \\ \nabla \cdot u &= 0 \qquad\qquad\qquad\qquad\qquad\qquad (x \in D,\ t > 0), \\ u|_S = 0 \ ;\ u &\to 0 \quad (|x| \to \infty), \\ u|_{t=0} &= a. \end{aligned} \tag{P}$$

The problem (P) is formally transformed into the integral equation :

$$u(t) = e^{-tL}a - \int_0^t e^{-(t-s)L}\left(P(u\cdot\nabla)u - f\right)(s)ds \tag{4.3}$$

where

$$L = A + B\ ;\ Bu = P(w\cdot\nabla)u + P(u\cdot\nabla)w,$$

and $\{e^{-tL}\ ;\ t \geq 0\}$ is the analytic semigroup in H generated by $-L$. (See [18].)

Given $a \in H$, a weakly continuous function $u : [0,\infty) \to H$ is called a weak solution of (P) if $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, $u(0) = a$, and satisfies

$$(u(t),\phi(t)) + \int_s^t \left[(\nabla u, \nabla\phi) + (w\cdot\nabla u + u\cdot\nabla w + u\cdot\nabla u, \phi)\right] d\tau \tag{4.4}$$

$$= (u(s),\phi(s)) + \int_s^t (u,\phi')d\tau + \int_s^t (f,\phi)d\tau$$

for all $t \geq s \geq 0$ and all $\phi \in C^1([0,\infty);H) \cap C^0([0,\infty);V)$.

For the existence of weak solutions, we already have the following result:

Theorem 4.1([18]). *Given $a \in H$ and $f \in L^2_{loc}([0,\infty);H)$, problem* (P) *possesses a weak solution u which, moreover, satisfies the energy inequality of the following form :*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 d\tau + 2\int_s^t (u\cdot\nabla w, u)d\tau \leq \|u(s)\|_2^2 + 2\int_s^t (f,u)d\tau \tag{SE}$$

for $s = 0$, a.e. $s > 0$, and all $t \geq s$.

In this section we shall prove

Theorem 4.2. *Let $C_0 < 1/2$, $a \in H$, and suppose f satisfies the assumption of Theorem 1.1. Then any weak solution u of problem* (P) *satisfying the energy inequality* (SE) *has the following decay properties :*

(i) $\|u(t)\|_2 \to 0$ *as* $t \to \infty$.

(ii) *If $\|e^{-tL}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then for an arbitrary $\varepsilon > 0$,*

$$\|u(t)\|_2 = O\left((\log t)^{\varepsilon - 1/2}\right).$$

(iii) *If $v^\infty = 0$ and $\|e^{-tL}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & if\ \ \alpha < 1/2 \\ O(t^{\varepsilon-1/2}) & if\ \ \alpha \geq 1/2 \end{cases}$$

where $\varepsilon > 0$ is arbitrary.

Theorems 4.1 and 4.2 mean that an exterior stationary flow w is globally asymptotically stable in energy sense provided $C_0 < 1/2$. The enrgy inequality (SE) was first deduced by

Leray [14] in the case $D = R^3$, $w = 0$ and $f = 0$. Part (i) of Theorem 3.2 is already proved in [18]. Heywood [8,9] and Galdi & Rionero [6, p.41] discuss the decay of local L^2-norms of strong solutions in case $C_0 < 1/2$. Contrary to these works, our Theorem 4.2 deals with global L^2-norms of weak solutions satisfying (SE). The decay properties of strong solutions in other function spaces are discussed in detail by Heywood [10] and Masuda [15]. For various problems related to the stability of fluid motions, we refer to [6] and references therein.

We begin the proof of Theorem 4.2 by establishing the following

Lemma 4.3. *Let $C_0 < 1/2$, and let L^* denote the adjoint operator of L. Then :*

(i) $\{e^{-tL}\ ;\ t \geq 0\}$ *and* $\{e^{-tL^*}\ ;\ t \geq 0\}$ *are contraction semigroups in H.*

(ii) $\|e^{-tL}a\|_2 \to 0$ *as* $t \to \infty$.

(iii) *The estimate*

$$\|E_\rho e^{-tL}P(u\cdot\nabla)u\|_2 \leq C\left(\rho^{1/2}+\rho^{1/4}\right)\|u\|_2^{1/2}\|\nabla u\|_2^{3/2}$$

holds for all $u \in V$ and $\rho > 0$, where E_ρ is the spectral measure associated to A.

(iv) *If $v^\infty = 0$, then for all $u \in V$ and $t > 0$,*

$$\|e^{-tL}P(u\cdot\nabla)u\|_2 \leq Ct^{-1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{3/2}.$$

Proof. As shown in [15], we have the estimate

$$|(\phi\cdot\nabla w, \phi)| \leq 2C_0\|\nabla\phi\|_2^2. \tag{4.5}$$

Since $(w\cdot\nabla\phi, \phi) = 0$, (4.5) implies

$$(L\phi, \phi) = (\phi, L^*\phi) \geq (1-2C_0)\|\nabla\phi\|_2^2 \tag{4.6}$$

for all $\phi \in D(L) = D(L^*) = D(A)$. Hence, we get (i) if $C_0 \leq 1/2$. To show (ii), suppose first that $a \in R(L)$ and hence $a = Lb$ for some $b \in D(L)$. Then $e^{-tL}a = -v'(t)$ with $v(t) = e^{-tL}b$. Since

$$v''(t) + Lv'(t) = 0,$$

we get

$$\|v'(t)\|_2^2 + 2(1-2C_0)\int_s^t \|\nabla v'\|_2^2 d\tau \leq \|v'(s)\|_2^2 \tag{4.7}$$

for all $t \geq s > 0$. This implies that $\|\nabla v'\|_2$ is in L^2 on the interval $[1,\infty)$. Next, from the Sobolev inequality :

$$\|f\|_6 \leq \frac{2}{\sqrt{3}}\|\nabla f\|_2 \tag{4.8}$$

we see that $\|Bv\|_2 \leq (\|w\|_\infty + C\|\nabla w\|_3)\|\nabla v\|_2$. Direct calculation thus gives

$$\begin{aligned}(v', v') &= -(v', Av) - (v', Bv) \leq \|\nabla v'\|_2\|\nabla v\|_2 + C\|v'\|_2\|\nabla v\|_2 \\ &\leq \|\nabla v'\|_2^2 + \|\nabla v\|_2^2 + \frac{1}{2}\|v'\|_2^2 + C\|\nabla v\|_2^2\end{aligned}$$

so that

$$\|v'\|_2^2 \leq C\left(\|\nabla v\|_2^2 + \|\nabla v'\|_2^2\right)$$

and hence $\|v'\|_2 \in L^2$ on $[1, \infty)$. From (4.7) it follows that

$$(t-1)\|v'(t)\|_2^2 \leq \int_1^t \|v'\|_2^2 ds \leq \int_1^\infty \|v'\|_2^2 ds < +\infty$$

and we conclude that

$$\|e^{-tL}a\|_2 = \|v'(t)\|_2 \to 0$$

as $t \to \infty$. This shows (ii) in case $a \in R(L)$. To complete the proof, it suffices to show that $R(L)$ is dense in H. But this follows from $N(L^*) = 0$ which is a consequence of the assumption $C_0 < 1/2$. The proof of (ii) is complete.

To show (iii), observe that (4.6) gives

$$\|\nabla e^{-tL^*}E_\rho\phi\|_2 \leq C\|L^*e^{-tL^*}E_\rho\phi\|_2^{1/2}\|E_\rho\phi\|_2^{1/2} \leq C\|L^*E_\rho\phi\|_2^{1/2}\|\phi\|_2^{1/2}.$$

By the definition of L we easily see that

$$\|L^*E_\rho\phi\|_2 \leq \|AE_\rho\phi\|_2 + C\left(\|w\|_\infty + \|\nabla w\|_3\right)\|\nabla E_\rho\phi\|_2.$$

The Hölder and Sobolev inequalities then imply

$$\begin{aligned}|(e^{-tL}P(u\cdot\nabla)u, E_\rho\phi)| &= |(u, u\cdot\nabla e^{-tL^*}E_\rho\phi)| \\ &\leq C\|u\|_4^2\|\nabla e^{-tL^*}E_\rho\phi\|_2 \leq C\left(\rho^{1/2} + \rho^{1/4}\right)\|u\|_2^{1/2}\|\nabla u\|_2^{3/2}\|\phi\|_2.\end{aligned}$$

This proves (iii). To show (iv), consider the equation

$$\frac{dv}{dt} + e^{i\theta}L^*v = 0\ ;\ v(0) = a$$

in the complexification of the Hilbert space H. The standard argument then gives

$$\|v(t)\|_2^2 + 2\mathrm{Re}\int_s^t (e^{i\theta}L^*v, v)d\tau = \|v(s)\|_2^2.$$

Estimating each term of

$$\mathrm{Re}(e^{i\theta}L^*v, v) = \mathrm{Re}(e^{i\theta}Av, v) + \mathrm{Re}(e^{i\theta}v, w\cdot\nabla v) + \mathrm{Re}(e^{i\theta}v, v\cdot\nabla w)$$

by (4.5) and the inequality

$$\int_D \frac{|v|^2}{|x|^2}dx \leq 4\|\nabla v\|_2^2,$$

we see that

$$\|v(t)\|_2^2 + 2\left[\cos\theta - 2C_0(1+2|\sin\theta|)\right]\int_s^t \|\nabla v\|_2^2 d\tau \le \|v(s)\|_2^2.$$

Hence

$$\|v(t)\|_2 \le \|a\|_2$$

provided $|\theta|$ is small enough. This means that $\{e^{-tL^*};\ t \ge 0\}$ is a bounded analytic semigroup in H [19] ; so we have

$$\|L^* e^{-tL^*} a\|_2 \le Ct^{-1}\|a\|_2.$$

Thus, (4.6) yields

$$\|\nabla e^{-tL^*}\phi\|_2^2 \le C\|L^* e^{-tL^*}\phi\|_2 \|e^{-tL^*}\phi\|_2 \le Ct^{-1}\|\phi\|_2^2.$$

We therefore obtain

$$\begin{aligned} |(e^{-tL}P(u\cdot\nabla)u, \phi)| &= |(u, u\cdot\nabla e^{-tL^*}\phi)| \\ &\le Ct^{-1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{3/2}\|\phi\|_2, \end{aligned}$$

which proves (iv).

Proof of Theorem 4.2. Since $(Lu, u) = \|\nabla u\|_2^2 + (u\cdot\nabla w, u)$, (4.6) and the energy inequality (SE) yield

$$\|u(t)\|_2^2 + 2(1-2C_0)\int_s^t \|\nabla u\|_2^2 d\tau \le \|u(s)\|_2^2 + 2\int_s^t (f, u) d\tau \tag{4.9}$$

for a.e. $s > 0$ and all $t \ge s$. Thus, if $C_0 < 1/2$, then as in Section 2, we see that $\|u\|_2 \in L^\infty$, $\|\nabla u\|_2^2 \in L^1$. As in Section 3 we get

$$\|u(t)\|_2^2 + g(t,s) \le \|u(s)\|_2^2 + h(t,s) \tag{4.10}$$

for a.e. $s > 0$ and all $t \ge s$, where

$$g(t,\tau) = C\int_\tau^t \rho\|u\|_2^2 d\sigma \quad ; \quad h(t,\tau) = C\left(\int_\tau^t \rho\|E_\rho u\|_2^2 d\sigma + \int_\tau^t \|f\|_2 d\sigma\right),$$

and the function $\rho(\tau) > 0$ of $\tau > 0$ is to be fixed later. Since

$$\frac{\partial g}{\partial \tau} = -C\rho(\tau)\|u(\tau)\|_2^2 \le -C\rho(\tau)\left[\|u(t)\|_2^2 + g(t,\tau) - h(t,\tau)\right],$$

assuming the existence of a function $F(\tau) > 0$ with $F' = C\rho F$, we obtain by Gronwall's lemma and (4.10),

$$F(t)\|u(t)\|_2^2 \le F(s)\|u(s)\|_2^2 - \int_s^t F\frac{\partial h}{\partial \tau} d\tau.$$

Letting $s \to 0$, we thus have

$$\begin{aligned} \|u(t)\|_2^2 &\leq CF(t)^{-1}\|a\|_2^2 + CF(t)^{-1}\int_0^t F'(\tau)\|E_\rho u\|_2^2 d\tau \\ &\quad + CF(t)^{-1}\int_0^t F(\tau)(1+\tau)^{-1}\|f\|_2(1+\tau)d\tau. \end{aligned} \tag{4.11}$$

Next, we take in (4.4) $\phi(\tau) = e^{-(t-\tau)L^*}E_\lambda\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, and set $s = 0$, to obtain

$$\begin{aligned} |(u(t), E_\lambda\psi)| &\leq \|\psi\|_2 \left[\|e^{-tL}a\|_2 + \int_0^t \|E_\lambda e^{-(t-\tau)L}P(u\cdot\nabla)u\|_2 d\tau\right] \\ &\quad + \|\psi\|_2 \int_0^t \|E_\lambda e^{-(t-\tau)L} f\|_2 d\tau. \end{aligned}$$

Applying Lemma 4.3 and Hölder's inequality yields

$$\|E_\lambda u(s)\|_2 \leq \|e^{-sL}a\|_2 + C\left(\lambda^{1/2} + \lambda^{1/4}\right)\left[\left(\int_0^s \|u\|_2^2 d\tau\right)^{1/4} + \int_0^s \|f\|_{\hat{V}^*} d\tau\right]. \tag{4.12}$$

We here set $\rho(\tau) = 2C/(\tau+e)\log(\tau+e)$ and $F = C(\log(\tau+e))^2$. Assertions (i) and (ii) follow from (4.11) and (4.12), by repeating the argument in the proof of (i) and (ii) of Theorem 1.2.

We finally prove (iii). As in Section 2, applying Gronwall's lemma to (4.9) gives

$$\|u(t)\|_2^2 + 2(1-2C_0)\int_s^t \|\nabla u\|_2^2 d\tau \leq C\left(\|u(s)\|_2^2 + \int_s^t \|f\|_2 d\tau\right)$$

for a.e. $s \in (0,t)$. So we get $\|u\|_2 \in L^\infty$, $\|\nabla u\|_2 \in L^2$, and

$$\begin{aligned} \|u(t)\|_2 &\leq Ct^{-1}\int_0^t \|u\|_2 ds + Ct^{-1}\int_0^t \left[\int_s^t \|f\|_2 d\tau\right]^{1/2} ds \\ &\leq Ct^{-1}\int_0^t \|u\|_2 ds + +Ct^{-1/2}\left[\int_0^t s\|f(s)\|_2 ds\right]^{1/2} \\ &\leq Ct^{-1}\int_0^t \|u\|_2 ds + Ct^{-1/2}. \end{aligned} \tag{4.13}$$

On the other hand, taking $\phi(\tau) = e^{-(t-\tau)L^*}\psi$, $\psi \in C_{0,\sigma}^\infty(D)$, in (4.4) and then applying Lemma 4.3 (iv) yields

$$\|u(\tau)\|_2 \leq \|e^{-(\tau-s)L}u(s)\|_2 + C\int_s^\tau (\tau-\sigma)^{-1/2}\left(\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} + \|f\|_{\hat{V}^*}\right) d\sigma. \tag{4.14}$$

The result now follows from (4.13) and (4.14) in the same way as in the proof of Theorem 1.1 (ii). The proof is complete.

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