数理解析研究所講究録
第 745 巻 1991 年 78-86

Hamiltonian formulation of two-dimensional motion of an ideal fluid and a finite-mode hydrodynamic system

T. Kambe 神部 勉
Department of Physics, University of Tokyo

1 Introduction

The fact that the total kinetic energy is conserved in the motion of an ideal fluid is a manifestation of the fundamental property of mechanics. However, restricting to two-dimensional motions, it is well-known that there exist an infinite number of invariants for the ideal fluid (see §2). Computer simulations of the fluid motions are carried out inevitably by means of *finite*-mode approximation to the exact infinite system. In those studies of two-dimensional motion performed so far, the above property of multiple invariants has not been considered seriously.

Recently, Zeilin [1] proposed a modified dynamical system , based on the SU(N) algeblas studied in the paper by Fairlie & Zachos [2]. This work has established connection between algebras of diffeomorphisms of the domain occupied by the flow and SU(N)-algebras in the limit $N \to \infty$. The Zeitlin's hydrodynamic system of the O(N^2)-mode truncation in Fourier space can be shown to have O(N) invariants. Accordingly, as the number of modes increases, the number of invariants increases arbitrarily.

2 Formulation from the hydrodynamics

2.1 Vorticity equation

Two-dimensional motion of an incompressible fluid in (x, y) plane is described by a streamfunction $\psi(x, y, t)$, giving the velocity $\mathbf{v} = (u, v)$ as

$$u = \partial\psi/\partial y\ , \qquad v = -\partial\psi/\partial x\ , \tag{1}$$

which satisfy the solenoidal relation:

$$\partial_x u + \partial_y v = 0 \ . \tag{2}$$

The vorticity

$$\omega = \partial_x v - \partial_y u = -(\partial_x^2 + \partial_y^2)\psi \tag{3}$$

is governed by the following evolution equation derived from the Euler's equation of motion for the velocity field:

$$\frac{\mathrm{D}}{\mathrm{D}t}\,\omega = \partial_t \omega + u\partial_x \omega + v\partial_y \omega = 0 \ , \tag{4}$$

which may be called again Euler equation. The above definition of u and v yields

$$\partial_t \omega = \frac{\partial(\psi,\omega)}{\partial(x,y)} = \{\psi,\omega\} \ , \tag{5}$$

where the right hand side is the Poisson bracket and the middle is the Jacobian. Since $\mathrm{D}/\mathrm{D}t$ stands for the Lagrange derivative, *i.e.* material derivative, the equation (4) represents that the vorticity ω is invariant with respect to each fluid particle in motion. The property (4) leads immediately to

$$\frac{\mathrm{D}}{\mathrm{D}t}\,\omega^n = 0 \tag{6}$$

for arbitrary integer n.

2.2 Motion on the torus T^2

Consider a fluid motion on the torus $\mathrm{T}^2 = \{x, y;\ \mathrm{mod}\ 2\pi\}$ with periodic boundary condition. It is not difficult to show that the equations (6) and (2) yield

$$\Omega_n = \int_D \omega^n(x,y,t)\,\mathrm{d}x\mathrm{d}y = const, \tag{7}$$

where D: $0 \le x, y \le 2\pi$. This means that there exist an infinite number of invariants for a system of infinite number of degree-of-freedom. The total kinetic energy is given by

$$K = \frac{1}{2}\int_D (u^2 + v^2)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_D \psi\,\omega\,\mathrm{d}x\mathrm{d}y, \tag{8}$$

which is an additional invariant.

2.3 Fourier representation

It is convenient to use the Fourier representation for the analysis on the torus T^2 with the Fourier bases,

$$e_{\mathbf{k}} = \exp(i\mathbf{k}\cdot\mathbf{x})\,, \qquad \text{where } \mathbf{x}=(x,y),\ \mathbf{k}=(k_x,k_y)\,,$$

where k_x and k_y are integers. The streamfunction ψ and vorticity ω are expanded as

$$\psi = \sum_{\mathbf{k}} \psi_{\mathbf{k}}(t)\, e_{\mathbf{k}}, \qquad \omega = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(t)\, e_{\mathbf{k}}\,.$$

Then the equations (3) and (5) lead to

$$\omega_{\mathbf{k}} = k^2\, \psi_{\mathbf{k}}, \tag{9}$$

$$\dot{\omega}_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{1}{q^2}\,\mathbf{p}\times\mathbf{q}\,\omega_{\mathbf{p}}\,\omega_{\mathbf{q}} = \frac{1}{q^2}\,\mathbf{p}\times\mathbf{q}\,\omega_{\mathbf{p}}\,\omega_{\mathbf{q}}\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\,. \tag{10}$$

where the two expressions on the right hand side are understood to be identical. This is the evolution equation of the vorticity $\omega_{\mathbf{k}}$ in Fourier space, here called again Euler equation. This interesting form of the equation will be reconsidered below.

The integral (7) gives

$$I_n = \frac{\Omega_n}{(2\pi)^2} = \sum_{\mathbf{k}_1}\cdots\sum_{\mathbf{k}_n} \omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\cdots\omega_{\mathbf{k}_n}\,, \quad (\mathbf{k}_1+\mathbf{k}_2+\cdots+\mathbf{k}_n=0\,)\,. \tag{11}$$

In particular for n=2, we have the enstrophy integral,

$$\frac{\Omega_2}{(2\pi)^2} = \sum_{\mathbf{k}} |\,\omega_{\mathbf{k}}\,|^2 = const\,. \tag{12}$$

The kinetic energy (8) is reduced to

$$H = \frac{K}{(2\pi)^2} = \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}=0} a^{\mathbf{pq}}\,\omega_{\mathbf{p}}\,\omega_{\mathbf{q}}\,, \tag{13}$$

where

$$a^{\mathbf{pq}} = \frac{1}{p^2}\,\delta(\mathbf{p}+\mathbf{q})\,. \tag{14}$$

3 Hamiltonian formulation

3.1 Algebraic structure

In order to derive the Euler equation (10) in Fourier space from a Hamiltonian formalism, let us first define a commutator (Kirillov bracket) by

$$\{f,\, g\}_K \equiv c^k_{pq}\omega_k \frac{\partial f}{\partial \omega_p}\frac{\partial g}{\partial \omega_q} \tag{15}$$

(the summation convention is understood for repeated indices) for two arbitrary functions of ω_k, where the structure constant c^k_{pq} has the two properties:

$$1) \qquad c^k_{pq} = -c^k_{qp} \ , \tag{16}$$

$$2) \quad c^s_{pk}\, c^q_{sr} + c^s_{kr}\, c^q_{sp} + c^s_{rp}\, c^q_{sk} = 0 \ . \tag{17}$$

The Kirillov bracket provided with these properties is characterized by (i) bilinearity with respect to f and g, (ii) antisymmetric relation: $\{f,\, g\} = -\{g,\, f\}$, and (iii) Jacobi identity:

$$\{\{f,\, g\},\, h\} + \{\{g,\, h\},\, f\} + \{\{h,\, f\},\, g\} = 0 \tag{18}$$

for any three functions f, g and h of ω_k. Hence this forms a Lie algebra. For the elements like $f = \omega_k$, the bracket (15) takes the form

$$\{\omega_p,\, \omega_q\}_K = c^k_{pq}\,\omega_k \ . \tag{19}$$

By this relation and the expression (13) for H, the Euler equation may be written in the following Hamiltonian form,

$$\dot{\omega}_k = \{H,\, \omega_k\}_K = a^{pr}\, c^q_{rk}\, \omega_p \omega_q \ . \tag{20}$$

Let us introduce the structure constant defined by

$$c^{\mathbf{k}}_{\mathbf{pq}} = (\mathbf{p}\times\mathbf{q})\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \ , \tag{21}$$

where the boldface indices $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{k}$ stand for 2-vectors with two integer components, *e.g.* $\mathbf{p} = (p_1,\, p_2)$. Using the definition (14), we recover the Euler equation (10):

$$\dot{\omega}_{\mathbf{k}} = \frac{1}{p^2}\,\delta(\mathbf{p}+\mathbf{r})\,\mathbf{r}\times\mathbf{k}\,\delta(\mathbf{q}-\mathbf{r}-\mathbf{k})\omega_{\mathbf{p}}\,\omega_{\mathbf{q}} = \frac{1}{q^2}\,\mathbf{p}\times\mathbf{q}\,\omega_{\mathbf{p}}\,\omega_{\mathbf{q}}\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \ . \tag{22}$$

3.2 Matrix formulation

The dynamical system has a matrix representation with some set of basis matrices L_i, satisfying the following commutation relation,

$$[L_p,\, L_q] = (\mathbf{p}\times\mathbf{q})\, L_{\mathbf{p}+\mathbf{q}}\ . \tag{23}$$

Then the Euler equation may be rewritten in the matrix form:

$$\dot{W} \;=\; [W,\, \Psi] \tag{24}$$

where

$$W \;=\; \omega_{\mathbf{l}}\, L_{\mathbf{l}}\ , \qquad \Psi \;= a^{\mathbf{lm}}\omega_{\mathbf{l}}\, L_{-\mathbf{m}}\ . \tag{25}$$

In fact, substituting (25) into (24), one obtains

$$\dot{\omega}_{\mathbf{i}}\, L_{\mathbf{i}} = a^{\mathbf{lm}}\omega_{\mathbf{k}}\,\omega_{\mathbf{l}}\,[L_{\mathbf{k}},\, L_{-\mathbf{m}}] = \frac{1}{l^2}\,\mathbf{k}\times\mathbf{l}\,\omega_{\mathbf{k}}\,\omega_{\mathbf{l}}\,\delta(\mathbf{i}-\mathbf{k}-\mathbf{l})\; \boldsymbol{L_{\mathbf{i}}}\ . \tag{26}$$

This is equivalent to (10). From the matrix equation (24), it is readily shown that Trace(W^n) is conserved for any integer n (Casimir functions) :

$$I_n = \mathrm{Tr}\,(W^n) = \sum_{\mathbf{k}_1}\cdots\sum_{\mathbf{k}_n}\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\cdots\omega_{\mathbf{k}_n}\ , \quad (\mathbf{k}_1+\mathbf{k}_2+\cdots+\mathbf{k}_n = 0\,)\ . \tag{27}$$

3.3 Finit-mode analogue

An attempt ot construct a finite-mode system closely connected with (10) has been made by Zeitlin [1]. This is based on the fact that there exists a special basis for SU(N)-algebras [2] in which the commutator takes the form,

$$[L_{\mathbf{p}},\, L_{\mathbf{q}}] \;=\; -2i\,\sin\frac{2\pi}{N}(\mathbf{p}\times\mathbf{q})\; L_{\mathbf{p}+\mathbf{q}|\mathrm{mod}N}\ . \tag{28}$$

Here $L_{\mathbf{p}}$ is a set of special $N\times N$ matrices defined by

$$L_{\mathbf{p}} = \alpha^{p_1p_2/2}\mathsf{G}^{p_1}\mathsf{H}^{p_2}\ ; \quad L_{-\mathbf{p}} \;=\; L^*_{\mathbf{p}}\ , \tag{29}$$

where the superscript * denotes taking the complex conjugate. For odd N, α is given as $e^{i4\pi/N}$ which is a primitive Nth root of unity. The 2-vector $\mathbf{p}$ is (p_1, p_2) with p_1 and p_2 being integers. A basis for the SU(N) algebras is built from the following two unitary unimodular matrices:

$$\mathsf{G} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \alpha^{N-1} \end{pmatrix}$$

$$\mathsf{H} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{30}$$

$$\mathsf{G}^N = \mathsf{H}^N = 1 \,, \qquad \mathsf{H}\,\mathsf{G} = \alpha\,\mathsf{G}\,\mathsf{H} \;.$$

The formula of matrix multiplication defined by

$$L_{\mathbf{p}}\, L_{\mathbf{q}} = \alpha^{\frac{1}{2}\mathbf{p}\times\mathbf{q}}\, L_{\mathbf{p}+\mathbf{q}|\mathrm{mod}N}$$

leads to the commutation relation (28). Renormalizing the generator $L_{\mathbf{p}}$ and taking the limit $N \to \infty$, the commutator (28) reduces to the relation (23).

The matrix $W = \omega_{\mathbf{i}}\, L_{\mathbf{i}}$ is a hermitean traceless matrix, hence there are $N-1$ functionally independent invariants $\mathrm{Tr}\, W^n$ (Casimir invariants) for $n = 2, \cdots, N$:

$$I_n^{(N)} = \mathrm{Tr}\,(W^n) = \sum_{\mathbf{k_1}+\cdots+\mathbf{k_n}=0|\mathrm{mod}N} \omega_{\mathbf{k_1}} \cdots \omega_{\mathbf{k_n}}\, \mathrm{Tr}\,(L_{\mathbf{k_1}} \cdots L_{\mathbf{k_n}}) \tag{31}$$

3.4 Examples

Let us illustrate the above results by two lowest-mode systems.

(A) $N = 3$ system

Minimal system is the su(3)-system in which $\alpha = e^{i4\pi/3}$: (i) take eight points on the plane with coordinates k_1, k_2 taking the values (-1, 0, +1) ; (ii) assign to each point except the origin (0, 0) the complex quantity $\omega_{\mathbf{k}}$; (iii) identify $\omega_{-\mathbf{k}} = \omega^*_{\mathbf{k}}$. As a result, we have three integrals of motion:

$$H = \frac{1}{2} \sum_{\mathbf{k}\neq 0} \frac{1}{k^2} \mid \omega_{\mathbf{k}} \mid^2 \qquad \text{(kinetic energy)} \,,$$

$$I_2^{(3)} = \frac{1}{2} \sum_{\mathbf{k}\neq 0} \mid \omega_{\mathbf{k}} \mid^2 \;,$$

$$I_3^{(3)} = \sum_{\mathbf{p},\mathbf{q}\neq 0} \cos \frac{2\pi}{3} (\mathbf{p} \times \mathbf{q})\, \omega_{\mathbf{p}}\, \omega_{\mathbf{q}}\, \omega_{-\mathbf{p}-\mathbf{q}|\mathrm{mod}3} \;.$$

(B) $N = 5$ system

Difference from the $N = 3$ system is to take 24 points on the plane with coordinates k_1, k_2 taking the values (-2,-1,0,+1,+2), and α is $e^{i4\pi/5}$ instead of $e^{i4\pi/3}$. There exist five invariants: energy integral H and $I_n^{(5)}$ $(n = 2, \cdots, 5)$, where

$$I_n^{(5)} = \sum_{\mathbf{k_1}+\cdots+\mathbf{k_n}=0|\mathrm{mod}5} \omega_{\mathbf{k_1}} \cdots \omega_{\mathbf{k_n}} \mathrm{Tr}\,(L_{\mathbf{k_1}} \cdots L_{\mathbf{k_n}})\ .$$

For example, $I_3^{(5)}$ has the same form as (32) except for 3 being replaced by 5.

A numerical test has been performed, in which only three modes of $\mathbf{k} =$ (0, 1), (1, 2), (2, 2) and their complex conjugate counterparts (hence 6 modes out of 24 modes) are given nonzero initial values. A double-precision calculation has shown that the relative errors of the values of the five invariant functions with respect to the initial values are

$$1.3 \times 10^{-15}\ \ (H)\ , \quad 1.2 \times 10^{-15}\ \ (I_2^{(5)})\ , \quad 17.3 \times 10^{-15}\ \ (I_3^{(5)})\ ,$$

$$0.4 \times 10^{-15}\ \ (I_4^{(5)})\ , \quad 4.4 \times 10^{-15}\ \ (I_5^{(5)})\ .$$

Figures 1 and 2 illustrate how the energy H and the fifth invariant $I_5^{(5)}$ stay at constant levels. Figure 3 shows the streamlines at the initial ($t = 0$) and final ($t = 10$) time.

The author wishes to acknowledge Mr. Y. Hattori for the computer calculation of the numerical test.

References

[1] V. Zeitlin (1990) *Finite-mode analogs of 2-D ideal hydrodynamics: coadjoint orbits and local canonical structure*, Institute of Atmospheric Physics (USSR Academy of Sciences, Moscow), Preprint No 4.

[2] D.B. Fairlie and C.K. Zachos (1989) *Infinite-dimensional algebras, sine brackets, and* SU(∞), Phys. Lett. B, **224**, 101-107.

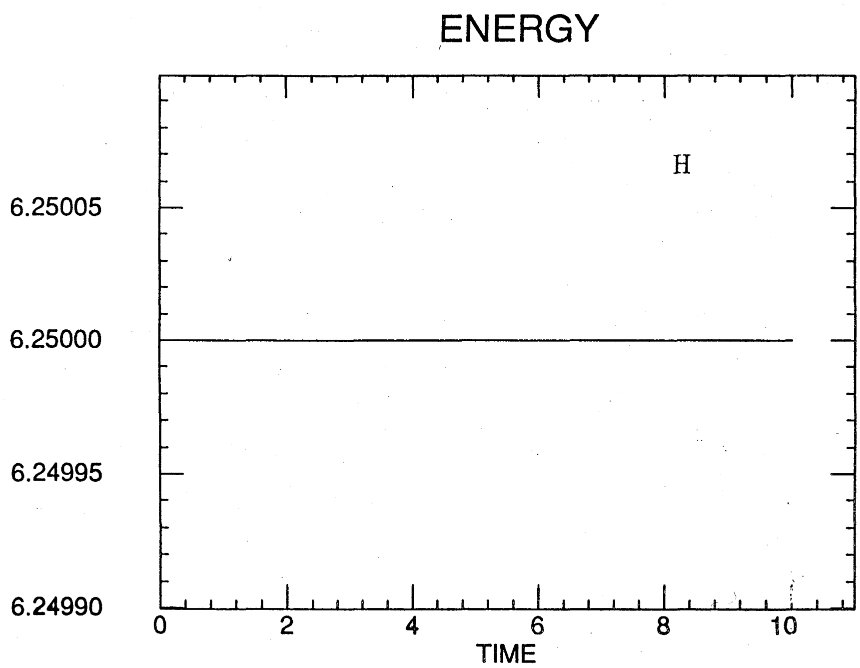


Figure 1

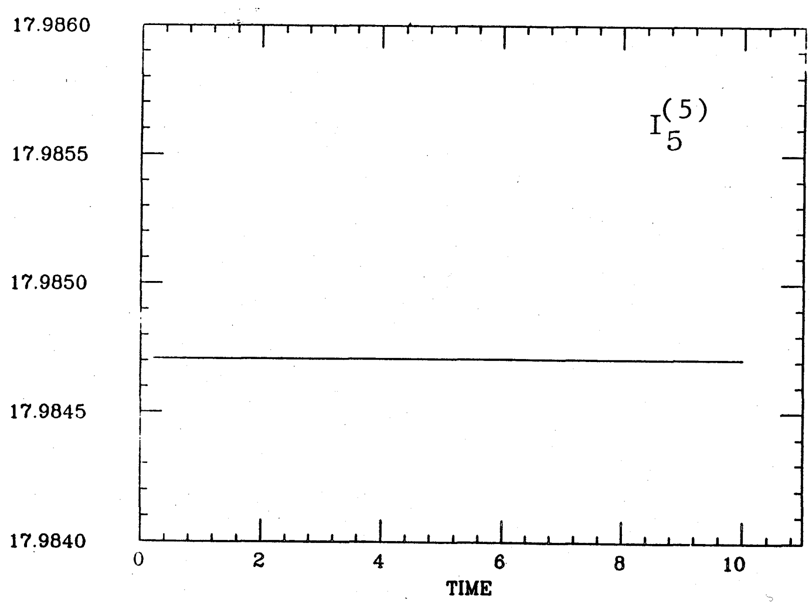
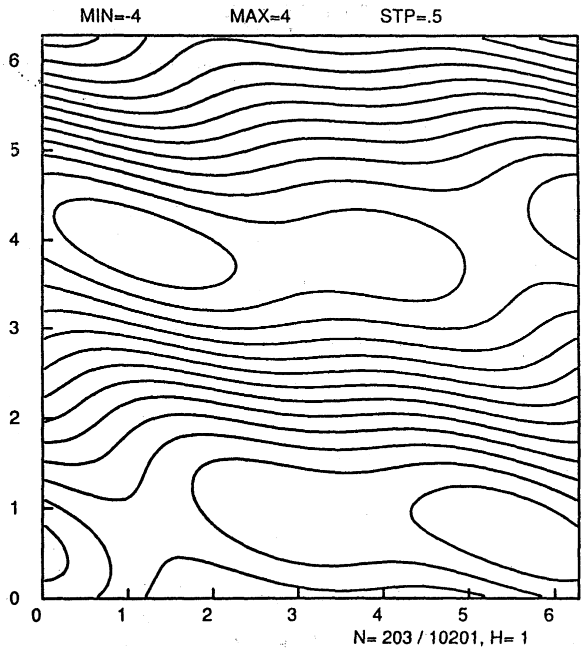
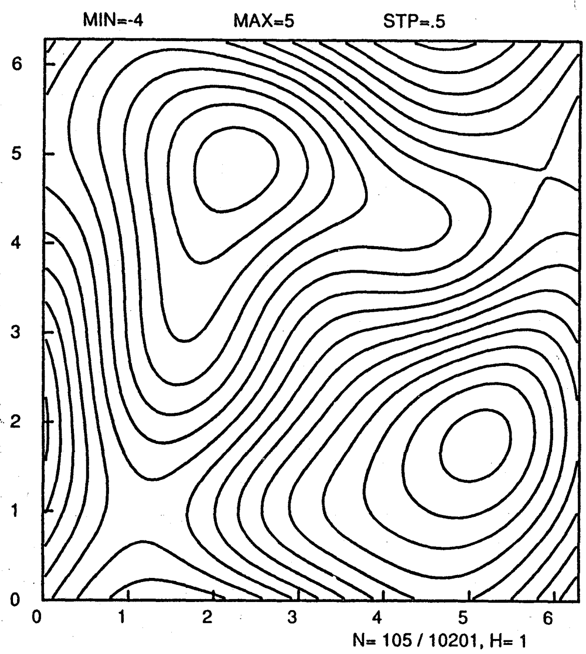


Figure 2



Stream Line (t=0)



Stream Line (t=10)

Figure 3