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On one dimensional nonlinear thermoelasticity

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In this note, I would like to report recent works by the auther and R. Racke, Bonn Univ. ([3], [5]), concerning a global existence of small and smooth solutions to one dimensional nonlinear thermoelastic equations in the case of a bounded reference configuration. Let us recall the equations of one dimensional nonlinear thermoelasticity. Let $(0,1)$ be a unit interval in $\mathbb{R}$, which is identified with the reference configuration $\mathcal{R}$. The thermoelastic motion is described by the deformation map: $x \in (0,1) \longmapsto X(t,x) \in \mathbb{R}$ and the absolute temperature $T(t,x) \in \mathbb{R}$ of the material point of coordinate $X(t,x)$, where t denotes time variable. Then, the equations of balance of linear momentum and balance of energy are given by (cf. [1]):

(B.M) $$X_{tt} = \tilde{S}_x + \rho_{\mathcal{R}} b,$$

(B.E) $$(\tilde{\varepsilon} + (\rho_{\mathcal{R}}/2)X_t^2)_t = (\tilde{S}X_t)_x + \tilde{q}_x + \rho_{\mathcal{R}} r,$$

where we use the following notation: The subscripts t and x denote differentiations with respect to t and x, respectively. $\rho_{\mathcal{R}}$ is the material density. The b and r are specific body force and heat supply, respectively. For simplicity, I assume that $\rho_{\mathcal{R}} = 1$ and that $b = r = 0$, below. $\tilde{\varepsilon}$ is the specific internal energy. $\tilde{q}$ is the heat flux. $\tilde{S}$ is the Piola-Kirchhoff stress tensor. According to 2nd Law of Thermodynamics and Coleman's theorem [2], I make the following assumptions.

Assumptions: (1) There exists a so called Helmholtz energy function $\psi(F,T)$ which is real-valued and in $C^{\infty}(G(B))$ such that

(A.1) $\quad \tilde{S} = S(X_x(t,x),T(t,x))$ and $\tilde{\varepsilon} = \varepsilon(X_x(t,x),T(t,x))$ where

(A.2) $\quad S(F,T) = (\partial\psi/\partial F)(T,F),\ \varepsilon(F,T) = \psi(F,T) - T(\partial\psi/\partial T)(F,T)\ (F = X_x),$

$\quad G(B) = \{\ (F,T) \in \mathbb{R}^2 \mid |F-1| + |T-T_0| < B,\ T > T_0/2\ \}.$

T_0 is a positive constant denoting the natural temperature of the reference body R and B is another positive constant. Moreover, I assume that

(A.3) $\quad (\partial^2\psi/\partial F^2)(F,T) > 0,\ (\partial^2\psi/\partial T^2)(F,T) < 0,\ (\partial^2\psi/\partial F\partial T)(F,T) \neq 0$

$\quad$ for $(F,T) \in G(B)$.

(2) There exists a positive function $Q(F,T) \in C^{\infty}(G(B))$ such that

(A.4) $\quad \tilde{q} = Q(X_x(t,x),T(t,x))T_x(t,x).$

And then, (B.M) and (B.E) are rewritten as follows: for $t > 0$ and $x \in (0,1)$,

(B.M)' $\quad X_{tt} = S(X_x,T)_x,$

(B.E)' $\quad (\varepsilon(X_x,T) + \frac{1}{2}X_t^2)_t = (S(X_x,T)X_t)_x + (Q(X_x,T)T_x)_x.$

If you use the entropy: $N(F,T) = -(\partial\psi/\partial T)(F,T)$, (B.E)' can be rewritten by:

(B.E)" $\quad TN(X_x,T)_t = (Q(X_x,T)T_x)_x.$

In fact, multiplying (B.M)' by X_t implies that $\frac{1}{2}(X_t^2)_t = S_xX_t$. Using the constitutive relations (A.2), you have the identity: $\varepsilon(X_x,T)_t = TN(X_x,T)_t + S(X_x,T)X_{tx}$. Since $(S(X_x,T)X_t)_x = S(X_x,T)X_t + S(X_x,T)X_{tx}$, (B.E)" follows from (B.M)' and (B.E)'. Obviously, (B.E)' follows also from (B.M)' and (B.E)". And then, the system (B.M)' and (B.E)' is equivalent to the system (B.M)' and (B.E)".

Put $u = X - x$ and $\theta = T - T_0$. As boundary conditions, I consider here the following four type: for $t > 0$ and $x = 0$ and 1,

(D.D) $u = 0$ and $\theta = 0$,

(D.N) $u = 0$ and $\theta_x = 0$,

(N.D) $u_x = 0$ and $\theta = 0$,

(N.N) $S = 0$ and $\theta_x = 0$.

Since S can be represented by using the Taylor expansion as follows: $S = S_1 u_x + N_1\theta$, (N.D) is equivalent to what $S = 0$ and $\theta = 0$ at $x = 0$ and 1. In (N.N) case, in addition to (A.1)-(A.3), I assume that

(A.5) $S(1,T_0) = 0$.

In other cases, you may assume without loss of generality that (A.5) is valid. In fact, you can consider

(B.M)" $X_{tt} = [S(X_x,T) - S(1,T_0)]_x$

instead of (B.M)' if (A.5) is not satisfied. But, in (N.N) case, if you consider (B.M)" instead of (B.M)', you must consider the boundary condition: $S(X_x,T) - S(1,T_0) = 0$ at $x = 0$ and 1 instead of (N.N) Since it is inhomogeneous, in general you can not expect to get the decay properties of solutions to linearized equations, and then the global existence theorem can not be expected in general.

As initial conditions, I put

(I.C) $X(0,x) = x + u_0(x)$, $X_t(0,x) = u_1(x)$, $T(0,x) = T_0 + \theta_0(x)$ in $(0,1)$,

where u_0, u_1 and θ_0 are given functions. In cases of (N.D) and (N.N), we assume that

(A.6) $\int_0^1 u_1(x)dx = 0$.

In fact, if you integrate (B.M)' under the boundary condition (N.D) or (N.N), you have $\int_0^1 X_t(t,x)dx = \int_0^1 u_1(x)dx$. Since what $X_t(t,x) \to 0$ as $t \to \infty$ is expected, (A.6) is needed. Since X does not appear in (B.M)' and (B.E)", if you put $X' = X - (\int_0^1 u_1(x)dx)t$, then X' and T satisfy (B.M)', (B.E)", boundary conditions (N.D) or (N.N) and

(I.C)' $X'(0,x) = x + u_0(x)$, $X_t'(0,x) = u_1(x) - \int_0^1 u_1(x)dx$,

$T(0,x) = T_0 + \theta_0(x)$.

Moreover, you have $\int_0^1 X_t'(t,x)dx = 0$. So, (A.6) is not an essential assumption.

Now, let us discuss the equilibrium state. In all the cases, $X = x$ and $T = T_0$ are solutions for initial data: $u_0 = u_1 = \theta_0 = 0$. In cases of (D.N) and (N.N), integrating (B.E)' on $(0,t) \times (0,1)$, you get

(1.1) $\int_0^1 \{\varepsilon(X_x(t,x),T(t,x)) + \frac{1}{2}X_t^2(t,x)\}dx = c(u_0,u_1,\theta_0)$ where

$c(u_0,u_1,\theta_0) = \int_0^1 \{\varepsilon(1+u_0'(x),T_0+\theta_0(x)) + \frac{1}{2}u_1(x)^2\}dx$, $u_0' = du_0/dx$,

as long as the solutions exist. If you expect that $X_t \to 0$, $X_x \to X_\infty$ and $T \to T_\infty$, X_∞ and T_∞ being constants, letting $t \to \infty$ in (1.1), you see that X_∞ and T_∞ should satisfy:

(1.2.a) $(X_\infty,T_\infty) = c(u_0,u_1,\theta_0)$,

(1.2.b) $(X_\infty,T_\infty) \in G(B)$.

In (N.N) case, in addition to (1.2.a) and (1.2.b), what $S = 0$ at $x = 0$ and 1 implies the condition:

(1.21c) $S(X_\infty,T_\infty) = 0$.

On the other hand, if you consider the map: $(1,T) \in G(B) \mapsto \varepsilon(1,T) \in \mathbb{R}$ in (D.N) case and the map: $(F,T) \in G(B) \longmapsto (S(F,T),\varepsilon(F,T)) \in \mathbb{R}^2$ in (N.N) case, respectively, the implicit function theorem tells you the unique existence of (X_∞,T_∞) satisfying (1.2) provided that $|u_0(x)|$, $|u_1(x)|$ and $|\theta_0(x)|$ are sufficiently small, especially $X_\infty = 1$ in (D.N) case. Because, $(\partial\varepsilon/\partial T)(1,T_0) = -T_0(\partial^2\psi/\partial T^2)(1,T_0) \neq 0$ in (D.N) case and the Jacobian $\partial(\varepsilon,S)/\partial(F,T)$ is equal to

$$-T_0(\partial^2\psi/\partial T^2)(1,T_0)(\partial^2\psi/\partial F^2)(1,T_0) + T_0(\partial^2\psi/\partial F\partial T)(1,T_0)^2 \neq 0$$

under the assumption (A.5) in (N.N) case.

I shall say that X and T will be global smooth solutions if X and T satisfy (B.M)', (B.E)' for $t \in (0,\infty)$ and $x \in (0,1)$, one of the boundary conditions: (D.D), (D.N), (N,D) and (N.N) for $t \in (0,\infty)$ and $x = 0$ and 1, and the initial condition (I.C) for $x \in (0,1)$, and if X and T belong to $C^2([0,\infty) \times [0,1])$ and $(X_x(t,x),T(t,x)) \in G(B)$ for all $(t,x) \in [0,\infty) \times [0,1]$.

Roughly spoken, the main result of my talk is the following.

Theorem: If initial data u_0, u_1 and θ_0 are sufficiently small and smooth and satisfy the suitable compatibility conditions, then there exists a unique pair of global smooth solutions $(X(t,x),T(t,x))$. Moreover, the following asymptotic behaviours hold true:

(D.D) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_0$ as $t \to \infty$,

(D.N) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_\infty$ as $t \to \infty$,

(N.D) $X_t(t,x) \to 0$, $X_x(t,x) \to 1$, $T(t,x) \to T_0$ as $t \to \infty$,

(N.N) $X_t(t,x) \to 0$, $X_x(t,x) \to T_\infty$, $T(t,x) \to T_0$ as $t \to \infty$.

Remark. The theorem was proved by M. Slemrod [4] in (D.N) and (N.D) cases, by R. Racke and the auther [3] in (D.D) case and by the auther in (N.N) case.

References.

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