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Continuous radial asymptotics for solutions to elliptic Fuchsian equations in 2 dimensions

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Abstract. Radial regularity at the origin of solutions to elliptic Fuchsian operators is studied in spaces $M(\Omega;\varrho)$ and $Z_d(\Omega;s)$ of distributions with continuous radial asymptotics by means of the techniques based on the Mellin transformation.

Introduction. We study regularity of solutions to singular linear equations $R(x_1, \ldots, x_n, x_1\frac{\partial}{\partial x_1}, \ldots, x_n\frac{\partial}{\partial x_n})u = w$, with $R(x,\xi)$ an elliptic symbol, on proper cones in the positive octant $\mathbf{R}^n_+$. Equations of this type appear i.a. in the study of Laplace-Beltrami operators on symmetric spaces. They fall within the scope of the singular operators considered by M. Kashiwara [5], R. Melrose [7] and M. Bony [1].

The study of solutions to singular elliptic equations is quite different from that of solutions to (non-singular) elliptic equations. In the latter case local regularity of solutions is completely controled by the behaviour of their Fourier transforms at infinity (first wave front set). In the case of singular operators the Fourier transformation is replaced by the Mellin transformation, however the growth order of the Mellin transform along the imaginary planes(second wave front set; see [11]) does not give information about the asymptotic expansions of solutions at the vertex of the corner. This information is contained in the boundary values of (the holomorphic extensions of) the Mellin transforms. Therefore in order to get a complete description of local regularities the two pieces of information should be coupled. This goal is acheaved by introducing the spaces $M(\Omega;\varrho)$ and $Z_d(\Omega;s)$. In contrast to solutions to Fuchsian equations in the sense of Baouendi-Goulaouic, the solutions u to $Ru=0$ do not expand into discrete powers of the radial variable. Instead, for $n=2$, we have "continuous"asymptotic expansions whose densities are distributions supported by several half lines parallel to the real axis. The densities are equal to the boundary values of the Mellin transforms times the factor $(2\pi i)^{-1}$. Moreover, they extend to holomorphic functions with logarithmic singularities situated in a discrete lattice in $\mathbf{C}$. This is resemblant of the resurgence phenomenon of Jean Ecalle and is investigated in a forthcoming paper [12].

The paper ends with an explicit example covering the case of the operator $\widetilde{\Delta} = (x_1\frac{\partial}{\partial x_1})^2 + (x_2\frac{\partial}{\partial x_2})^2$.

A more detailed exposition together with complete proofs is to be found in papers [15] and [16].

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1. Notation and basic facts on the Mellin transformation.

Throughout the paper we use the following vector notation: if $a, b \in \mathbf{R}^n, a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$ then $a < b$ $(a \leq b$, resp.) denotes $a_j < b_j (a_j \leq b_j$, resp.) for $j = 1, \dots, n$. We denote $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : 0 < x\}, \mathbf{R}_- = \{x \in \mathbf{R} : x < 0\}, I = (0, t] = \{x \in \mathbf{R}^n_+ : x \leq t\}$ where $t \in \mathbf{R}^n_+$. We also write $\mathbf{1} = (1, \dots, 1) \in \mathbf{R}^n$.

$\mathbf{Z}$ is the set of integers and $\mathbf{N}_0$ stands for the set of non-negative integers. If $x \in \mathbf{R}^n_+$ and $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ we write $x^z = x_1^{z_1} \dots x_n^{z_n}$. Vector notation is also used for differentiations. Namely we write

$$\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}),\ x\frac{\partial}{\partial x} = (x_1\frac{\partial}{\partial x_1}, \dots, x_n\frac{\partial}{\partial x_n})$$

and if $\nu \in \mathbf{N}_0^n$ then

$$(\frac{\partial}{\partial x})^\nu = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}}, (x\frac{\partial}{\partial x})^\nu = (x_1\frac{\partial}{\partial x_1})^{\nu_1} \dots (x_n\frac{\partial}{\partial x_n})^{\nu_n}.$$

For points $a \in \mathbf{R}^n$ we write $a = (a_1, a')$ where $a \in \mathbf{R}, a' \in \mathbf{R}^{n-1}$, similarly for $\zeta \in \mathbf{C}^n, \zeta = (\zeta_1, \zeta'), \zeta_1 \in \mathbf{C}, \zeta' \in \mathbf{C}^{n-1}$, we also consider sets $W \subset \mathbf{C}^n$ of the form $W = W^1 \times W'$ where $W^1 \subset \mathbf{C}, W' \subset \mathbf{C}^{n-1}$. For a set $W \subset \mathbf{C}^n$ and a vector $a \in \mathbf{R}^n$ we write $W + a = \{z \in \mathbf{C}^n : z - a \in W\}$.

For an open set $V \subset \mathbf{R}^n, C_0^\infty(V)$ is the space of compactly supported C^∞ functions on $V, D'(V)$ is the space of distrbutions on V.

$S(\mathbf{R}^n)$ denotes the Schwartz space of rapidly decreasing functions, $S'(\mathbf{R}^n)$ is the space of tempered distributions. $\mathcal{A}(V)$ stands for the space of analytic functions on an open set $V \subset \mathbf{R}^n, \mathcal{O}(W)$ denotes the space of holomorphic functions on an open set $W \subset \mathbf{C}^n$. More generally $\mathcal{A}(\mathbf{R}^{n-1}; S'(\mathbf{R}))$ $(\mathcal{O}(\mathbf{C}^{n-1}; S'(\mathbf{R}))$, resp.) denotes the space of analytic (holomorphic, resp.) functions on $\mathbf{R}^{n-1}(\mathbf{C}^{n-1}$resp.) with values in $S'(\mathbf{R}^n)$ i.e. functions $\mathbf{R}^{n-1} \ni x \longmapsto T(x) \in S'(\mathbf{R})$ such that for any $\sigma \in S(\mathbf{R})$ the function $\mathbf{R}^{n-1} \ni x \longmapsto T(x)[\sigma] \in \mathbf{C}$ is analytic. We also make use of the isomorphism $S'(\mathbf{R}^n) \simeq S'(\mathbf{R}^{n-1}; S'(\mathbf{R}))$ where the right-hand side is the space of continuous linear mappings on $S(\mathbf{R}^{n-1})$ with values in $S'(\mathbf{R})$. Similarly, we consider $S'(\mathbf{R}^n) \simeq S'(\mathbf{R}, S'(\mathbf{R}^{n-1}))$. The isomorphisms can be regarded as S' versions of the Schwartz kernel theorem ([2]).

For the sake of completenes below we present an outline of the main facts on the Mellin transformations which are used in this paper (for details cf. [13, 4]).

Fix $t \in \mathbf{R}^n_+$. Let $I = (0, t]$. By $C^\infty(I)$ we denote the set of complex functions on I which are restrictions to I of smooth functions on $\mathbf{R}^n_+$.

Let $\alpha \in \mathbf{R}^n$. By $M_\alpha = M_\alpha(I)$ we denote the space of the functions $\varphi \in \mathbf{C}^\infty(I)$ such that for every $\nu \in \mathbf{N}_0^n$

$$\varrho_{\alpha,\nu}(\varphi) = \sup_{x\in I}|x^{\alpha+1}(x\frac{\partial}{\partial x})^\nu \varphi(x)|$$

is finite, with the topology given by the seminorms $\varrho_{\alpha,\nu}, \nu \in \mathbf{N}_0^n$. The space $M_{(\omega)} = M_{(\omega)}(I)$ for $\omega \in (\mathbf{R} \cup \{\infty\})^n$ is the inductive limit

$$M_{(\omega)} = \underrightarrow{\bigcup_{\alpha<\omega}} M_\alpha(I)$$

The space $M' = \bigcup_{\omega\in\mathbf{R}^n} M'_{(\omega)} \subset D'(\mathbf{R}_+^n)$, where $M'_{(\omega)}$ is the dual of $M_{(\omega)}$, is called the space of Mellin (transformable)distributions. We define the Mellin transform of $u \in M'_{(\omega)}$:

$$\mathcal{M}u(z) = u[x^{-z-1}] \quad \text{for} \quad z \in \mathbf{C}^n,\ \mathrm{Re} z < \omega,$$

$\mathcal{M}u$ is holomorphic for Re $z < \omega$. By the Mellin transform of u we shall regard any holomorphic extension of the function defined above.

The Mellin transformation introduced above satisfies the following operational identities. If $u \in M'_{(\omega)}, a \in \mathbf{C}^n$ then

$$\mathcal{M}(x^a u)(z) = \mathcal{M}(z-a) \quad \text{for } \mathrm{Re}\, z < \omega + \mathrm{Re}\, a.$$

If $\nu \in \mathbf{N}_0^n, |\nu| = 1$ then

$$\mathcal{M}\Big((\frac{\partial^\nu}{\partial x})u\Big)(z) = (z^\nu + 1)\mathcal{M}(z+\nu) \quad \text{for } \mathrm{Re}\, z < \omega - \nu.$$

In the sequel $\Gamma \subset \mathbf{R}_+^n$ will denote a proper cone i.e. a cone such that $\overline{\Gamma} \cap \overline{\mathbf{R}_+^n} = \{0\}$. We shall write M'_Γ for the set of Mellin distributions supported by Γ. We also introduce the cut -off functions subordinated to a proper cone:

Definition A (see [15]). Let $\delta\mathring{x} \in \mathbf{R}_+^n$. By a conical cut-off function at $(0;\delta\mathring{x})$ we understand any function $\kappa \in C^\infty(\mathbf{R}_+^n)$ of the form $\kappa = \varphi\cdot\tilde{\kappa}$ where $\varphi \in C_0^\infty(\mathbf{R}^n), \varphi \equiv 1$, in a neighbourhood of zero and $\tilde{\kappa} \in C^\infty(\mathbf{R}_+^n)$ is homogeneous of order zero, supported by a proper cone and $\tilde{\kappa}(\delta\mathring{x}) \neq 0$.

Definition B (see[15]). Let $\delta\mathring{x} \in \mathbf{R}_+^n$ and suppose $u \in D'(\mathbf{R}_+^n)$. We say that $u \in M'$ 2-locally at $(0;\delta\mathring{x})$ if there exists a conical cut-off function κ at $(0;\delta\mathring{x})$ such that $\kappa u \in M'$.

2. Generalized Taylor formula for distributions in $Z_d(\Omega; s)$.

Below we define the spaces $M(\Omega;\rho)$ and $Z_d(\Omega;s)$ of Mellin distributions with continuous radial asymptotics at the origin. Since we are interested in the behaviour with respect to the radial variable we introduce "radial" coordinates $S:\mathbf{R}^n_+ \to \mathbf{R}^n_+$, $x = S(y)$ where

$$\begin{aligned} x_1 &= y_1 \\ x_j &= y_1 y_i \quad \text{for} \quad j = 2,\dots,n. \end{aligned}$$

The coordinates S are related (see Proposition 1 in [15]) by means of the formula

$$(\mathcal{M}u)\circ A^{-1}(\zeta) = \mathcal{M}(u\circ S)(\zeta)$$

valid for $u \in M'_\Gamma$, to the linear transformation $A: \mathbf{C}^n \to \mathbf{C}^n, \zeta = Az$ given by

$$\begin{aligned} \zeta_1 &= z_1 + \dots + z_n \\ \zeta_j &= z_j \qquad \text{for} \quad j = 2,\dots,n. \end{aligned}$$

Definition 1. Let Ω^1 be an $\mathbf{R}_-$ connected open subset of $\mathbf{C}$, i.e. a subset such that together with any point $\mathring{\zeta}_1 \in \Omega^1$ it contains the half-line $\mathring{\zeta}_1 + \overline{\mathbf{R}_-}$. Also suppose that for any $r \in \operatorname{Re}\Omega^1 \overset{df}{=} \{\operatorname{Re}\zeta_1 : \zeta_1 \in \Omega^1\}$ the set $\Lambda_r = \{\zeta_1 \in \mathbf{C}\backslash\Omega^1 : \operatorname{Re}\zeta_1 \leqslant r\}$ is compact in $\mathbf{C}$. Let $\rho : \operatorname{Re}\Omega^1 \to \mathbf{R}$ be a non-decreasing function. We say that a Mellin distribution u in M'_Γ belongs to $M(\Omega;\rho)$ where

$$\Omega = A^{-1}(\Omega^1 \times \mathbf{C}^{n-1})$$

if the function $H(\zeta) = \mathcal{M}u \circ A^{-1}(\zeta)$ satisfies the conditions

i) $H \in \mathcal{O}(\Omega^1 \times \mathbf{C}^{n-1})$,

ii) For any open neighbourhood W of $\Lambda = \mathbf{C}\backslash\Omega^1$

$$|H(a+ib)| \leqslant C(1+ \parallel b \parallel)^{\rho(a_1)} \text{ for } a+ib \in (\mathbf{C}\backslash W)\times \mathbf{C}^{n-1}$$

where $C = C(W,a)$ is locally bounded in $a \in \operatorname{Re}\Omega^1 \times \mathbf{R}^{n-1}$.

In the case where $\Omega = \mathbf{C}^{n-1}$ and $\rho \equiv s \in \mathbf{R}$ is a constant function we also consider the space $D'(\mathbf{R}; M(\mathbf{C}^{n-1};s))$ of $M'(\mathbf{C}^{n-1};s)$-valued distributions on $\mathbf{R}$ such that for any compact set $K \subset \mathbf{R}$

$$|\mathcal{M}'T[\varphi](a'+ib')| \leqslant C(1+\parallel b' \parallel)^s \sum_{|\alpha|\leqslant m} \sup|D^\alpha\varphi(b_1)| \quad \text{for } b' \in \mathbf{R}^{n-1},$$

$\varphi \in C_0^\infty(\mathbf{R})$ supp $\varphi \subset K$, with $C = C(K,a'), m = m(K,a')$ locally bounded in $a' \in \mathbf{R}^{n-1}$.

Next we impose more constraints on the set Ω^1 and on the behaviour of the Mellin transforms near the boundary of Ω^1.

Definition 2. Let $s \in \mathbf{R}$ and let $\Omega^1 = \mathbf{C} \setminus \bigcup_{j=1}^{k} L^j$ where $L^j = \theta_j + \overline{\mathbf{R}_+}$ where $\theta_j \in \mathbf{C}$. Denote $\mathbf{B} = \{\mathrm{Im}\theta_j, i = 1, \dots, k\}$ and set $\Omega = A^{-1}(\Omega^1 \times \mathbf{C}^{n-1})$. We write $u \in Z_d(\Omega; s)$ if $u \in M'_\Gamma$ for some proper cone $\Gamma \subset \mathbf{R}^n_+$ and the function $H(\zeta) = \mathcal{M}u \circ A^{-1}(\zeta)$ satisfies the conditions

i) $H \in \mathcal{O}(\Omega^1 \times \mathbf{C}^{n-1})$,

ii) For every open neighbourhood W of $\bigcup_{j=1}^{k} L^j$

$$|H(a+ib)| < C(1+ \| b \|)^s \text{ for } a+ib \in (\mathbf{C} \setminus W) \times \mathbf{C}^{n-1}$$

locally uniformly with respect to $a \in \mathbf{R}^n$,

iii) $|H(a+ib)| < \tilde{C} \frac{(1+\|b'\|)^s}{(\mathrm{dist}(b_1, \mathbf{B}))^m}$

for b_1 close to $\mathbf{B}$ and $\zeta' = a' + ib' \in \mathbf{C}^{n-1}$, for some constants $0 < \tilde{C} = \tilde{C}(a), m = m(a) \in \mathbf{R}$ locally bounded for $a \in \mathbf{R}^n$.

We also denote $Z_d(\Omega; -\infty) = \bigcap_{s \in \mathbf{R}} Z_d(\Omega; s)$.

Let H be a function satisfying i) - iii) in Definition 2. For a fixed $\zeta' \in \mathbf{C}^{n-1}$ and every $j = 1, \dots, k$ define

$$\Xi^j_{\zeta'} = \lim_{\substack{b_1 \to \mathrm{Im}\theta_j \\ b_1 > \mathrm{Im}\theta_j}} H(\cdot + ib_1, \zeta') - \lim_{\substack{b_1 \to \mathrm{Im}\theta_j \\ b_1 < \mathrm{Im}\theta_j}} H(\cdot + ib_1, \zeta')$$

where the limit is taken in the sense of distributional convergence in $D'(\theta_j + \mathbf{R})$. By iii) the limit clearly exists and defines a holomorphic function

$$\mathbf{C}^{n-1} \ni \zeta_1 \longmapsto \Xi^j_{\zeta^1} \in D'(\theta_j + \mathbf{R})$$

which satisfies the estimates

$$|\Xi^j_{a'+ib'}[\varphi]| \leqslant C \sup_{|\alpha| \leqslant m} |D^\alpha \varphi|(1+ \| b' \|)^s \quad \text{for} \quad \varphi \in C_0^\infty(\theta_j + \mathbf{R})$$

locally uniformly in $a' \in \mathbf{R}^n$. Define

$$T^j = \frac{1}{2\pi i}(\mathcal{M}')^{-1} \Xi^j$$

where $(\mathcal{M}')^{-1}$ is the inverse Mellin transformation in variables ζ'.
Then it follows from the above that

$$T^j \in D'(L^j; M(\mathbf{C}^{n-1}; s)).$$

Let $L^j_r = L^j \cap \{\mathrm{Re}\,\zeta_1 \leqslant r\} (r \in \mathbf{R})$ and let T^j_r be any distribution in $D'(\theta_j + \mathbf{R}; Z_d(\mathbf{C}^{n-1}; s))$ with support in L^j_r which coincides with T^j on $\theta_j + \{a \in \mathbf{R} : a < r\}$. Finally let

$$\Omega_r = A^{-1}(\{\mathrm{Re}\,\zeta_1 < r\} \cup \Omega^1 \times \mathbf{C}^{n-1})$$

Theorem 1(Generalized Taylor formula). Let $u \in Z_d(\Omega; s)$. Then for any $r \in \mathbf{R}$ there exists $R_r \in M(\Omega_r; s)$ such that

$$u \circ S = \sum_{j=1}^{k} T_r^j[y_1^\theta \chi(y_1)] + R_r \circ S \tag{1}$$

where χ is in $C_0^\infty(\mathbf{R})$ $\chi \equiv 1$ in a neighbourhood of zero and for every fixed $y_1 > 0$ and $j = 1, \ldots, k$, y_1^θ denotes the test function $L^j \ni \theta \mapsto y_1^\theta \in \mathbf{C}$ (Note that T_r^j are regarded as distributions in the variable θ). Conversely, if for any $r \in \mathbf{R}$ formula (1) holds for some $T_r^j \in D'(L_r^j; M(\mathbf{C}^{n-1}; s))(j = 1, \cdots, k)$ and $R_r \in M(\{\sum_{j=1}^{n} \mathrm{Re} z_j < r\}; s)$ then $u \in Z_d(\Omega; s)$.

Moreover if (1)holds for some $\mathring{r} \in \mathbf{R}$ then it holds for every $r \leqslant \mathring{r}$.

The proof of the theorem can be found in [16]; see also [12] and [13].

Remark. If estimations in the supremum norm in Definition 2 are replaced by those in L^2 norm we obtain an analogue of Theorem 1 in terms of the weighted Soboles spaces $SP(s, s')$ (see [11] and [1] for the definition).

Remark. Different variants of decomposition (1) can be used to extend the classical concept of differentiability.

Corollary 1. The spaces $Z_d(\Omega; \rho)$ and $M(\Omega; \rho)$ are 2-local, i.e. for any conical cut-off function κ if $u \in Z_d(\Omega; \rho)(M(\Omega; \rho)$resp.) then $\kappa u \in Z_d(\Omega; \rho)(M(\Omega; \rho)$resp.).

The proof for the space $M(\Omega; \rho)$ can be found in [15]. The proof for $Z_d(M; \rho)$ follows from that for $M(\Omega; \rho)$ and Theorem 1.

3. The radial characteristic set $\mathrm{char}_{\alpha}^{*} P$ of a polynomial.

We start by stating the following property of the classical Cauchy transformation.

Proposition 1. Let $\mathbf{C}^{n-1} \ni \zeta' \longmapsto T_{\zeta'} \in E'(\mathbf{R})$ be a distribution valued holomorphic function which is rapidly decreasing as a function of $\mathrm{Im}\zeta'$, locally uniformly in $\mathrm{Re}\zeta'$. Suppose that $T_{\zeta'}$ restricted to an interval $(0, \mathring{b})$, $\mathring{b} > 0$ is a function $T_{\zeta'}(\gamma_1)$ for $\zeta' \in \mathbf{C}^{n-1}$ and for $j = 0, 1$ and some $l \in \mathbf{N}_0$

$$|||\frac{\partial^j}{\partial_{\gamma_1^j}} T_{a'+i\cdot}(\gamma_1)|||_l \leqslant \frac{C_l}{\gamma_1^p} \qquad \gamma \in (0, \mathring{b})$$

locally uniformly with respect to $a' \in \mathbf{R}^{n-1}$, where $|||\sigma|||_l = \sup_{x \in \mathbf{R}^{n-1}} (1+\|x\|)^l (\sum_{|\alpha| \leqslant l} |D^\alpha \sigma(x)|)$ for $\sigma \in S(\mathbf{R}^{n-1})$. Then for $\mathring{a}_1 \leqslant a_1 < 0$ and small $b_1 > 0$

$$|||\mathcal{C}^- T_{a'+i\cdot}(a_1 + ib_1)|||_l \leqslant \frac{\tilde{C}_l}{b_1^{\tilde{p}}}$$

locally uniformly in $a' \in \mathbf{R}^{n-1}$, where $\hat{p} = \max(p, 1+\tilde{p}), \tilde{p} = \sup_{\zeta^1} \text{order } T_{\zeta'}$ and

$$\mathcal{C}^- T_{\zeta'}(\zeta_1) = -\frac{1}{2\pi} T_{\zeta'}\left[\frac{1}{\zeta_1 - i\gamma}\right] \quad \text{for} \quad \zeta' \in \mathbf{C}^{n-1}, \text{Re}\zeta_1 < 0.$$

The proof can be found in [16].

Before passing to the definition of a radial chartacteristic set of a polynomial we recall some properties of the Mellin transforms of conical cut-off functions and of the related Cauchy transforms. Details and proofs are to be found in [15].

Proposition 2. Let κ be a conical cut off function of Definition A. Denote

$$\kappa'(y') = \tilde{\kappa}(1, y'),$$

$$K'(\zeta') = \mathcal{M}'(\kappa')(\zeta') \quad \text{for} \quad \zeta' \in \mathbf{C}^{n-1}$$

$$K(\zeta) = (\mathcal{M}\kappa) \circ A^{-1}(\zeta) \quad \text{for} \quad \zeta \in (\mathbf{C}\backslash\{0\}) \times \mathbf{C}^{n-1}$$

Then
i) $K \in \mathcal{O}((\mathbf{C}\backslash\{0\}) \times \mathbf{C}^{n-1})$,
ii) For every $a \in \mathbf{R}$ the function

$$\mathbf{R}^n \ni b \longmapsto (a_1 + ib_1)K(a + ib)$$

is in $\mathcal{S}(\mathbf{R}^n)$ locally uniformly with respect to $a \in \mathbf{R}^n$,
iii) $K(\zeta) = -\frac{K'(\zeta')}{\zeta_1} + \tilde{K}(\zeta) \quad \text{with} \quad \tilde{K} \in \mathcal{O}(\mathbf{C}^n)$.
Moreover $\kappa' \in C_0^\infty(\mathbf{R}_+^{n-1})$ and
i′) $K' \in \mathcal{O}(\mathbf{C}^{n-1})$,
ii′) For every $a' \in \mathbf{R}^{n-1}$ the function

$$\mathbf{R}^{n-1} \ni b' \longmapsto K'(a' + ib')$$

is in $\mathcal{S}(\mathbf{R}^{n-1})$ locally uniformly with respect to $a' \in \mathbf{R}^{n-1}$.

Theorem 2. Let $T \in \mathcal{S}'(\mathbf{R}^n)$ and fix $\overset{*}{a} \in \mathbf{R}^n$. Fix a conical cut off function κ as in Definition B, and let κ', K' and K be defined in Proposition 2. Denote

$$\tilde{\mathcal{C}}^{\pm}(\zeta) = \tilde{\mathcal{C}}^{\pm}T(\zeta) = \frac{1}{(2\pi)^n} T[K(\zeta - \overset{*}{a} - i\gamma)] \quad \text{for} \quad \pm \text{Re}\zeta_1 > \pm\overset{*}{a}_1, \zeta' \in \mathbf{C}^{n-1}$$

and

$$(2) \quad \tilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta') = (\tilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\zeta') = \frac{1}{(2\pi)^{n-1}} T[K'(\zeta' - \overset{*}{a}' - i\gamma')] \in \mathcal{S}'(\mathbf{R}) \quad \text{for} \quad \zeta' \in \mathbf{C}^{n-1}$$

(in (2) T is regarded as an element of $\mathcal{S}'(\mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}))$ under a canonical isomorphism $\mathcal{S}'(\mathbf{R}^n) \simeq \mathcal{S}'(\mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}))$).

Then

$$\tilde{\mathcal{C}}^{\pm}T \in \mathcal{O}(\{\pm\mathrm{Re}\zeta_1 > \pm\overset{*}{a}_1\} \times \mathbf{C}^{n-1}),$$
$$(\tilde{\mathcal{C}}'T)_{\overset{*}{a}_1} \in \mathcal{O}(\mathbf{C}^{n-1}; S'(\mathbf{R}))$$

and in the sense of convergence in $S'(\mathbf{R}^n)$

$$\lim_{\substack{a\to\overset{*}{a}\\ a_1>\overset{*}{a}_1}} \tilde{\mathcal{C}}^{-}T(a+i\cdot) - \lim_{\substack{a\to\overset{*}{a}\\ a_1<\overset{*}{a}_1}} \tilde{\mathcal{C}}^{+}T(a+i\cdot) = (\tilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\overset{*}{a}' + i\cdot)$$

(here $(\tilde{\mathcal{C}}'T)_{\overset{*}{a}_1}(\overset{*}{a}' + i\cdot) \in S'(\mathbf{R}^{n-1}; S'(\mathbf{R}))$ is regarded as an element of $S'(\mathbf{R}^n)$).

Corollary 2. Let H be a function holomorphic on an open set $U \subset \mathbf{C}^n$. Fix $\overset{*}{a} \in \mathbf{R}^n$ and suppose that the function $b \longmapsto H(\overset{*}{a} + ib)$, defined for $b \in \mathbf{R}^n$ such that $\overset{*}{a} + ib \in U$, extends to a distribution in $S'(\mathbf{R}^n)$ which we denote by $H_{\overset{*}{a}}$. Further suppose that there exists an open set $U^1 \subset \mathbf{C}$ such that for every $\zeta_1 \in U^1$ the function $b' \longmapsto H_{\zeta_1}(\overset{*}{a}' + ib')$, defined for $b' \in \mathbf{R}^{n-1}$ such that $(\zeta_1, \overset{*}{a}' + ib') \in U$, extends to a distribution $H_{\zeta_1,\overset{*}{a}'}$ in $S'(\mathbf{R}^{n-1})$ and the distribution valued function

$$U^1 \ni \zeta_1 \longmapsto H_{\zeta_1,\overset{*}{a}'} \in S'(\mathbf{R}^{n-1})$$

is holomorphic on U^1. Finally assume that there exists a regularization $\tilde{H}_{\overset{*}{a}_1,\overset{*}{a}'} \in S'(\mathbf{R}; S'(\mathbf{R}^{n-1}))$ of the function $b_1 \longmapsto H_{\overset{*}{a}_1+ib_1,\overset{*}{a}'} \in S'(\mathbf{R}^{n-1})$, defined for $b_1 \in \mathbf{R}$ with $\overset{*}{a}_1 + ib_1 \in U^1$, such that $\tilde{H}_{\overset{*}{a}_1,\overset{*}{a}'} = H_{\overset{*}{a}}$ under the canonical isomorphism $S'(\mathbf{R}, S'(\mathbf{R}^{n-1})) \simeq S'(\mathbf{R}^n)$.

Then the function

$$\tilde{\mathcal{C}}'_{\zeta_1}(\zeta') = \frac{1}{(2\pi)^{n-1}} H_{\zeta_1,\overset{*}{a}'}[K'(\zeta' - \overset{*}{a}' - i\gamma')], \quad (\zeta_1, \zeta') \in U^1 \times \mathbf{C}^{n-1}$$

is holomorphic on $U^1 \times \mathbf{C}^{n-1}$, and for every fixed $\zeta' \in \mathbf{C}^{n-1}$ the distribution $\mathcal{C}'_{\overset{*}{a}_1}(\zeta') \in S'(\mathbf{R})$ is a regularization of the function

$$b_1 \longmapsto \tilde{\mathcal{C}}'_{\overset{*}{a}_1+ib_1}(\zeta')$$

defined for $b_1 \in \mathbf{R}$ such that $\overset{*}{a}_1 + ib_1 \in U^1$. Moreover the function

$$\tilde{\Psi}(\zeta) = \begin{cases} \tilde{\mathcal{C}}^{-}(\zeta) & \text{for } \mathrm{Re}\zeta_1 < \overset{*}{a}_1, \quad \zeta' \in \mathbf{C}^{n-1} \\ \tilde{\mathcal{C}}^{+}(\zeta) + \tilde{\mathcal{C}}'_{\zeta_1}(\zeta') & \text{for } \mathrm{Re}\zeta_1 > \overset{*}{a}_1, \quad \zeta_1 \in U^1,\ \zeta' \in \mathbf{C}^{n-1} \end{cases}$$

extends to a holomorphic function Ψ on $(\{\mathrm{Re}\zeta_1 < \overset{*}{a}_1\} \cup U^1) \times \mathbf{C}^{n-1}$ (here $\tilde{\mathcal{C}}^{\pm}(\zeta) = (\tilde{\mathcal{C}}^{\pm}H_{\overset{*}{a}})(\zeta)$ as in Th. 2).

We recall the definition of the radial characteristic set $\mathrm{char}_{\overset{*}{\alpha}}P$ introduced in [15]. Let P be a polynomial in $\mathbf{C}^n$ and $\overset{*}{\alpha} \in \mathbf{R}^n$. Denote $\overset{*}{a} = A\overset{*}{\alpha}$ and $\mathcal{P}(\zeta) = \mathrm{P} \circ A^{-1}(\zeta)$, and let $(\frac{1}{\mathcal{P}})_{\overset{*}{a}}$ be a regularization to a distribution in $S'(\mathbf{R}^n)$ of the function $b \longmapsto \frac{1}{\mathcal{P}(\overset{*}{a}+ib)}$.

Definition 3. Let Ω^1 be the biggest set (as in Def.1) such that the function

$$(3) \quad \tilde{C}^-(\zeta) = \frac{1}{(2\pi)^n}(\frac{1}{\mathcal{P}})_{\overset{*}{a}}[F(a+i\gamma)K(\zeta - \overset{*}{a} - i\gamma)] \text{for } \mathrm{Re}\zeta_1 < \overset{*}{a}_1, \zeta' \in \mathbf{C}^{n-1}$$

extends to a holomorphic function on $\Omega \bigcap \tilde{\Omega}^1 \times \mathbf{C}^{n-1}$ for any $F \in \mathcal{O}(\tilde{\Omega}^1 \times \mathbf{C}^{n-1})$ (with $\tilde{\Omega}^1$ as in Def. 1) such that for any open set W of $\mathbf{C}\backslash\tilde{\Omega}^1$ there exist constants C and M such that

$$|F(a+ib)| \leqslant C(1+\|b\|)^M \quad \text{for} \quad a+ib \in (\mathbf{C}\backslash W) \times \mathbf{C}^{n-1}$$

locally uniformly in $a \in \mathrm{Re}\tilde{\Omega}^1 \times \mathbf{R}^{n-1}$. We define $\mathrm{char}_{\overset{*}{\alpha}} P = \mathbf{C}^n \backslash A^{-1}(\Omega^1 \times \mathbf{C}^{n-1})$.

In Theorem 3 below we compute the set $\mathrm{char}_{\overset{*}{\alpha}} P$ for a class of polynomials in two complex variables. We start with notation and preliminaries.

Let P $= \sum_{|\rho| \leqslant m} a_\rho z^\rho$ be a polynomial in $\mathbf{C}^2$ with complex coefficients. We assume that the vector $\gamma = (-1, 1)$ is non characteristic for $\mathcal{P}$ i.e. $P_m(\gamma) \neq 0$ where $P_m(z) = \sum_{|\rho|=m} a_\rho z^\rho$.

Define $\mathcal{P}(\zeta_1, \zeta_2) = P(\zeta_1 - \zeta_2, \zeta_2)$ and write $\mathcal{P}(\zeta_1, \zeta_2) = a_m(\zeta_1)\zeta_2^m + \cdots + a_1(\zeta_1)\zeta_2 + a_0(\zeta_1)$. Observe that $a_m(\zeta_1)$ is a constant function

$$a_m = P_m((-1,1)) = \sum_{|\rho|=m} a_\rho(-1)^{\rho_1} \neq 0.$$

Represent $\mathcal{P}$ as

$$(4) \qquad \mathcal{P}(\zeta_1, \zeta_2) = a_m \prod_{j=1}^{m} (\zeta_2 - c_j(\zeta_1))$$

where $c_1(\zeta_1), \cdots, c_m(\zeta_1)$ are the complex roots of $\mathcal{P}$ with ζ_1 regarded as a parametr. Define the discriminant of $\mathcal{P}$

$$\Delta = \prod_{j<k} (c_j(\zeta_1) - c_k(\zeta_1)).$$

If $\mathcal{P}$ has no multiple polynomial factors then it follows from Lemmas A.12 and A.13 in [2] that Δ is a non-zero polynomial in ζ_1 and for every $\overset{\circ}{\zeta}_1$ such that $\Delta(\overset{\circ}{\zeta}_1) \neq 0$ the functions $c_j(\zeta_1), j = 1, \ldots, m$ are holomorphic in a neighbourhood of $\overset{\circ}{\zeta}_1$. Further, since a_m is a constant function, in a neighbourhood of every point $\overset{\circ}{\zeta}_1$ such that $\Delta(\overset{\circ}{\zeta}_1) = 0$ the $c_j(\zeta_1)$ have expansions into Puiseux series, i.e. a series of the form

$$(5) \qquad c_j(\zeta_1) = \sum_{k=0}^{\infty} a_k((\zeta_1 - \overset{\circ}{\zeta}_1)^{1/p})^\kappa$$

for some $p \in \mathbf{N}$.

Fix $\overset{*}{a} \in \mathbf{R}^2$. We shall consider the functions $c_j, j = 1, \ldots, m$ defined as follows. For $\mathrm{Re}\zeta_1$ close to $\overset{*}{a}_1$, $c_j(\zeta_1)$ are the holomorphic functions satisfying (4) and we choose the following extension of c_j to $\{\mathrm{Re}\zeta_1 > \overset{*}{a}_1\}$. Let $\theta_\nu(\nu = 1, \ldots, N)$ be all points in $\mathbf{C}$ such that $\Delta(\theta_\nu) = 0$ and for some $j = 1, \ldots, m$, c_j has Puiseux expansion at θ_ν with $p > 1$ and $\mathrm{Re}\theta_\nu > \overset{*}{a}_1$.

At the points θ_ν, c_j has value $c_j(\theta_\nu)$. For $\overset{\circ}{\zeta}_1 \in \mathbf{R}_+ + \theta_\nu$, we define $c_j(\overset{\circ}{\zeta}_1) \overset{\mathrm{df}}{=} c_j^+(\overset{\circ}{\zeta}_1) = \lim\limits_{\substack{\zeta_1 \to \overset{\circ}{\zeta}_1 \\ \mathrm{Im}\zeta_1 > \mathrm{Im}\overset{\circ}{\zeta}_1}} c_j(\zeta_1)$. We also define $c_j^-(\overset{\circ}{\zeta}_1) = \lim\limits_{\substack{\zeta_1 \to \overset{\circ}{\zeta}_1 \\ \mathrm{Im}\zeta_1 < \mathrm{Im}\overset{\circ}{\zeta}_1}} c_j(\zeta_1)$.

Denote by $B_\mu(\mu = 1, \ldots, M)$ all points in $\mathbf{R}$ such that for some $j = 1, \ldots, m$

$$\mathrm{Re}c_j(\overset{*}{a}_1 + iB_\mu) = \overset{*}{a}_2. \tag{6}$$

For j satisfying (6) we define: $\mathrm{sgn}(j;\mu) = +$ if for $a_1 > \overset{*}{a}_1$ close to $\overset{*}{a}_1$, $b \longmapsto \mathrm{Re}c_j(a_1 + ib)$ is an increasing function in a neighbourhood of $B_{j,\mu}$. Otherwise we put $\mathrm{sgn}(j;\mu) = -$. Finally for $\zeta_1 \in \mathbf{C}$ we denote

$$I^o(B_\mu) = \{j : \mathrm{formula(6)holds}\}$$

$$I^+(\theta_\nu) = \{j : c_j \text{ has a Puiseux expansion at } \theta_\nu \text{ with } p > 1$$

$$\text{and } \mathrm{Re}c_j(\overset{*}{a}_1 + ib_1) > \overset{*}{a}_2 \text{ for } b_1 > \mathrm{Im}\theta_\nu \text{ close to } \mathrm{Im}\theta_\nu\}$$

Theorem 3. Fix $\overset{*}{\alpha} \in \mathbf{R}^2$ and let $\overset{*}{a} = A\overset{*}{\alpha}$. Under the notation and assumptions introduced above denote

$$L_\mu = \mathbf{R} + iB_\mu \quad \text{for} \quad \mu = 1, \ldots, M$$

$$\tilde{L}_\nu = \overline{\mathbf{R}}_+ + \theta_\nu \quad \text{for} \quad \nu = 1, \ldots, N$$

$$\mathbf{L} = \bigcup_{\mu=1}^{M} L_\mu \cup \bigcup_{\nu=1}^{N} \tilde{L}_\nu, \quad \mathbf{L}_{\overset{*}{a}} = L \cap \{\mathrm{Re}\zeta_1 \geqslant \overset{*}{a}_1\}.$$

Then $\mathrm{char}_{\overset{*}{\alpha}} P = A^{-1}(\mathbf{L}_{\overset{*}{a}} \times \mathbf{C})$. Moreover, for any $F \in \mathcal{O}(\tilde{\Omega}_1 \times \mathbf{C})$, such that the function $\mathbf{R} \ni \gamma_2 \longmapsto F(\zeta_1, a_2 + i\gamma_2)$ is polynominally bounded at ∞ locally uniformly in ζ_1 and a_2, the differences of the boundary values of the Cauchy transform (3) are distributions $\Xi^\mu_{\zeta_2} F$ on the lines $L_\mu \cap \tilde{\Omega}^1(\mu = 1, \ldots, M)$ with support in $L_\mu \cap \{\mathrm{Re}\zeta_1 \geqslant \overset{*}{a}_1\}$, and distributions $\tilde{\Xi}^\nu_{\zeta_2}$ on the lines $\{\mathbf{R} + \theta_\nu\} \cap \tilde{\Omega}^1(\nu = 1, \ldots, N)$ with support in $\tilde{L}_\nu \cap \{\mathrm{Re}\zeta_1 \geqslant \overset{*}{a}_1\}$ such that for any $l \in \mathbf{N}$

$$|||\Xi^{\mu(\nu)}_{a_2+i\cdot} F[\varphi]|||_l \leqslant C_p \sup \sum_{|\alpha| \leqslant p} |D^\alpha \varphi| \quad \text{for} \quad \varphi \in C_0^\infty(L^\mu \cap \tilde{\Omega}^1) \tag{7}$$

($\varphi \in C_0^\infty((\mathbf{R}+\theta_\nu)\cap\tilde{\Omega}^1)$ resp.) for some constants $C_p = C_p(a_2)$, $p = p(a_2)$ locally bounded in $a_2 \in \mathbf{R}$. Eplicitly we have

$$(8) \qquad \Xi^\mu_{\zeta_1} F(\zeta_2) = \frac{-1}{a_m} \sum_{j \in I^0(B_\mu)} \operatorname{sgn}(j;\mu) \frac{K'(\zeta_2 - c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1)) F(\zeta_1, c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1))}{\prod_{\substack{q=1\\q\neq j}}^m (c_j^{\operatorname{sgn}(j;\mu)}(\zeta_1) - c_q^{\operatorname{sgn}(j;\mu)}(\zeta_1))}$$

for $\zeta_1 \in (\mathbf{R}_+ + \overset{*}{a}_1 + iB_\mu) \cap \tilde{\Omega}^1$, $\mu = 1, \ldots, M$ with $\Delta(\zeta_1) \neq 0$

$$(9) \qquad \tilde{\Xi}^\nu_{\zeta_2} F(\zeta_1) = \frac{-1}{a_m} \sum_{j \in I^+(\theta_\nu)} \left(\frac{K'(\zeta_2 - c_j^+(\zeta_1)) F(\zeta_1, c_j^+(\zeta_1))}{\prod\limits_{\substack{q=1\\q\neq j}}^m (c_j^+(\zeta_1) - c_q^+(\zeta_1))} - \frac{K'(\zeta_2 - c_j^-(\zeta_1)) F(\zeta_1, c_j^-(\zeta_1))}{\prod\limits_{\substack{q=1\\q\neq j}}^m (c_j^-(\zeta_1) - c_q^-(\zeta_1))} \right)$$

for $\zeta_1 \in (\mathbf{R}_+ + \theta_\nu) \cap \tilde{\Omega}^1$, $\nu = 1, \ldots, N$.

Proof. In view of Corollary 2 we are interested in the holomorphic extensions in variable ζ_1 of the function

$$\tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) = \frac{1}{2\pi i} \int_{\operatorname{Re}\theta = \overset{*}{a}_2} \frac{K'(\zeta_2 - \theta) F(\zeta_1, \theta)}{\mathcal{P}(\zeta_1, \theta)} d\theta$$

defined for $\zeta_1 = \overset{*}{a}_1 + ib_1$ with $b_1 \neq B_\mu$ for $\mu = 1, \ldots, M$. Since the function $\mathbf{C} \ni \theta \longmapsto K'(\zeta_2 - \theta) F(\zeta_1, \theta)$ is rapidly decreasing along the imaginary axis locally uniformly in ζ_1 and ζ_2, it follows that the integral over the line $\operatorname{Re}\theta = \overset{*}{a}_2$ may be replaced by an integral over $\operatorname{Re}\theta = r$ (for big $r > 0$) if we add the suitable residuum terms. To this end denote for $\zeta_1 \in \mathbf{C}$

$$I_r^+(\zeta_1) = \{j : r > \operatorname{Re} c_j(\overset{*}{a}_1 + ib_1) > \overset{*}{a}_2 \text{ for } b_1 > \operatorname{Im}\zeta_1, \text{ close to } \operatorname{Im}\zeta_1\}.$$

In view of (4) we have

$$(10) \qquad \tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) = \frac{-1}{a_m} \sum_{j \in I_r^+(\zeta_1)} \frac{K'(\zeta_2 - c_j(\zeta_1)) F(\zeta_1, c_j(\zeta_1))}{\prod\limits_{\substack{q=1\\q\neq j}}^m (c_j(\zeta_1) - c_q(\zeta_1))} + \frac{1}{2\pi i a_m} \int_{\operatorname{Re}\theta = r} \frac{K'(\zeta_2 - \theta) F(\zeta_1, \theta)}{\prod\limits_{j=1}^m (\theta - c_j(\zeta_1))} d\theta$$

The integral in the second summand is holomorphic (as a function of ζ_1 for a fixed ζ_2) in the set $\hat{\Omega}_r = \{\zeta_1 \in \mathbf{C} : \mathrm{Re} c_j(\zeta_1) < r \text{ for } j = 1, \ldots, m\}$ hence it follws that (10) gives on extension of $\tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2)$ to $\hat{\Omega}_r \cap \tilde{\Omega}_1 \cap \{\mathrm{Re}\zeta_1 \geqslant \overset{*}{a}_1\} \backslash \mathbf{L}$. Since the functions c_j are locally bounded we observe (by pushing r to $+\infty$) that all singularities of the extension are contained in the residuum terms, and the computation of the "jumps"of $\tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2)$ is now simple. It follows from Corollary 2 that the holomorphic extension of $\tilde{\mathcal{C}}^-$ is given by

$$
\psi(\zeta) = \begin{cases} \tilde{\mathcal{C}}^-(\zeta) & \text{for } \mathrm{Re}\zeta_1 < \overset{*}{a}_1, \zeta_2 \in \mathbf{C} \\ \tilde{\mathcal{C}}^+(\zeta) + \tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2) & \text{for } \mathrm{Re}\zeta_1 > \overset{*}{a}_1, \zeta_1 \in (\mathbf{C} \backslash \mathbf{L}_{\overset{*}{a}}) \cap \tilde{\Omega}^1, \zeta_2 \in \mathbf{C} \end{cases} \tag{11}
$$

where $\tilde{\mathcal{C}}^{\pm}(\zeta) = \frac{1}{(2\pi)^2}\left(\frac{1}{P}\right)_{\overset{*}{a}}[F(\overset{*}{a} + i\gamma)K(\zeta - \overset{*}{a} - i\gamma)]$ for $\pm\mathrm{Re}\zeta_1 > \pm\overset{*}{a}_1$. Thus for $\mathrm{Re}\zeta_1 > \overset{*}{a}_1$ the jumps of $\psi(\cdot, \zeta_2)$ coincide with those of $\tilde{\mathcal{C}}'_{\zeta_1}(\zeta_2)$ which gives formulas (8) and (9). It remains to prove that $\Xi^{\mu}_{\zeta_2}, \Xi^{\nu}_{\zeta_2}$ are distributions on the respective lines. To this end we shall modify the function $\psi(\zeta)$ to a function $\tilde{\psi}(\zeta)$ which has the same jumps as ψ but whose growth properties are easier to investigate. In view of iii) in Proposition 2 we can write

$$
K(\zeta) = K'(\zeta_2)K^1(\zeta_1) + \tilde{K}(\zeta)
$$

where K^1 is a modified Cauchy kernel in variable ζ_1 and $\tilde{K} \in \mathcal{O}(\mathbf{C})$ is such that $\tilde{K}(a_2 + i\cdot) \in \mathrm{S}(\mathbf{R})$ locally uniformly in $a_2 \in \mathbf{R}$.
Then we have

$$
\tilde{\mathcal{C}}^{\pm}(\zeta) = \psi_1(\zeta) + \psi_2(\zeta)
$$

where.

$$
\psi_1(\zeta) = \frac{1}{2\pi}\tilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)[K^1(\zeta_1 - \overset{*}{a}_1 - i\cdot)] \quad \text{for } \mathrm{Re}\zeta_1 \neq \overset{*}{a}_1,
$$
$$
\psi_2(\zeta) = \frac{1}{(2\pi)^2}\left(\frac{1}{P}\right)_{\overset{*}{a}}[\tilde{K}(\zeta - \overset{*}{a} - i\gamma)] \quad \text{for} \quad \zeta \in \mathbf{C}^2.
$$

Since ψ_2 is an entire function on $\mathbf{C}^2$ we are interested in ψ_1 . Let χ be a $C_0^\infty(\mathbf{R})$ function which is 1 in a neighbourhood of the points $B_\mu(\mu = 1, \ldots, M)$ and $\mathrm{Im}\theta_\nu(\nu = 1, \ldots, N)$. Write for Re $\zeta_1 \neq \overset{*}{a}_1$

$$
\psi_3(\zeta) = \frac{1}{2\pi}\chi\tilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)[K^1(\zeta_1 - \overset{*}{a}_1 - i\cdot)],
$$
$$
\psi_4(\zeta) = \frac{1}{2\pi}(1-\chi)\tilde{\mathcal{C}}'_{\overset{*}{a}_1}(\zeta_2)[K^1(\zeta_1 - \overset{*}{a}_1 - i\cdot)].
$$

Again $\psi_4(\cdot, \zeta_2)$ is holomorphic in complex neighbourhoods of the points $\overset{*}{a}_1 + iB_\mu(\mu = 1, \ldots, M)$ and $\overset{*}{a}_1 + i\mathrm{Im}\theta_\nu(\nu = 1, \ldots, N)$ so we are reduced to ψ_3. Inserting

$$
K^1(\zeta_1) = -\frac{1}{\zeta_1} + \overline{K}(\zeta_1)
$$

where $\overline{K} \in \mathcal{O}(\mathbf{C})$, in the definition of ψ_3 we find that modulo a holomorphic factor we are led to consider the function

$$\psi_5(\zeta) = \frac{1}{2\pi}\chi\widetilde{C}'_{\overset{*}{a}_1}(\zeta_2)\left[\frac{1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\right] \quad \text{for } \operatorname{Re}\zeta_1 \neq \overset{*}{a}_1.$$

Summing up we find that about the points $\overset{*}{a}_1 + iB_\mu, \overset{*}{a}_1 + \mathrm{i}\operatorname{Im}\theta_\nu$ the function ψ given by (11) has the same jumps as the function

$$\Phi(\zeta_1,\zeta_2) = \begin{cases} \frac{1}{2\pi}\chi\widetilde{C}'_{\overset{*}{a}_1}(\zeta_2)\left[\frac{1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\right] & \text{for } \operatorname{Re}\zeta_1 < \overset{*}{a}_1, \zeta_2 \in \mathbf{C} \\ \frac{1}{2\pi}\chi\widetilde{C}'_{\overset{*}{a}_1}(\zeta_2)\left[\frac{1}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}\right] + \tilde{C}'_{\zeta_1}(\zeta_2) & \text{for } \operatorname{Re}\zeta_1 > \overset{*}{a}_1, \zeta_2 \in \mathbf{C} \end{cases}$$

Next from (10) we find that Φ may be replaced by

$$\tilde{\psi}(\zeta_1,\zeta_2) = \begin{cases} \frac{1}{2\pi}\int_{\mathbf{R}} \chi(\gamma_1)\frac{E(\overset{*}{a}_1 + i\gamma_1, \zeta_2)}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}d\gamma_1 & \text{for } \operatorname{Re}\zeta_1 < \overset{*}{a}_1 \\ \frac{1}{2\pi}\int_{\mathbf{R}} \chi(\gamma_1)\frac{E(\overset{*}{a}_1 + i\gamma_1, \zeta_2)}{\zeta_1 - \overset{*}{a}_1 - i\gamma_1}d\gamma_1 + E(\zeta_1,\zeta_2) & \text{for } \operatorname{Re}\zeta_1 < \overset{*}{a}_1 \end{cases}$$

where

(12)
$$E(\zeta_1,\zeta_2) = \frac{-1}{a_m}\sum_{j\in I^+(\zeta_1)} \frac{K'(\zeta_2 - c_j(\zeta_1))F(\zeta_1, c_j(\zeta_1))}{\prod_{\substack{j=1 \\ q\neq j}}^{m}(c_j(\zeta_1) - c_q(\zeta_1))}$$

and

$$I^+(\zeta_1) = \{j : \operatorname{Re}c_j(\overset{*}{a}_1 + ib_1) > \overset{*}{a}_2 \quad \text{for} \quad b_1 > \operatorname{Im}\zeta_1, \text{close to } \operatorname{Im}\zeta_1\}.$$

In view of the properties of H the assertion (7) now follows from Proposition 1 and Köthe theorem [3].

Corollary 3. Formulas (8) and (9) demonstrate the occurence of a "coupled " resurgence effect in the spirit of J. Ecalle [6,8]. This phenomenon is studied in the forthcoming paper [17].

Another remarkable feature of the spectral distributions $\Xi^{\mu(\nu)}_{\zeta_2}F$ is the following:

Corollary 4. The distribution valued holomorphic functions

$$\mathbf{C} \ni \zeta_2 \longmapsto \Xi^{\mu}_{\zeta_2}F, \ \mathbf{C} \ni \zeta_2 \longmapsto \Xi^{\nu}_{\zeta_2}F$$

are rapidly decreasing in $\operatorname{Im}\zeta_2$, locally uniformly in $\operatorname{Re}\zeta_2$ even though the function $\mathbf{C} \ni \zeta_2 \longmapsto F(\zeta_1,\zeta_2)$ may grow polynominally in $\operatorname{Im}\zeta_2$.

4. Singular elliptic operators on $\mathbf{R}_+^n$ with regular singularities.

Let U be an open set $0 \in U \subset \mathbf{R}^n$. Consider a linear partial differential operator R on U of order m with smooth coefficients of the form

$$R = R(x, x\frac{\partial}{\partial x}) = R(x_1, \cdots, x_n, x_1\frac{\partial}{\partial x_1}, \cdots, x_n\frac{\partial}{\partial x_n}).$$

Denoting $P(z) = R(0, z)$ (=the principal Mellin symbol of R at zero) we may write R as

$$R = P(x\frac{\partial}{\partial x}) - Q(x, x\frac{\partial}{\partial x})$$

where

$$Q(x, x\frac{\partial}{\partial x}) = x_1 Q^1(x, x\frac{\partial}{\partial x}) + \cdots + x_n Q^n(x, x\frac{\partial}{\partial x})$$

and $Q^1, \cdots, Q^n$ are differential operators of order m.

We assume the following "ellipticity" condition:

For every $\overset{\circ}{\alpha} \in \mathbf{R}^n$ there exist $C_1 < \infty$ and $C_2 > 0$ such that

$$|P(\overset{\circ}{\alpha} + i\beta)| > C_2(1 + \|\beta\|)^m \quad \text{for} \quad \|\beta\| > C_1.$$

We recall the following existence and regularity result for solutions of the equation $Ru = w$ in the spaces $M(\Omega_\omega; \rho)$ where $\Omega_\omega = \{z \in \mathbf{C}^n : \sum_{j=1}^n \mathrm{Re} z_j < \sum_{j=1}^n \omega_j\}$ and $\varrho(a_1) \equiv s \in \mathbf{R}$ -a constant function.

Theorem 4 (Theorem 2 in [10]; see also [9]). Let R be as above and let $\delta\overset{\circ}{x} \in \mathbf{R}_+^n$. Suppose $w \in M(\Omega_\omega; s)2-$locally at $(0; \delta\overset{\circ}{x})$ for some $\omega \in \mathbf{R}^n, s \in \mathbf{R}$. Then for an arbitrary $\alpha \in \mathbf{R}^n$ with $\sum_{j=1}^n \alpha_j < \sum_{j=1}^n \omega_j$ there exists $u \in M(\Omega_\alpha; s - m)2-$locally at $(0; \delta\overset{\circ}{x})$ such that $Ru = w$ in a local canonical neighbourhood of $\delta\overset{\circ}{x}$ at 0.

In the statement of Th.2 in [10] we assumed in addition that $s \geq m$. We may get rid of this assumption by applying, for instance, the Petree inequality as in Lemma 1 in [16].

We shall now establish more refined regularity results in general spaces $M(\Omega; s)$ where Ω is an arbitrary set as in Definition 1.

Theorem 5. Let $w \in M(\Omega; s)$ 2-locally at $(0, \delta\overset{\circ}{x})$. Then for every $\overset{*}{\alpha} \in \mathbf{R}^n$ with $\Omega_{\overset{*}{\alpha}} \subset \Omega$ there exists $u_{\overset{*}{\alpha}} \in M(\Omega \setminus \bigcup_{j=1}^{\infty} \{\mathrm{char}_{\overset{*}{\alpha}} P + \mathbf{j}\}; s - m)$, where $\mathbf{j} = (j, 0, \cdots, 0) \in \mathbf{N}_0^n$ 2-locally at $(0; \delta\overset{\circ}{x})$ and $Ru_{\overset{*}{\alpha}} = w$ 2-locally at $(0; \delta\overset{\circ}{x})$.

The proof is given [11].

In dimension 2 we have a better result with the spaces $M(\Omega; s)$ replaced by $Z_d(\Omega; s)$:

Theorem 5′. Let $n = 2$ and let $w \in Z_d(\Omega; s)$ -2 locally at $(0; \delta\overset{\circ}{x})$ $\delta\overset{\circ}{x} \in \mathbf{R}_+^2$. Then for every $\overset{*}{\alpha} \in \mathbf{R}^2$ with $\Omega_{\overset{*}{\alpha}} \subset \Omega$ there exists $u_{\overset{*}{\alpha}} \in Z_d(\Omega \setminus \bigcup_{j=0}^{\infty} \{\mathrm{char}_{\overset{*}{\alpha}} P + \mathbf{j}\}; s - m)$ 2-locally at $(0, \delta\overset{\circ}{x})$ and $Ru_{\overset{*}{\alpha}} = w$ 2-locally at $(0, \delta\overset{\circ}{x})$.

The proof can be found in [16].

Finally we state the following 2-locall regularity result proved in [15].

Theorem 6. Let $\delta\overset{\circ}{x} \in \mathbf{R}^n_+$. Suppose $w \in M(\widetilde{\Omega};\tilde{\varrho})$ and $u \in M(\Omega;\varrho)$ 2-locally at $(0;\delta\overset{\circ}{x})$ and $Ru = w$ in a local conical neighbourhood of $(0,\delta\overset{\circ}{x})$. Then

1) for any $\varepsilon > 0\, u \in M(\Omega\cap\widetilde{\Omega};\varrho(-\infty)+\varepsilon)$ 2 – locally at $(0;\delta\overset{\circ}{x})$ if $\tilde{\varrho} - m \leq \varrho(-\infty) \overset{\text{def}}{=} \lim_{a_1\to-\infty}\varrho(a_1)$

2) $u \in M(\Omega\cap\widetilde{\Omega};\tilde{\varrho}-m)$ 2 – locally at $(0;\delta\overset{\circ}{x})$ if $Q \equiv 0$.

Example. Let $0 \neq \overset{*}{\alpha} \in \mathbf{R}^2$ and denote by $u_{\overset{*}{\alpha}} \in \mathfrak{M}'_{\overset{*}{\alpha}}(\mathbf{R}^2)$ (see [10] and [15]) the solution of the equation $Pu_{\overset{*}{\alpha}} = \delta_{(1,1)}$ on $\mathbf{R}^2_+$ where $P(x\frac{\partial}{\partial x}) = (x_1\frac{\partial}{\partial x_1})^2 + (x_2\frac{\partial}{\partial x_2})^2$, such that

$$\mathcal{M}^{\overset{*}{\alpha}}u_{\overset{*}{\alpha}}(\beta) = \frac{1}{(\overset{*}{\alpha}_1 + i\beta_1)^2 + (\overset{*}{\alpha}_2 + i\beta_2)^2} \in L^1(\mathbf{R}^2).$$

We have $\mathcal{P}(\zeta_1,\zeta_2) = (\zeta_1-\zeta_2)^2 + \zeta_2^2 = 2(\zeta_2 - c_1(\zeta_1))(\zeta_2 - c_2(\zeta_1))$
where $c_1(\zeta_1) = \frac{1+i}{2}\zeta_1$, $c_2(\zeta_1) = \frac{1-i}{2}\zeta_1$ and $\Delta = i\zeta_1$. Further from (6) we find

$$B_1 = \overset{*}{a}_1 - 2\overset{*}{a}_2, \quad B_2 = 2\overset{*}{a}_2 - \overset{*}{a}_1$$

and since c_1 and c_2 are regular there are no points θ_ν.
Denote

$$\begin{aligned} &L_1 = \{\zeta_1 \in \mathbf{C} : \zeta_1 = a_1 + i(\overset{*}{a}_1 - 2\overset{*}{a}_2), a_1 \geqslant \overset{*}{a}_1\},\\ &L_2 = \{\zeta_1 \in \mathbf{C} : \zeta_1 = a_1 + i(2\overset{*}{a}_2 - \overset{*}{a}_1), a_1 \geqslant \overset{*}{a}_1\},\\ &\mathbf{L} = L_1 \cup L_2. \end{aligned}$$

Then it follows from Theorems 1 and 3 (also see Example 3 in [15]) and Theorem 5 that

$$\kappa u_{\overset{*}{\alpha}} \in Z_d(\Omega;-\infty)$$

where $\Omega = A^{-1}((\mathbf{C}\backslash\mathbf{L})\times\mathbf{C})$ and $\kappa = \varphi\cdot\tilde{\kappa}$ is any proper conical cut off function. Moreover by Theorem 3 and Proposition 1 the spectral distributions $\Xi^{1(2)}_{\zeta_2}$ equal

$$\Xi^1_{\zeta_2}[\varphi] = a^1(\zeta_2)\delta_{(\overset{*}{a}_1,\overset{*}{a}_1-2\overset{*}{a}_2)}[\varphi] + \\ + \frac{1}{2i}\int_{L_1}\frac{K'(\zeta_2 - \frac{1+i}{2}\zeta_1)}{\zeta_1}\varphi(\zeta_1)d\zeta_1 \quad \text{for} \quad \varphi \in C_0^\infty(\mathbf{R}+iB_1),$$

$$\Xi^2_{\zeta_2}[\varphi] = a^2(\zeta_2)\delta_{(\overset{*}{a}_1,2\overset{*}{a}_2-\overset{*}{a}_1)}[\varphi] + \\ + \frac{1}{2i}\int_{L_2}\frac{K'(\zeta_2 - \frac{1-i}{2}\zeta_1)}{\zeta_1}\varphi(\zeta_1)d\zeta_1 \quad \text{for} \quad \varphi \in C_0^\infty(\mathbf{R}+iB_2)$$

where $K'(\zeta_2) = \mathcal{M}'(\tilde{\kappa}(1,y_2))(\zeta_2)$ and $a^{1(2)}(\zeta_2)$ are some entire functions which are the Mellin transforms of some distributions $\tilde{T}^{1(2)}$ in $M(\mathbf{C};-\infty)$ vanishing near zero. Then $\tilde{T}^{1(2)}$ are smooth functions $\tilde{T}^{1(2)}(y_2)$ vanishing for y_2 close to zero. Computing the inverse Mellin transforms of $\Xi^{1(2)}$ with respect to ζ_2 we find

$$\begin{aligned} T_1[\varphi](y_2) &= \frac{1}{2\pi i}\tilde{T}^1(y_2)\delta_{(\overset{*}{a}_1,\overset{*}{a}_1-2\overset{*}{a}_2)}[\varphi]- \\ &-\frac{1}{4\pi}\tilde{\kappa}(1,y_2)\int\limits_{L_1}\frac{y_2^{\frac{1+i}{2}\zeta_1}}{\zeta_1}\varphi(\zeta_1)d\zeta_1 \quad \text{for} \quad \varphi\in C_0^\infty(\mathbf{R}+iB_1), \\ T_2[\varphi](y_2) &= \frac{1}{2\pi i}\tilde{T}^2(y_2)\delta_{(\overset{*}{a}_1,2\overset{*}{a}_2-\overset{*}{a}_1)}- \\ &-\frac{1}{4\pi}\tilde{\kappa}(1,y_2)\int\limits_{L_2}\frac{y_2^{\frac{1-i}{2}\zeta_1}}{\zeta_1}\varphi(\zeta_1)d\zeta_1 \quad \text{for} \quad \varphi\in C_0^\infty(\mathbf{R}+iB_2) \end{aligned}$$

Thus from Theorem 3 we get the following Taylor formula for $u_{\overset{*}{\alpha}}$:
For any $r > \overset{*}{a}_1$

$$\begin{aligned} (13)\qquad & \kappa\cdot u_{\overset{*}{\alpha}}(y_1,y_1\cdot y_2) = \\ &= \frac{1}{2\pi i}y_1^{\overset{*}{a}_1+i(\overset{*}{a}_1-2\overset{*}{a}_2)}\tilde{T}^1(y_2)+ \\ &+\frac{1}{2\pi i}y_1^{\overset{*}{a}_1+i(2\overset{*}{a}_2-\overset{*}{a}_1)}\tilde{T}^2(y_2)- \\ &-\frac{1}{4\pi}\tilde{\kappa}(1,y_2)\chi(y_1)\left(\int_{\overset{*}{a}_1+iB_1}^{\overset{*}{a}_1+r+iB_1}\frac{y_2^{\frac{1+i}{2}\zeta_1}}{\zeta_1}y_1^{\zeta_1}d\zeta_1+\right. \\ &\left.+\int_{\overset{*}{a}_1+iB_2}^{\overset{*}{a}_1+r+iB_2}\frac{y_2^{\frac{1-i}{2}\zeta_1}}{\zeta_1}y_1^{\zeta_1}d\zeta_1\right) \\ &+R_r(y_1,y_1\cdot y_2) \end{aligned}$$

where $R_r \in M(\sum_{j=1}^{2}\mathrm{Re}z_j < r;-\infty)$(in particular $\frac{R(y_1,y_1\cdot y_2)}{y_1^r}$ is bounded at zero).

Remark. Actually it can be proved that $u_{\overset{*}{\alpha}}(y_1,y_1\cdot y_2)$ is a generalized analytic function in variable y_1, which in our case means that in formula (13) we may take $r=+\infty$ and $R_r\equiv 0$.

Remark. Observe that if $\overset{*}{\alpha}\to 0$ then the lines L_1, L_2 tend towards the half-line $\overline{\mathbf{R}}_+$ and the spectral densities T^1+T^2 tend towards that for $u_0 = \frac{1}{4\pi}\ln\left((\ln x_1)^2+(\ln x_2)^2\right)$. Indeed for $a_1>0$ we have

$$\frac{1}{a_1}\left(y_2^{\frac{1+i}{2}a_1}+y_2^{\frac{1-i}{2}a_1}\right)=\frac{2}{a_1}y_2^{\frac{a_1}{2}}\cos\left(\frac{a_1}{2}\ln y_2\right)$$

which agrees with the explicit formula given in Example 1 in [15].

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