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DIFFERENTIAL CALCULUS ON QUANTUM SPHERES

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1. Introduction

Quantum groups appear in several areas of mathematics and physics, such as theory of deformations of simple Lie groups, generalizations of the Pontriagin theory of duality, theory of paragroups of Ocneanu, quantum inverse scattering method and conformal field theory.

The theory of quantum groups is developed in many different approaches. Here we use C^*-algebraic approach. The aim of this paper is to give a short overview of the basic results on quantum spheres. In Section 2, following [12]-[14] we shall give the definitions of compact quantum space, compact matrix quantum group and (as an example) quantum SU(2) groups. In Section 3 we present the general notions concerning the actions of compact matrix quantum groups on compact quantum spaces and study in details the particular example of quantum spheres, which are generalizations of two-dimensional sphere S^2 endowed with a standard right action of SU(2). In Section 4 we present the classification of differential structures on quantum spheres, which generalize differential structure on S^2.

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Such generalization is unique, but exists only on one quantum sphere. For others, there exists a bigger differential structure, presented in [8]. A part of the results from Sections 3-4 is contained in [7]-[8]. We are going to present the proofs of others elsewhere.

Related material is also contained in [2],[11],[4]-[6],[3],[10].

In all the formulas we sum over repeated indices, which aren't taken in brackets (Einstein's convention).

2. Compact quantum spaces and groups

In this Section we want to make it clear that the C^*-approach provides a natural generalization of compact topological spaces and compact matrix topological groups. Here we present only the most introductory part of the theory, including the classification of the quantum SU(2) groups (cf [12]-[15]).

Let X be a compact space and C(X) denotes set of all continuous functions on X with complex values. For $f,h \in C(X)$, $\lambda \in \mathbb{C}$, $x \in X$ we set

$$(f+h)(x) = f(x)+h(x),\ (\lambda \cdot f)(x) = \lambda \cdot f(x),\ (f \circ h)(x) = f(x)h(x),$$

$$f^*(x) = \overline{f(x)} \quad , \quad \|f\| = \sup_{x \in X} |f(x)| .$$

Then $(C(X), +, \cdot, \circ, *, \|.\|)$ is a commutative C^*-algebra with unity I ($I(x) = 1$ for $x \in X$). Moreover, the above correspondence

(compact spaces) $\longleftrightarrow$ (commutative C^*-algebras with unity)

is one to one (up to homeomorphisms of compact spaces and C^*-isomorphisms of C^*-algebras).

If $\varphi : X \to Y$ is a continuous mapping of compact spaces, then $\varphi^* : C(Y) \to C(X)$ denotes a unital ($\varphi^* I = I$) C^*-homomorphism defined by formula $\varphi^* f = f \circ \varphi$ $(f \in C(Y))$. It occurs that the correspondence

$$\begin{pmatrix} \text{continuous mappings} \\ \text{from X to Y} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{unital } C^*\text{-homomorphisms} \\ \text{from C(X) to C(Y)} \end{pmatrix}$$

is one to one. Moreover, if $\Psi : Y \to Z$ is also a continuous mapping of compact spaces then $(\Psi \circ \varphi)^* = \varphi^* \circ \Psi^*$.

Let us also remind that we can identify $C(X) \otimes C(Y)$ and $C(X \times Y)$ by formula

$$(f \otimes g)(x,y) = f(x)\, g(y)\,,\ f \in C(X), g \in C(Y),\ x \in X\,,\ y \in Y\,.$$

Therefore, the cartesian product of compact spaces corresponds to the tensor (spatial) product of related commutative C^*-algebras with unity. Thus the theory of compact spaces is the same as theory of commutative C^*-algebras with unity. So, considering all (not necessarily commutative) C^*-algebras with unity we obtain objects more general then compact spaces, which we call quantum compact spaces. Quantum compact spaces are nothing more but abstract objects, which are in one to one correspondence with C^*-algebras with unity. For quantum space Z, we denote the corresponding C^*-algebra by C(Z). The above ideas (in more general, locally compact setting) are contained in [12]. In the following we assume that all quantum spaces are compact, C^*-algebras possess unity and C^*-homomorphisms are unital.

In order to endow quantum space with a group structure, we have to add some structure to the corresponding C^*-algebra. In order to do that, first let us consider the following

EXAMPLE 1 [13]

Let G be a compact subgroup of GL(N,$\mathbb{C}$). Let $A = C(G)$. We define $u_{ij} \in C(G)$, i,j = 1,2,...N, by formula $u_{ij}(g) = g_{ij}$, where for $g \in G \subset GL(N,\mathbb{C})$, g_{ij} denotes the matrix element of g standing in i-th row and j-th column. Thus u_{ij} are matrix elements of the fundamental representation of G. It can be checked that the pair (A,u), where $u = (u_{ij})_{i,j=1}^{N} \in M_{N\times N}(A)$, has the following properties:

1) *-algebra (with identity) $\mathcal{A}$ generated by u_{ij} is dense in A.

2) there exists a C^*-homomorphism $\Phi : A \to A \otimes A$ such that

$$\Phi u_{ij} = u_{ik} \otimes u_{kj} \quad , \quad i,j = 1,2,\dots N.$$

(Einstein's convention !)

3) There exists a linear antimultiplicative mapping $\kappa : \mathcal{A} \to \mathcal{A}$ such that

a) $[\kappa(u_{ij})]_{i,j=1}^{N}$ is the inverse of $(u_{ij})_{i,j=1}^{N}$

b) $\kappa(\kappa(a^*))^* = a$ for $a \in \mathcal{A}$

Indeed, 1) follows from the fact that the functions u_{ij} separate points of G . Next, let $p: G\times G \to G$ be the multiplication and $j: G \to G$ be taking inverse in G. Then setting $\Phi = p^* : C(G) \longrightarrow C(G) \otimes C(G)$ we obtain

$$(\Phi u_{ij})(g,h) = u_{ij}(g\cdot h) = (gh)_{ij} = g_{ik}h_{kj} = u_{ik}(g)u_{kj}(h) = (u_{ik} \otimes u_{kj})(g,h)$$

$$(i,j = 1,\dots N \quad , \quad g,h \in G),$$

and 2) holds. It can be also proved, that $\mathcal{X} = j^* \restriction \mathcal{A}$ satisfies all conditions of 3). In the following, the pair constructed above will be denoted by $(C(G), u_G)$. Generalizing this example we obtain

DEFINITION 1 [13]

We say that (A,u) is a (compact matrix) quantum group if A is a C^*-algebra (with unity), $u = (u_{ij})_{i,j=1}^{N} \in M_{N \times N}(A)$ is an $N \times N$ matrix with entries in A and conditions 1)-3) are satisfied.

REMARK. In the following , we abbreviate compact matrix quantum groups with quantum groups. According to previous considerations, we should rather write that a quantum group is an abstract object corresponding to the pair (A,u), but for simplicity and following [13], we just identify these two objects.

The mappings $\Phi, \mathcal{X}$ are uniquely defined.

The following theorem shows that Example 1 is in fact general for comutative A.

THEOREM 1 [13]

1. Let (A,u) be a quantum group with commutative A. Then

$$G = \left\{ \left(\chi(u_{ij}) \right)_{i,j=1}^{N} : \chi - \text{character on } A \right\} \tag{1}$$

is a compact subgroup of $GL(N, \mathbb{C})$. Moreover, up to C^*-isomorphism, $(A, u) = (C(G), u_G)$.

2. $(C(G), u_G)$ are not C^*-isomorphic for different G.

Thus the correspondence $G \leftrightarrow (C(G), u_G)$ between compact subgroups of $GL(N, \mathbb{C})$ and quantum groups with commutative

A is one to one (up to C^*-isomorphisms). This justifies the identification $G = (C(G), u_G)$.

For a general quantum group G = (A,u) we also use the notation A = C(G).

EXAMPLE 2 [14]

Let $\mu \in [-1,1] \setminus \{0\}$. We define A as the universal C^*-algebra generated by α, γ satisfying the following relations

$$\alpha^*\alpha + \gamma^*\gamma = I, \quad \alpha\gamma = \mu\gamma\alpha, \quad \gamma^*\gamma = \gamma\gamma^*$$
$$\alpha\alpha^* + \mu^2\gamma\gamma^* = I, \quad \alpha\gamma^* = \mu\gamma^*\alpha$$

and set

$$u = \begin{pmatrix} \alpha, & -\mu\gamma^* \\ \gamma, & \alpha^* \end{pmatrix}$$

It can be proved [14] that then $S_\mu U(2) = (A, u)$ is a quantum group. For $\mu = 1$, A is commutative, hence due to (1)

$$S_1U(2) = \left\{ \begin{pmatrix} \chi(\alpha), & -\overline{\chi(\gamma)} \\ \chi(\gamma), & \overline{\chi(\alpha)} \end{pmatrix} : \chi\text{-character on } C(S_1U(2)) \right\} =$$
$$= \left\{ \begin{pmatrix} a, & -\bar{b} \\ b, & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\} = SU(2).$$

Thus $S_\mu U(2)$ are called quantum SU(2) groups (cf Theorem 2). C^*-algebra $C(S_\mu U(2))$ can be faithfully embedded in B(H) where H is the Hilbert space with the orthonormal basis φ_{nk}, n = 0,1,2,..., k-integer, in such a way that

$$\alpha\varphi_{nk} = \sqrt{1-\mu^{2n}}\,\varphi_{n-1,k}, \qquad \gamma\varphi_{nk} = \mu^n\varphi_{n,k+1}.$$

For each quantum group there exists a unique *-character $e: \mathcal{A} \to \mathbb{C}$ such that $e(u_{ij}) = \delta_{ij}$, $i,j = 1, \dots N$. In Example 1, $e(f) = f(e_0)$, where $f \in \mathcal{A}$ and e_0 is the neutral element of G.

Now we will present the basic notions of the representations theory of quantum groups, which is very similar to that of compact groups.

DEFINITION 2 (cf [13])

Let G = (A,u) be a quantum group. We say that $v = (v_{ij})_{i,j=1}^{R} \in M_{R\times R}(A)$ is a representation of G iff

1) $\Phi v_{ij} = v_{ik} \otimes v_{kj}$

2) v is invertible

REMARK. For commutative A, we may write $v_{ij} \in C(G)$ and identify v with a continuous mapping from G into $M_{R\times R}(\mathbb{C})$ $\left(v(g) = (v_{ij}(g))_{i,j=1}^{R} \right)$. Then condition 1) means that $v(gh) = v(g)v(h)$, while condition 2) is equivalent to the invertibility of $v(g)$, $g, h \in G$.

Let $v = (v_{ij})_{i,j=1}^{R}$, $w = (w_{kl})_{k,l=1}^{S}$ be representations of quantum group G. We say that v is irreducible if it is irreducible as matrix. If R = S and there exists $Q \in GL(R, \mathbb{C})$ such that $v = QwQ^{-1}$ then we say that v and w are equivalent and write $v \simeq w$. We define tensor product $v \otimes w \in M_{RS\times RS}(A)$ by formula

$$(v \otimes w)_{ik,jl} = v_{ij} w_{kl} \in A, \quad i,j = 1, \dots R, \quad k,l = 1, \dots S.$$

Tensor products and direct sums of representations are representations. For commutative A

$v = QwQ^{-1}$ iff $v(g) = Qw(g)Q^{-1}$ for each $g \in G$

and

$$(v \textcircled{T} w)(g) = v(g) \otimes w(g) \quad , g \in G \ .$$

In the general case each representation is equivalent to a unitary one. Moreover, each representation is a direct sum of irreducible representations. Let us denote the set of all nonequivalent irreducible representations of G by $\{u^\alpha\}_{\alpha \in \hat{G}}$. We put $u^0 = (I)$ (trivial representation). There exist unique continuous linear functionals $\varphi_\alpha \in A'$, $\alpha \in \hat{G}$, such that

$$\varphi_\alpha (u^\beta_{ij}) = \delta_{\alpha\beta} \delta_{ij} \quad , \quad \alpha, \beta \in \hat{G} \quad , \quad i,j = 1,2,\dots \dim u^\beta .$$

We have a general theory of quantum groups, including Haar measure, Peter-Weyl theory [13], quantum SU(N) groups, Tannaka-Krein theory [15], differential calculus [16], dual [12],[9] and double groups [9]. Up to now, general theory of non-compact quantum groups is not known although we have interesting notions and examples (cf [12],[17],[9]).

It occurs that quantum SU(2) groups are the unique nonisomorphic quantum groups which have the same representation theory as SU(2):

THEOREM 2 (cf [15],p.75)

Let G = (A,w) be a quantum group. Then the following conditions are equivalent.

1. Set of nonequivalent irreducible representations of G may be numbered by $\hat{G} = \{0, 1/2, 1, \dots\}$ in such a way that

a) $\dim u^\alpha = 2\alpha + 1 \ , \quad \alpha \in \hat{G}$

b) $u^\alpha \textcircled{T} u^\beta \simeq \bigoplus_{j=|\alpha-\beta|}^{\alpha+\beta} u^j$

c) $w \simeq u^{1/2}$

2. G is similar to $S_\mu U(2)$ for some $\mu \in [-1,1] \setminus \{0\}$, i.e. there exists a C^*-isomorphism $i : C(S_\mu U(2)) \longrightarrow A$ such that $(i(u_{kl}))_{k,l=1}^{2} \simeq w$.

Moreover the constant μ is unique.

We may choose $u^0 = (I)$, $u^{1/2} = u$,

$$u^1 = (u^1_{ij})^1_{i,j=-1} = \begin{pmatrix} \alpha^{*2} & , -(\mu^2+1)\alpha^*\gamma , & -\mu\gamma^2 \\ \gamma^*\alpha^* & , I-(\mu^2+1)\gamma^*\gamma , & \alpha\gamma \\ -\mu\gamma^{*2} & , -(\mu^2+1)\gamma^*\alpha , & \alpha^2 \end{pmatrix} \qquad (2)$$

3. Symmetries of quantum spaces. Quantum spheres

Here we present some notions concerning actions of quantum groups on quantum spaces. As an example we present the classification of generalizations of the two-dimensional sphere S^2 endowed with the standard right action of SU(2) (which is the covering group of SO(3)).

Let X be a compact space, $G \subset GL(N, \mathbb{C})$ compact matrix group and $\varsigma : X \times G \longrightarrow X$ continuous mapping. Let us denote $\delta = \varsigma^* : C(X) \longrightarrow C(X) \otimes C(G)$. Then ς is the action of G on X iff the following conditions are satisfied

a) $(\delta \otimes id)\delta = (id \otimes \Phi_G)\delta$

b) $\langle (I \otimes b)\delta(a) : a \in C(X),\ b \in C(G) \rangle = C(X) \otimes C(G)$

($\langle Z \rangle$ denotes the closure of the linear span of the set Z contained in a C^*-algebra).

Indeed, a) means that

$$\rho(\rho(x,g),h)=\rho(x,gh),\ x\in X,\ g,h\in G.$$

Assuming a) one can prove that b) is equivalent to $\rho(x,e_0)=x\ ,x\in X$.

Therefore we can introduce the following

DEFINITION 3.

Let X be a quantum space and G be a quantum group. We say that the C^*-homomorphism $\delta : C(X)\longrightarrow C(X)\otimes C(G)$ describes an action of G on X iff the conditions a)-b) are fulfilled.

From now on we assume that G is any quantum group and (X,δ) is any pair satisfying the conditions of the above Definition 3.

We say that a vector space $W\subset C(X)$ corresponds to a representation v of G iff there is a basis $e_i\ ,\ i=1,\dots \dim v$, in W such that $\delta e_i = e_k\otimes v_{ki}$.

It occurs that C(X) can be decomposed into vector spaces corresponding to irreducible representations of G :

THEOREM 3

Let δ describe an action of a quantum group G on a quantum space X . We set $W_\alpha=(id\otimes\rho_\alpha)\delta C(X)$ (ρ_α were introduced in Section 2). Then

1. $C(X)=\overline{\bigoplus_{\alpha\in\hat{G}} W_\alpha}$

2. For each $\alpha\in\hat{G}$ there exist sets I_α and vector spaces $W_{\alpha i}\subset C(X)$, $i\in I_\alpha$, corresponding to u^α such that

$$W_\alpha=\bigoplus_{i\in I_\alpha} W_{\alpha i}\quad .$$

Cardinal numbers C_α of I_α don't depend on the choice of I_α and $W_{\alpha i}$. Moreover, each vector space $W \subset C(X)$ corresponding to u^α, $\alpha \in \hat{G}$, is contained in W_α.

We call C_α multiplicities of u^α in the spectrum of δ.

Let us now consider the special case of the standard right action $\varsigma : S^2 \times SU(2) \longrightarrow S^2$ of SU(2) on S^2. Denoting $X = S^2$ and $\delta = \varsigma^*$ we have that

a) multiplicities in the spectrum of δ are

$$C_0 = C_1 = C_2 = \ldots = 1 \quad , \quad C_{1/2} = C_{3/2} = C_{5/2} = \ldots = 0$$

b) W_1 generates the C*-algebra with unity C(X).

Indeed, a) is a classical result. Then $W_l = W_{l1}$ is the linear span of spherical harmonics Y_{lm}, $m = -l, \ldots l$, $l = 0, 1, 2, \ldots$. Condition b) follows from the fact that cartesian coordinates on $S^2 \subset \mathbb{R}^3$ (which are combinations of Y_{1m}, $m = -1, 0, 1$) separate points of S^2 and therefore generate C*-algebra $C(S^2)$.

The above justifies the following

DEFINITION 4.

We say that the pair (X, δ) is a quantum sphere if X is a quantum space, $\delta : C(X) \longrightarrow C(X) \otimes C(S_\mu U(2))$ describes an action of $S_\mu U(2)$ on X and conditions a)-b) are satisfied.

REMARK. Fixing $l \in \{0, 1, 2, \ldots\}$ and choice of u^l we can define generalized spherical harmonics e_{lm}, m = -l,...l, as any basis in W_l such that $\delta e_{lm} = e_{lk} \otimes u^{(l)}_{km}$. Thus we can also define quantum sphere as any pair (B, δ) where B is a C^*-algebra and $\delta : B \longrightarrow B \otimes C(S_\mu U(2))$ is a C^*-homomorphism for which there exist $e_{lm} \in B$, m = -l,...l, l = 0,1,2,... , such that

a) e_{lm} are linearly independent and span a dense subset in B

b) $\delta e_{lm} = e_{ls} \otimes u^{(l)}_{sm}$

c) e_{1m}, m = -1,0,1, generate B.

Quantum spheres are classified in the following way:

THEOREM 4 (cf [7])

For $\mu = \pm 1$ we have precisely one quantum sphere $(S^2_{\mu 0}, \delta_{\mu 0})$.

For $\mu \in (-1,1) \setminus \{0\}$ we have a continuum of quantum spheres $(S^2_{\mu c}, \delta_{\mu c})$, $c \in [0, \infty]$.

The classical case (S^2, δ^*) corresponds to $\mu = 1$, $c = 0$.

$C(S^2_{\mu c})$ can be in each case realized as C^*-subalgebra of $C(S_\mu U(2))$ generated by

$$e_m = s_k u^1_{km} \qquad , m = -1,0,1,$$

where

$(s_{-1}, s_0, s_1) = (\sqrt{c}, 1, \sqrt{c})$ for $c < \infty$, $(s_{-1}, s_0, s_1) = (1, 0, 1)$ for $c = \infty$

and u^1_{km} are given in (2). In this embedding

$$\delta_{\mu c} = \Phi \restriction_{C(S^2_{\mu c})} : C(S^2_{\mu c}) \longrightarrow C(S^2_{\mu c}) \otimes C(S_\mu U(2)).$$

According to Section 1 $\mathcal{A}$ is the dense $*$-subalgebra of $C(S_\mu U(2))$, generated by α, γ. There is also a distinguished dense $*$-algebra $\mathcal{A}_c \subset C(S^2_{\mu c})$ generated by e_{-1}, e_0, e_1. It can be also written as

$$\mathcal{A}_c = \bigoplus_{l=0}^{\infty} W_l = \operatorname{lin}\{ e_{lm} : m = -l, \dots l \ , \ l = 0, 1, \dots \}.$$

Then we have $\delta_{\mu c} \mathcal{A}_c \subset \mathcal{A}_c \otimes \mathcal{A}$, $\mathcal{A}_c \subset \mathcal{A}$.

REMARK. We can also define $C(S^2_{\mu c})$ as universal C^*-algebras generated by e_{-1}, e_0, e_1 satisfying

$$a_{lm} e_l e_m = \rho I ,$$

$$b_{lm,k} e_l e_m = \lambda e_k , \quad k = -1,0,1,$$

where

$\lambda = 1-\mu^2$, $\rho = (1+\mu^2)^2 \mu^{-2} c + 1$ (for $c = \infty$: $\lambda = 0$, $\rho = (1+\mu^2)^2 \mu^{-2}$)

and nonzero values of a_{lm} , $b_{lm,k} \in \mathbb{C}$ are:

$$a_{-11} = 1+\mu^2 , \quad a_{1-1} = 1+\mu^{-2} , \quad a_{00} = 1 ,$$

$$b_{0-1,-1} = b_{10,1} = 1 , \quad b_{-10,-1} = b_{01,1} = -\mu^2 ,$$

$$b_{-11,0} = 1+\mu^2 , \quad b_{1-1,0} = -(1+\mu^2) , \quad b_{00,0} = 1-\mu^2 .$$

We also introduce the coefficients $c_{kl,r}$, k,l = -1,0,1, r = -2,...2, as

$$c_{-1-1,-2} = c_{11,2} = c_{-10,-1} = c_{01,1} = c_{00,0} = 1 ,$$

$$c_{0-1,-1} = c_{10,1} = \mu^2 , \quad c_{-11,0} = -\mu^{-2} , \quad c_{1-1,0} = -\mu^2 ,$$

$$c_{kl,r} = 0 \quad \text{for} \quad r \neq k+l \qquad \text{(cf [8])}.$$

Warning: The definition of quantum sphere given here is more restrictive then in [7]. Moreover, generalized spherical harmonics are chosen in [7] in some special way.

4. Differential calculus on quantum spheres

Let ϱ be the standard right continuous action of SU(2) on S^2 and $\tau = \varrho^* : C(S^2) \longrightarrow C(S^2) \otimes C(SU(2))$. Moreover, we can define a right coaction r of SU(2) on $C(S^2)$ by formula

$$(r_g f)(x) = f(\varrho(x,g)) \ , \ f \in C(S^2) \, , \ g \in SU(2) \ , \ x \in S^2 .$$

The last two mappings are related by $(id \otimes \chi_g)\tau = r_g$, where $g \in SU(2)$ and χ_g is the corresponding character on C(SU(2)). We know that S^2 is a manifold, ϱ is smooth and $\mathcal{A}_0 \subset C^\infty(S^2)$.

We set $\mathcal{B} = \mathcal{A}_0$. Let $S^\wedge = \bigoplus_{n=0}^{\infty} S^{\wedge n}$,

where $$S^{\wedge n} = \mathcal{B} \underbrace{d\mathcal{B} \wedge d\mathcal{B} \wedge \dots \, d\mathcal{B}}_{n \text{ times}}$$

is the bimodule of external forms on S^2 of n-th degree , which are generated by $\mathcal{B}$. Then $d : S^\wedge \longrightarrow S^\wedge$. Let $* : S^\wedge \longrightarrow S^\wedge$ be the complex conjugation:

$$(a_0 \, da_1 \wedge \dots \, da_n)^* = a_0^* \, d(a_1^*) \wedge \dots \, d(a_n^*) \ , \quad a_0, a_1, \dots a_n \in \mathcal{B}.$$

We have moreover the right shifts $R_g : S^\wedge \to S^\wedge$, $g \in SU(2)$:

$$R_g \, (a_0 \, da_1 \wedge \dots \, da_n) = (r_g a_0) \, d(r_g a_1) \wedge \dots \, d(r_g a_n), \ a_0, a_1, \dots a_n \in \mathcal{B}.$$

Alternatively we can also consider a unique linear mapping

$$\tau^\wedge : S^\wedge \longrightarrow S^\wedge \otimes \mathcal{A}$$

(*-algebra $\mathcal{A} \subset C(SU(2))$ was introduced in Section 1) such that

$$(id \otimes \chi_g) \, \tau^\wedge = R_g \quad , \ g \in SU(2).$$

It is easy to check that $(S^\wedge, \tau^\wedge, d, *)$ satisfies

1) $S^\wedge = \bigoplus_{n=0}^{\infty} S^{\wedge n}$ is a graded algebra such that $S^{\wedge 0} = \mathcal{B}$ and the unity of $S^{\wedge 0}$ is the unity of $S^\wedge$

2) $\delta^\wedge : S^\wedge \longrightarrow S^\wedge \otimes \mathcal{A}$ is a graded homomorphism such that
$(id \otimes e)\delta^\wedge = id$, $(\delta^\wedge \otimes id)\delta^\wedge = (id \otimes \Phi)\delta^\wedge$, $\delta^{\wedge 0} = \delta_{\restriction \mathcal{B}}$

3) * is a graded antilinear involution such that
$(\theta \wedge \theta')^* = (-1)^{kl}\theta'^* \wedge \theta^*$, $\theta \in S^{\wedge k}$, $\theta' \in S^{\wedge l}$

($\wedge$ denotes multiplication in $S^\wedge$),

$$(\delta^\wedge)^* = (* \otimes *)\delta^\wedge \ ,$$

* on $S^{\wedge 0}$ reduces itself to the standard * .

4) $d : S^\wedge \longrightarrow S^\wedge$ is a linear mapping such that

a) $d(S^{\wedge n}) \subset S^{\wedge (n+1)}$, $n = 0, 1, 2, \ldots$

b) $d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta'$, $\theta \in S^{\wedge k}$, $\theta' \in S^\wedge$

c) $d* = *d$

d) $(d \otimes id)\delta^\wedge = \delta^\wedge d$

e) $dd = 0$

5) $S^{\wedge n} = \operatorname{span}\{a_0 da_1 \wedge \ldots da_n : a_0, a_1, \ldots a_n \in \mathcal{B}\}$
(we omit $\wedge$ if one from multipliers belongs to $S^{\wedge 0}$).

In the following we assume that $\mu \in [-1,1] \setminus \{0\}$, $c \in [0,\infty]$ ($c = 0$ for $\mu = \pm 1$), $\mathcal{A} \subset C(S_\mu U(2))$ is the distinguished *-algebra, $\mathcal{B} = \mathcal{A}_c \subset C(S^2_{\mu c})$, $\delta = \delta_{\mu c}$.

DEFINITION 5

We say that $\underline{S}^\wedge = (S^\wedge, \delta^\wedge, d, *)$ is an external algebra on $S^2_{\mu c}$, invariant w.r.t. $\delta_{\mu c}$ iff conditions 1)-5) are satisfied.

The above choice of axioms is motivated by [1] and [16]. We don't introduce (and don't know if it is in self-consistent way possible) any condition replacing

$$\theta \wedge \theta' = (-1)^{kl} \theta' \wedge \theta , \quad \theta \in S^{\wedge k} , \theta' \in S^{\wedge l} ,$$

which is not good for non-commutative $\mathcal{B}$. However, we can introduce 'dimensionality' conditions as follows.

We define L as the free left module over $\mathcal{B}$ with basis ω_i, i = = -1,0,1. Let $\Lambda : L \to L \otimes \mathcal{A}$ be defined by $\Lambda(a_i \omega_i) = \delta_{\mu c}(a_i)(\omega_k \otimes u^1_{ki})$, where $a_{-1}, a_0, a_1 \in \mathcal{A}_c$. Assume that $\underline{S}^\wedge$ is an external algebra on $S^2_{\mu c}$, invariant w.r.t. $\delta_{\mu c}$. Then the homomorphism of left modules $j : L \to S^{\wedge 1}$ given by $j(\omega_i) = de_i$ satisfies $(j \otimes id)\Lambda = \delta^{\wedge 1} j$ (we defined Λ in such a way that this equality holds).

An element $\tau \in L$ is called Λ-invariant iff $\Lambda \tau = \tau \otimes I$. The set of Λ-invariant elements is equal

$$L^{inv} = \mathbb{C}\, a_{kl} e_k \omega_l$$

(values of a_{kl} are given at the end of Section 3).

DEFINITION 6

Let μ, c, $\mathcal{B}$, $\underline{S}^\wedge$, L, L^{inv}, j be as above. We say that $\underline{S}^\wedge$ is $\binom{2}{\cdot}$-dimensional iff

a) Ker j $= \mathcal{B} L^{inv}$, Range j $= S^{\wedge 1}$.

b) there exists a $\delta^{\wedge 2}$-invariant basis of the left module $S^{\wedge 2}$, consisting from one element.

REMARK. 1. Condition a) means that de_k generate left module $S^{\wedge 1}$ with one constraint $a_{kl} e_k de_l = 0$.

2. Conditions a)b) are satisfied (for $\mu = 1$, $c = 0$) by $(S^\wedge, \delta^\wedge, d, *)$ introduced at the beginning of the present Section.

In that case we can replace e_{-1}, e_0, e_1 by another basis in W_1, consisting from cartesian coordinates x_1, x_2, x_3. In this basis a) means that dx_k, k = = 1,2,3, generate the left module $S^{\wedge 1}$ with one constraint $x_k dx_k = 0$, whereas b) is satisfied by $(1/2)\,\varepsilon_{ijk}\, x_i\, dx_j \wedge dx_k \in S^{\wedge 2}$.

THEOREM 5

For $\mu \in [-1,1] \setminus \{0\}$, c = 0 there exists unique $\binom{2}{\cdot}$-dimensional external algebra $\underline{S}^{\wedge}$ on the quantum sphere $S^2_{\mu c}$, invariant w.r.t. $\mathcal{E}_{\mu c}$.

For $\mu \in (-1,1) \setminus \{0\}$, $c \in (0,\infty]$ there are no $\binom{2}{\cdot}$-dimensional external algebras on the quantum spheres $S^2_{\mu c}$, invariant w.r.t. $\mathcal{E}_{\mu c}$.

The same facts hold if we restrict ourselves to $S^{\wedge 0} \oplus S^{\wedge 1}$ or $S^{\wedge 0} \oplus S^{\wedge 1} \oplus S^{\wedge 2}$ (without * or with *) instead of $\underline{S}^{\wedge}$ (in Definitions 5-6 and in this theorem).

Moreover $\underline{S}^{\wedge}$ for c = 0 has the following properties:

a) one-element $\mathcal{E}^{\wedge 2}$-invariant basis in the left module $S^{\wedge 2}$ can be chosen as

$$\omega_\bullet = d_{kl}\, e_k\, b_{mn,l}\, de_m \wedge de_n \quad ,$$

b) $S^{\wedge k} = \{0\}$, $k > 2$,

c) the following formulae hold:

$$d_{kl}(de_k)e_l = 0$$
$$b_{kl,r}(de_k)e_l = (1-\mu^2)\,de_r - b_{kl,r}\, e_k\, de_l \quad , \quad r = -1,0,1,$$
$$c_{kl,r}(de_k)e_l = c_{kl,r}\, e_k\,[de_l + \mu^{-2}(1-\mu^2)\, b_{mn,l}\, e_m\, de_n],$$
$$r = -2, \dots 2,$$
$$\omega_\bullet e_r = e_r \omega_\bullet \quad , \quad r = -1,0,1,$$

$$a_{kl}\, de_k \wedge de_l = 0\,,$$
$$b_{kl,r}\, de_k \wedge de_l = e_r\,\omega_\bullet\,, \quad r=-1,0,1,$$
$$c_{kl,r}\, de_k \wedge de_l = \mu^{-2}(1+\mu^2)^{-2}(\mu^6-1)\, c_{kl,r}\, e_k e_l\, \omega_\bullet\,, \quad r=-2,\dots 2,$$
$$(de_k)^* = de_{-k}\,, \quad k=-1,0,1,$$
$$\omega_\bullet^* = -\omega_\bullet\,,$$
$$\delta^{\wedge 1} de_k = de_m \otimes u^1_{mk}\,, \quad k=-1,0,1,$$
$$\delta^{\wedge 2}\, \omega_\bullet = \omega_\bullet \otimes I\,.$$

REMARK. Thus the quantum sphere $(S^2_{\mu 0}, \delta_{\mu 0})$ is distinguished. What concerns other quantum spheres the following result seems to be interesting:

THEOREM 6 (cf [8])

Let $\mu \in (-1,1)\setminus\{0\}$, $c \in [0,\infty]$. There exists an external algebra $\Gamma_c^\wedge = (\Gamma_c^\wedge, \Phi^\wedge_{\Gamma_c}, d, *)$ on $S^2_{\mu c}$, invariant w.r.t. $\delta_{\mu c}$ such that $\operatorname{Ker} j = \{0\}$, $\operatorname{Range} j = S^{\wedge 1}$, i.e. de_{-1}, de_0, de_1 form a basis in the left module $\Gamma_c^{\wedge 1}$.

The following question remains open: Is there any uniqueness result in that case?

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