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On the transcendency of the values of modular functions at algebraic points

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§0. In this short note we point out a certain properties of the abelian variety defined over $\bar{\mathbf{Q}}$.In the main theorem we state the caracterization of the abelian variety of CM type by its property of periods. There are many applications of this theorem to get the transcendecy criterion for the special value of a modular function. As a typical example we can show the answer for the problem studied by Y.Morita (cf. [M]).

§1. We use the following notations.

A:g-dimensional abelian variety defined over $\bar{\mathbf{Q}}$ with a polarization

$End_0 A = (End A) \otimes \mathbf{Q}$,

$\omega_1, \cdots, \omega_g$: a basis system of holomorphic 1-forms on A defined over $\bar{\mathbf{Q}}$,

$\gamma_1, \cdots, \gamma_{2g}$: a basis system of $H_1(A, \mathbf{Z})$,

$\mathfrak{S}_g$: the Siegel upper half space of degree g.

We suppose A is given as the complex torus $\mathbf{C}^g/\Omega_2\mathbf{Z}^g + \Omega_1\mathbf{Z}^g$ for a certain period matrix ${}^t(\Omega_2, \Omega_1)$ in $M(2g, g, \mathbf{C})$. Set $\Lambda = \Omega_2\mathbf{Z}^g + \Omega_1\mathbf{Z}^g$ and let $E(x, y)$ be the Riemann form corresponding to the polarization on A. Let χ be the integral skew symmetric matrix obtained by putting $\chi_{ij} = E(e_i, e_j)$ for the canonical generator system $\{e_i\}$ of Λ. We may change the period matrix ${}^t(\Omega_2, \Omega_1)$ by $M^t(\Omega_2, \Omega_1)$ with a transformation M of $GL(2g, \mathbf{Z})$. By this procedure we may suppose the polarization χ is given by a $2g \times 2g$ matrix

$$\begin{pmatrix} O & \Delta \\ -\Delta & O \end{pmatrix}$$

with a certain integral diagonal $g \times g$ matrix Δ. Concerning this polarization the period relation is expressed as

$$ {}^t\begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} \chi \begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} = O$$

$$\sqrt{-1}\,{}^t\begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} \chi \begin{pmatrix} \bar{\Omega}_2 \\ \bar{\Omega}_1 \end{pmatrix} > 0.$$

Hence we obtain a point $\Omega = \Omega_2(\Delta\Omega_1)^{-1}$ of $\mathfrak{S}_g$.

The modular group

$$\Gamma = Sp(2g, \mathbf{Z}, \Delta) = \{M \in M(2g, \mathbf{Z}) | {}^t g\chi g = \chi\} \tag{1.1}$$

acts on $\mathfrak{S}_g$ by

$$M \circ \Omega = (A\Omega + B\Delta^{-1})(C\Omega + D\Delta^{-1})^{-1}\Delta^{-1} \tag{1.2}$$

,where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{Z}, \Delta)$. In the following we regard the Γ-equivalence class represented by Ω as the reduced period matrix of A. Our main theorem is stated as the following.

Main Theorem.
Let A be a g-dimensional polarized abelian variety defined over $\bar{\mathbf{Q}}$, and let Ω be the reduced period matrix of A. If Ω is an algebraic point of $\mathfrak{S}_g$, then A is an abelian variety of CM type.

Remark 1.1.. A simple abelian variety A is said to be of CM type if $End_0\, A$ is isomorphic to a certain number field of degree $2 \times \dim A$. When A is not simple, A is said to be of CM type if every simple component is of CM type.

Remark 1.2. When A is simple and defined over $\bar{\mathbf{Q}}$, the period $\int_{\gamma_i} \omega_k$ does not vanish for every i and k (cf. [**W-W**]).

Remark 1.3. Let Ω be a reduced period matrix of a polarized abelian variety (A, χ). The abelian variety A is of CM type if and only if Ω is an isolated fixed point of a certain element of $Sp(2g, \mathbf{Q}, \Delta)$,(cf.[**H-I**]). So we call it a CM-point.

We also have the following modified main theorem.

Theorem M'.
Let A be a g-dimensional polarized abelianvariety defined over $\bar{\mathbf{Q}}$. Let ω_i and γ_j $(1 \le i \le 1, 1 \le j \le 2g)$ be as above. Suppose the ratio $\dfrac{\int_{\gamma_i} \omega_k}{\int_{\gamma_j} \omega_k}$ is an algebraic number for any indices i, j and k whenever the denominator does not vanish. Then A is an abelian variety of CM type.

§2. As the applications of these two main theorems, we can show the following transcendency theorems.

1)The transcendency of the Igusa-Rosenhein modular mapping.

Let $\lambda : \mathfrak{S}_2 \longrightarrow \mathbf{P}^3$ be the Igusa-Rosenhein modular mapping defined by

$$\lambda(\Omega) = [\xi_0, \cdots, \xi_3]$$
$$= [\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}]$$

(cf. [I]). It is a modular mapping with respect to $\Gamma(2)$, the principal congruence subgroup of $Sp(2, \mathbf{Z})$ with level 2. Moreover λ gives a birational correspondence between $\mathfrak{S}/\Gamma(2)$ and $\mathbf{P}^2$.

THEOREM A-1. *Suppose Ω is an algebraic point of $\mathfrak{S}_2$, then $\lambda(\Omega)$ is an algebraic point of $\mathbf{P}^3$ if and only if Ω is a CM-point.*

Remark 2.1. The mapping λ is the inverse of the period mapping for the family of the curves of genus 2 given by the Legendre normal form

$$C(\xi) : y^2 = x \prod_{i=0}^{3} (x - \xi_i)$$

, where $[\xi_0, \cdots, \xi_3]$ is a parameter on $\mathbf{P}^3$.

2) The Picard modular mapping.

We use the following notations in this argument.

$\zeta = \exp(2\pi i/3), K = \mathbf{Q}(\zeta)$, $\mathcal{O}_K = \mathbf{Z} \bigoplus \mathbf{Z}\zeta$,

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Let D be the domain defined by

$$D = \{\eta \in \mathbf{P}^2(\mathbf{C}) : {}^t\bar{\eta} H \eta < 0\} = \{(u, v) \in \mathbf{C}^2 : 2\Re v + |u|^2 < 0\}$$

(by putting $v = \eta_1/\eta_0, u = \eta_2/\eta_0$), this is biholomorphically equivalent to the 2 dimensional hyperball. Let Γ be the modular group defined by

$$\Gamma = U(H, \mathcal{O}_K) = \{g \in M(3, \mathcal{O}_k) :^t \bar{g} H g = H\}.$$

We also consider the modular group with the level structure by $\sqrt{-3}$

$$\Gamma(\sqrt{-3}) = \{g \in \Gamma : g \equiv 1_3(mod.(\sqrt{-3}))\}.$$

Set

$$\Omega = \Omega(u, v)$$

$$= \begin{pmatrix} (u^2 + 2\omega^2 v)/(1-\omega) & \omega^2 u & (\omega u^2 - \omega 2v)/(1-\omega) \\ \omega^2 u & -\omega^2 & u \\ (\omega u^2 - \omega^2 v)/(1-\omega) & u & (u^2 + 2v)/(\omega - \omega^2) \end{pmatrix}$$

this Ω gives an embedding of D into $\mathfrak{S}_3$. Using above notations we define the mapping $\lambda : D \longrightarrow \mathbf{P}^2$ by $\lambda(u, v) = [\xi_0, \xi_1, \xi_2] =$

$$[\vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix} (\Omega), \vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} (\Omega), \vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{1}{6} & \frac{2}{3} \end{bmatrix} (\Omega)].$$

Remark 2.2. The mapping λ is the inverse of the period mapping for the following family of algebraic curves of genus 3 (cf. [S])

$$C(\xi) : y^3 = x \prod_{i=0}^{2} (x - \xi_i),$$

where $[\xi_i]$ is a parameter in

$$\mathbf{P}^2 - \{\prod_{i=0}^{2} \xi_i \prod_{j,k=0}^{2} (\xi_j - \xi_k) \neq 0\}.$$

Moreover λ induces the biholomorphic correspondence between the compactification of $D/\Gamma(\sqrt{-3})$ and $\mathbf{P}^2$.

THEOREM A-2. *Suppose the point $P = (u, v)$ varies on the algebraic points of D. Then $\lambda(u, v)$ is an algebraic point if and only if P is an isolated fixed point of an element in*

$$U(H, K) = \{g \in M(3, K) :^t \bar{g} H g = H\}.$$

3) The inverse of the Schwarz function for the Gauss hypergeometric differential equation.

Let $F(\alpha,\beta,\gamma)$ be the Gauss hypergeometric function and $D(\alpha,\beta,\gamma)$ be the correspnding hypergeometric differential equation:

$$x(x-1)y'' + \{\gamma + (1+\alpha+\beta)x\}y' - \alpha\beta y = 0 \tag{2.1}$$

Always the parameters α, β and γ are supposed to be rational numbers. Set

$$\lambda = 1-\gamma, \mu = \beta - \alpha, \nu = \gamma - \alpha - \beta$$

Let N be the least common multiplier of the denominators of α, β and γ. Put

$$1-\alpha = A/N, \alpha + 1 - \gamma = B/N, \beta = C/N$$

We assume the following condition for λ, μ and ν

$$\begin{cases} 1/\lambda, 1/\mu, 1/\nu \in \mathbf{Z} \cup \{0\}, \\ |\lambda| + |\mu| + |\nu| < 1. \end{cases} \tag{2.2}$$

Set

$$I(\infty, z) = \Gamma(\gamma)\{\Gamma(\alpha)\Gamma(\beta)\}^{-1} \int_0^\infty x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx \tag{2.3}$$

$$I(1, z) = \Gamma(\gamma)\{\Gamma(\alpha)\Gamma(\beta)\}^{-1} \int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx. \tag{2.3'}$$

Under the condition (2.2) $I(\infty, z)$ and $I(1,z)$ are the independent solution of (2.1). Moreover the Schwarz function

$$\sigma(z) = I(\infty, z)/I(1, z)$$

has a single valued inverse function defined on a bounded domain D (namely it is biholomorphically equivalent to the upper half plane H).

Here we use the notations according to **[Wo]**. The hypergeometric function $F(\alpha,\beta,\gamma : z) = I(\infty, z)$ can be considered as a period integral on the algebraic curve

$$X(N, z) : y^N = x^A(1-x)^B(1-zx)^C.$$

Let us denote its Jacobian variety by $Jac(X(N,z))$. Set

S= the system of linearly independent differential forms of the form $\omega_n = \frac{P(x)dx}{y^n}$ for a certain positive integer n.

Then we have $r = \sharp S = \varphi(N)$. So we put $S = \{\omega^{(1)}, \cdots, \omega^{(r)}\}$ and $\omega^{(i)} = \omega_{n_i}$.

Remark 2.3. We have $n_i \neq n_j$ for $i \neq j$ under the condition (2.2). Let ζ be the primitive N-th root of unity, and set $K = \mathbf{Q}(\zeta)$. We obtain a lattice Λ in $\mathbf{C}^r$ by putting

$$\Lambda = \{(\rho_i(a) \int_0^1 \omega^{(i)} + \rho_i(b) \int_0^\infty \omega^{(i)})_{1 \leq i \leq r} : a, b \in \mathcal{O}_K\},$$

where $\mathcal{O}_K$ is the ring of integers in K and ρ_i is the automorphism of K with $\rho_i(\zeta) = \zeta^{n_i}$. Then we obtain an abelian variety $T = \mathbf{C}^r/\Lambda$. According to Wolfart $Jac(X(N,z))$ is isogenous to

$$T \bigoplus \sum_{D|N, D \neq N} X(D,z).$$

THEOREM A-3. *Suppose τ is algebraic. Then $\lambda(\tau)$ is an algebraic number if and only if T is an abelian variety of CM type.*

Remark 2.4. According to Takeuchi (cf. [T]) there are 46 arithmetic triangle group obtained as the monodromy group of (2.1). In this case the monodromy group,to be considered acting on the upper half plane, is commensurable with a group of the form $SL(2, \mathcal{O}_k)$, where $\mathcal{O}_k$ is the ring of integers for a certain real algebraic field k. For these cases the above CM condition is equivalent to be a fixed point of an element of $GL(2,k)$.

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