数理解析研究所講究録
第 754 巻 1991 年 206-219

The number of orthogonal permutations

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概要

A problem on maximal clones in universal algebra leads to the natural concept of orthogonal orders and their characterization. Two (partial) orders on the same set P are orthogonal if they share only trivial endomorphisms i.e. if the identity selfmap of P is the sole non-constant selfmap preserving (i.e. compatible with) both orders. We start with a neat and easy characterization of orthogonal pairs of chains (i.e. linear or total orders) and then proceed to the study of the number $q(k)$ of chains on $\{0,1,\ldots,k-1\}$ orthogonal to the natural chain $0<1\ldots<k-1$. We obtain a recurrence formula for $q(k)$ and prove that the ratio $q(k)/k!$ (of such chains among all chains) goes to $e^{-2}=0.1353\ldots$ as $k\to\infty$. Results are formulated in terms of permutations.

1. Introduction

1.1.

Let k be an integer, $k>2$ and $\boldsymbol{k}:=\{0,1,\ldots,k-1\}$. For a positive integer n an *n-ary k-valued logic function* is a map $f:\boldsymbol{k}^n\to\boldsymbol{k}$ assigning a value from $\boldsymbol{k}$ to every n-tuple $(a_1,\ldots,a_n)$ over $\boldsymbol{k}$. For example, for $i=1,\ldots,n$ and $a\in\boldsymbol{k}$ the i-th *projection* (or trivial function) e_i^n and the constant c_a^n are defined by setting $e_i^n(a_1,\ldots,a_n):=a_i$ and $c_a^n\ (a_1,\ldots,a_n):=a$ for all $a_1,\ldots,a_n\in\boldsymbol{k}$. Denote $P_k^{(n)}$ the set of all n-ary k-valued logic functions and put $P_k:=\cup_{n=1}^{\infty}P_k^{(n)}$. A composition closed subset of P_k containing all projections is a *clone* on $\boldsymbol{k}$. Clones may be seen as multiple-valued analogs of transformation monoids (whereby the projections replace the neutral element) and they are basic for universal algebra, the propositional calculus of k-valued logics (or k-valued switching functions), theoretical computer science and automata theory. The set L_k of clones on $\boldsymbol{k}$, ordered by inclusion, is an (algebraic) lattice. The dual atom (or co-atoms, i.e. clones covered by the clone P_k), called *maximal* (or precomplete) clones, are known. In the difficult problem of basis classification (known only for $k=2$ [Jab52] and $k=3$ [Miy71] and some other clones, cf. [MSLR87]) a subproblem is to find all sets of maximal clones intersecting in a proper clone and maximal with respect to this property (i.e. if we add any maximal clone to the set, the intersection will be the least clone J_k of all projections). We address this problem in a very special case.

1.2.

Let $\leq$ be a (partial) *order* on $\boldsymbol{k}$ (i.e. a reflexive, antisymmetric and transitive binary relation on $\boldsymbol{k}$). The order is *bounded* if it has a least element o and a greatest element e (i.e. $o \leq x \leq e$ holds for all $x \in \boldsymbol{k}$). A function $f \in P_k^{(n)}$ is $\leq$*-isotone* (monotone, order preserving or order-compatible) if $f(a_1, \dots, a_n) \leq f(b_1, \dots, b_n)$ whenever $a_1 \leq b_1, \dots, a_n \leq b_n$. Denote $Pol \leq$ the set of all $\leq$-isotone $f \in P_k$. It is easy to see that $Pol \leq$ is a clone. Martiniuk [Mar60] showed that $Pol \leq$ is a maximal clone if and only if $\leq$ is bounded. Let $\leq$ and $\leq'$ be two orders on $\boldsymbol{k}$. Denote T the set of all projections and constants on $\boldsymbol{k}$. It is easy to see that T is a clone and that $Pol \leq \cap\, Pol \leq' \supseteq T$.

The discussion of Subsection 1.1 leads to the following problem: when is $Pol \leq \cap\, Pol \leq' = T$? This problem actually reduces to the following simpler problem (cf [DMRSS90]). A unary $\leq$-isotone operation is an *endomorphism* of $\leq$ and $End \leq\; := P_k^{(1)} \cap Pol \leq$. The orders $\leq$ and $\leq'$ are *orthogonal* if

$$End \leq \cap\; End \leq' = T^{(1)} := \{c_a^1 : a \in \boldsymbol{k}\} \cup \{e_1^1\},$$

i.e. if the identity selfmap is the only non-constant joint endomorphism of both $\leq$ and $\leq'$. Clearly $End \leq\; = End \geq$ and therefore $\leq$ and $\leq'$ are orthogonal exactly if $\geq$ and $\leq'$ are orthogonal. In other words, $\leq$ and $\leq'$ are orthogonal if and only if $\{\leq, \leq'\}$ is a semirigid relational system [LaPo84]. In [DMRSS90] we found a pair of orthogonal orders of height 1 for all $k > 5$ (with the exception of $k = 7$ but this can be fixed by another construction).

If we ask the question for bounded orders, chains (linear or total orders) are the simplest bounded orders to investigate. The above result easily yields the existence of 4 chains $\leq_1, \dots, \leq_4$ on $\boldsymbol{k}$ such that $\cap_{i=1}^4 End \leq_i = T^{(1)}$. A computer program found all pairs of chains orthogonal to $0 < 1 < \dots < k-1$ for $k \leq 7$ (cf. Tables 1–3) and this lead directly to a very simple characterization of orthogonal chains in Lemma 11 below. Now it was natural to ask about the number $q(k)$ of chains orthogonal to the natural chain $0 < 1 < \dots < k-1$. Our results for this enumeration problem, obtained in May–July 1990, are presented below. The fourth author presented the results of [DMRSS90] and work in progress at the CMS Summer Meeting (Halifax, N.S., Canada, June 1-3, 1990) and this lead to M. Haiman's independent results [Hai90] mentioned at the conclusion of this paper.

2. The number $q(k)$

2.1. Permutations and chains

We prefer to work with permutations rather than chains.

Definition 1. A permutation σ (i.e. a injective selfmap) of $\boldsymbol{k}$ induces the following chain (linear order relation) $\mathcal{R}(\sigma)$ on $\boldsymbol{k}$

$$\sigma(0) \sqsubset \sigma(1) \sqsubset \dots \sqsubset \sigma(k-1).$$

For example, the identity e_k induces the natural order $\mathcal{R}(e_k)$:

$$0 < 1 < \dots < k-1.$$

Example 2. We represent a permutation σ by the k-tuple $(\sigma(0)\sigma(1)\dots\sigma(k-1))$. For example, (021) represents the permutation $\binom{012}{021}$, and therefore $\mathcal{R}(021)$ stands for the order $0 \sqsubset 2 \sqsubset 1$.

Definition 3. Permutations σ and τ are *orthogonal* if the chains $\mathcal{R}(\sigma)$ and $\mathcal{R}(\tau)$ are orthogonal, i.e. if

$$End(\mathcal{R}(\sigma)) \cap End(\mathcal{R}(\tau)) = T^{(1)}.$$

A permutation can be regarded as a "renaming" of elements of $\boldsymbol{k}$. From this it follows that permutations σ and τ are orthogonal iff $\sigma\tau^{-1}$ and e are orthogonal. Thus the set of permutations orthogonal to an arbitrary permutation τ can be obtained if one knows the set $Q(k)$ of permutations orthogonal to the identity permutation e_k. Put

$$R(k) := S_k \setminus Q(k), q(k) := |Q(k)| \text{ and } r(k) := |R(k)|$$

where S_k denotes the symmetric group of all permutations of $\boldsymbol{k}$. We have $q(k)+r(k) = k!$.

Definition 4. A *segment* E of a permutation σ is a set of consecutive elements in σ:

$$\{\sigma(i), \sigma(i+1), \ldots, \sigma(i+l-1)\}.$$

and E is *nontrivial* if $1 < l < k$.

Example 5. The nontrivial segments of the permutation $e_4 = (0123)$ are $\{\{0,1\}, \{0,1,2\}, \{1,2\}, \{1,2,3\}, \{2,3\}\}$. The nontrivial segments of the permutation $\sigma := (2031)$ are $\{\{0,2\}, \{0,2,3\}, \{0,3\}, \{0,1,3\}, \{1,3\}\}$.

Lemma 6. *Two permutation σ and τ of $\boldsymbol{k}$ are orthogonal if and only if they share no nontrivial segment.*

Proof. ($\Leftarrow$). Let σ and τ be not orthogonal. Then a non-constant selfmap h of $\boldsymbol{k}$ different from e is both $\mathcal{R}(\sigma)$-isotone and $\mathcal{R}(\tau)$-isotone. Since $\mathcal{R}(\sigma)$ is a chain, clearly h is not a permutation and so $1 < |h^{-1}(a)| < k$ for some $a \in \boldsymbol{k}$. It is easy to see that $h^{-1}(a)$ is a segment of both σ and τ.

($\Rightarrow$). Let E be a common nontrivial segment of σ and τ, and a be an arbitrary element of E. Define a function h as follows:

$$h(x) := \begin{cases} x & \text{if } x \notin E, \\ a & \text{if } x \in E. \end{cases}$$

It is easy to see that h is nontrivial and both $\mathcal{R}(\sigma)$-isotone and $\mathcal{R}(\tau)$-isotone. Hence the permutations σ and τ are not orthogonal. □

Example 7. It is easy to check that $q(2) = q(3) = 0$. The permutations e and σ in Example 5 share no nontrivial segments. The permutation σ and its reverse $\sigma' = (1302)$ are the only permutations orthogonal to the permutation e_4 and so $q(4) = 2$. For example, the cyclic permutation $\tau := (1230)$ shares nontrivial segments $\{1,2\}$, $\{1,2,3\}$ and $\{2,3\}$ with e_k.

2.2. A recursive formula for $q(k)$

Definition 8. A *natural segmentation* is a nontrivial partition π of $\boldsymbol{k}$ into intervals. A permutation σ is *compatible* with π if each interval of π is a segment of σ.

Denote $R(k,s)$ the set of all permutations of $\boldsymbol{k}$ compatible with some natural segmentation having exactly s segments. Further put $R^*(k,2) := R(k,2)$ and

$$R^*(k,s) := R(k,s) \setminus \cup_{r=2}^{s-1} R^*(k,r)$$

for all $s \geq 3$.

Lemma 9.
(1) $R(k)=\cup_{s=2}^{k-1}R(k,s)=\cup_{s=2}^{k-1}R^*(k,s)$,
(2) $R^*(k,s)\cap R^*(k,s')=\phi$ *if and only if* $s\neq s'$.

Note that $\sigma\in R(k)$ belongs to $R^*(k,s)$ if and only if s is the least size of a natural segmentation π such that σ is compatible with π.

Definition 10. For a natural segmentation π of $\mathbf{k}$ with at least s intervals denote by $R^*(k,s;\pi)$ the set of all permutations in $R^*(k,s)$ compatible with π and put $r^*(k,s;\pi):=|R^*(k,s;\pi)|$.

Example 11. Consider the segmentation $\pi:=\{\{01\},\{23\}\}$. Then

$$R^*(4,2;\pi)=\{(0123),(0132),(1023),(1032),(2301),(2310),(3201),(3210)\}.$$

For an order $\leq$ on $\mathbf{k}$ we say that $E\subset\mathbf{k}$ precedes $E'\subset\mathbf{k}$ in $\leq$ if $a\leq b$ for all $a\in E$ and $b\in E'$.

Lemma 12. *Let E and E' be segments of a permutation σ.*
(1) If E and E' are not disjoint then their intersection $E\cap E'$ is also a segment of σ.
(2) If E and E' are disjoint then either E precedes E' or E' precedes E in $R(\sigma)$.

Proof. Obvious. □

Definition 13. Let $\pi:=\{E_0\dots,E_{s-1}\}$ be a natural segmentation of $\mathbf{k}$ with $E_i=[a_i,a_{i+1}-1]$ $(i=0,\dots,s-1)$ and $0=a_0<a_1<\dots<a_s=k$. Let σ be a permutation compatible with π and let $E_{j_0},\dots,E_{j_{s-1}}$ be the blocks of π as they appear in σ (from left to right). We denote the permutation $(i_0\dots i_{s-1})$ of $\mathbf{s}$ by σ^π and call it the *intersegment permutation* induced by σ and π.

2.2. 1 Evaluation of $R(k,2)$.

Definition 14. *For $0<j<k$ denote $\pi_j=(0\dots j-1|j\dots k-1)$ the natural segmentation $\{\{0,\dots,j-1\}\ \{j,\dots,k-1\}\}$.*

Lemma 15.
(1) $|R^*(k,2;\pi_j)|=2\cdot j!(k-j)!$ *for all* $0<j\leq k-1$.
(2) $|R^*(k,2;\pi_{j_1})\cap R^*(k,2;\pi_{j_2})\cap\ldots\cap R^*(k,2;\pi_{j_{t-1}})|$
$=2\cdot j_1!(j_2-j_1)!\cdot\ldots\cdot(k-j_{t-1})!$ *for all* $0<j_1<j_2\ldots<j_{t-1}\leq k-1$.

Proof. (1) A permutation σ compatible with π_j is determined by 1) the intersegment permutation $\sigma^{\pi_j}\in S_2$, 2) a permutation of $\{0,\dots,j-1\}$ and 3) a permutation of $\{j,\dots,k-1\}$.

(2) Let $\sigma\in R^*(k,2;\pi_{j_1})\cap R^*(k,2;\pi_{j_2})\ldots\cap R^*(k,2;\pi_{j_{t-1}})$. By Lemma 12 the permutation σ is compatible with the natural segmentation

$$\pi=\{0,1,\dots,j_1-1\},\dots,\{j_{t-1},\dots,k-1\}.$$

Put $i:=\sigma^\pi(0)$. We have that $i\in\{0,t-1\}$ because, were $0<i<t-1$ then $\{j_1,\dots,k-1\}$ would not be a segment of σ. First consider the case $i=0$. Using $\sigma\in R(k,2,\pi_{j_s})$ for $s=1,\dots,t-1$, an easy induction shows that $\sigma^\pi=01\cdots(t-1)$. Similarly, if $i=t-1$ we get $\sigma^\pi=(t-1)(t-2)\cdots0$. The formula follows in the same way as in (1). □

Example 16.

$$R^*(5,2;\pi_2)\cap R^*(5,2;\pi_3)=\{(01234),(01243),(10234),(10243),(34201),(34210),(43201),(43210)\}.$$

Lemma 17. $r^*(k,2) = 2\sum_{s=2}^{k}(-1)^s \sum_{n_1+n_2+\ldots+n_s=k} n_1!n_2!\cdots n_s!$,
(where the second sum is over positive integers $n_1, \ldots, n_s$*).*

Proof. We apply Lemma 15 and the "principle of inclusion and exclusion (the sieve formula)" to the union of $R^*(k,2;j)$'s.

$$|R^*(k,2)| = \sum_{t=1}^{k-1}(-1)^{t+1} \sum_{0<j_1<\ldots<j_t\leq k-1} |R^*(k,2;\pi_{j_1}) \cap \cdots \cap R^*(k,2;\pi_{j_t})|$$

$$= \sum_{t=1}^{k-1}(-1)^{t+1}2 \sum_{0<j_1<j_2\ldots<j_t\leq k-1} j_1! \cdots (k-j_t)!.$$

By putting $s := t+1$, $n_1 := j_1$, $n_2 := j_2 - j_1$, $\ldots$ and $n_s := k - j_t$ we have the desired formula. □

Lemma 18. *Let* π *and* π' *be distinct natural segmentations with* s *and* s' *segments where* $3 \leq s \geq s'$. *If a permutation* σ *is compatible with both* π *and* π' *then:*

(1) The induced intersegment permutation σ^π *(on* s*) is not in* $Q(s)$*.*

(2) The permutation σ *is not in* $R^*(k,s;\pi)$*, i.e. it is contained in* $R(k,s'')$ *for some* $s'' < s$*.*

Proof. 1) Since $s \geq s'$ and $\pi \neq \pi'$, there is an segment E' of π' not contained in any segment of π. Let $\pi = \{E_0, \ldots E_{s-1}\}$ where the segments are listed in their natural order. Put $L := \{l \in s : E' \cap E_l \neq \phi\}$ and $i = \min L$, $j = \max L$. Then $i < j$ and $E' \subseteq E_i \cup E_{i+1} \cup \ldots \cup E_j$.

a) First consider the case $i = 0$ and $j = s-1$. Since π' is a proper partition, at least one of the sets $E_0 \setminus E'$ and $E_{s-1} \setminus E'$ is non-empty, say $E_0 \setminus E' \neq \phi$. However, then E_0 is an initial or terminal segment of σ and so $\{1, \ldots, s-1\}$ a segment of σ^π and $\sigma^\pi \notin Q(s)$ due to $s \geq 3$.

b) Thus let $i \neq 0$ or $j \neq s-1$. The permutation σ is compatible with the nontrivial natural segmentation $E_0, \ldots, E_{i-1}, E_i \cup \ldots \cup E_j, E_{j+1} \ldots, E_{s-1}$. Thus the set $\{i, \ldots, j\}$ is a nontrivial segment of the permutation σ^π, and hence σ^π is not in $Q(s)$ proving 1). Moreover, in both cases a) and b) the assertion (2) is easily verified. □

Corollary 19. *Let* π *and* π' *be distinct natural segmentations of* k *with* s *segments where* $s \geq 3$.

(1) Let $\sigma \in R(k,s)$ *be compatible with* π. *Then* $\sigma \in R^*(k,s)$ *if and only if* $\sigma^\pi \in Q(s)$.

(2) $R^*(k,s;\pi) \cap R^*(k,s;\pi') = \phi$.

Proof. 1) Let $\pi = \{E_0, \ldots, E_{s-1}\}$ where $E_0, \ldots, E_{s-1}$ are in natural order. ($\Rightarrow$). Suppose $\sigma^\pi \notin Q(s)$. Then there is a nontrivial segment $\{i, \ldots, j\}$ of σ^π and σ is compatible with the non-trivial segmentation $E_0, \ldots, E_{i-1}, E_i \cup \ldots \cup E_j, E_{j+1}, \ldots, E_{s-1}$ having less than s segments proving $\sigma \notin R^*(k,s;\pi)$.

($\Leftarrow$). Let $\sigma \notin R^*(k,s;\pi)$. By definition then $s \geq 3$ and $\sigma \in R(k,s')$ for some $s' < s$. Denote π' the corresponding natural segmentation. According to Lemma 18 we have $\sigma^\pi \notin Q(s)$.

(2) If $\sigma \in R^*(k,s;\pi)$ then by what we have proved $\sigma^\pi \in Q(s)$, and hence by Lemma 18 (1) (with $s' = s$) we have $\sigma \notin R(k,s,\pi') \supset R^*(k,s;\pi')$. □

Definition 20. Put $g(k,s) := \sum_{n_1+n_2+\ldots+n_s=k} n_1!n_2!\cdots n_s!$, (where we sum over positive integers $n_1, \ldots, n_s$). Note that $g(k,1) := k!$ and $g(k,k) = 1$ and $r^*(k,2) = 2\sum_{s=2}^{k}(-1)^s g(k,s)$.

2.2. 2 Evaluation of $r^*(k,s)$ for $s \geq 3$.

Recall that by Definition 13 every permutation σ compatible with π induces the intersegment-permutation σ^π of the segments of π.

Lemma 21. *Let* $s > 2$. *Then*
(1) If π *is a segmentation of* k *with segments* E_i *of size* n_i $(i = 0, \ldots, s-1)$ *then* $r^*(k,s;\pi) = q(s)\, n_1! \cdots n_s!$.
(2) $r^*(k,s) = q(s)g(k,s)$.

Proof. (1) A permutation σ compatible with π is determined by σ^π and the permutations of E_i $(i = 0, \ldots, s-1)$. Now $\sigma \in R^*(k,s;\pi)$ if and only if $\sigma^\pi \in Q(s)$. Therefore the number of permutations in $R^*(k,s;\pi)$ is given by $q(s)n_1! \cdot \ldots \cdot n_s!$.

(2) By Corollary 19 the sets $R^*(k,s;\pi)$ and $R^*(k,s;\pi')$ are disjoint for distinct segmentations π and π'. Therefore

$$r^*(k,s) = \sum_{n_1+\ldots+n_s=k} q(s)n_1! \cdot \ldots \cdot n_s! = q(s)g(k,s).$$

□

The following will serve for a recursive formula for $q(k)$.

Theorem 22.

$$k! = \sum_{s=2}^{k}((-1)^s 2 + q(s))g(k,s).$$

Proof. By Lemmas 17 and 21

$$\begin{aligned} k! &= r(k) + q(k) = r(k,2) + \sum_{s=3}^{k-1} r^*(k,s) + q(k) \\ &= \sum_{s=2}^{k}(-1)^s 2 \cdot g(k,s) + \sum_{s=3}^{k-1} q(s)g(k,s) + q(k). \end{aligned}$$

Since $q(2) = 0$ and $g(k,k) = 1$, the above equation becomes

$$k! = \sum_{s=2}^{k}(-1)^s 2 \cdot g(k,s) + \sum_{s=2}^{k} q(s)g(k,s) = \sum_{s=2}^{k}((-1)^s 2 + q(s))g(k,s).$$

□

Corollary 23.

$$q(k) = k! - (-1)^k 2 - \sum_{s=2}^{k-1}((-1)^s 2 + q(s))g(k,s).$$

Example 24. We recalculate $q(3)$. Since $3 = 2+1$, we have $g(3,2) = 2(1!2!) = 4$ and using $q(2) = 0$ from Example 7

$$q(3) = 3! + 2 - (2 + q(2))g(3,2) = 8 - 2 \cdot 4 = 0.$$

Now we recalculate $q(4)$. We have

$$q(4) = 4! - (2 + q(2))g(4,2) + (-2 + q(3))g(4,3) = 4! - 2g(4,2) + 2g(4,3).$$

From $4 = 3+1 = 2+2$ we get $g(4,2) = 2 \cdot 3! + 2!2! = 16$. Similarly from $4 = 2+1+1$ we obtain $g(4,3) = 3 \cdot 2! = 6$ and so

$$q(4) = 24 - 2 - 32 + 12 = 2,$$

the same value we found in Example 7.

Using $g(5,2) = 72$, $g(5,3) = 30$ and $g(5,4) = 8$ we obtain

$$q(5) = 5! + 2 - (2 \cdot 72 - 2 \cdot 30 + 4 \cdot 8) = 6.$$

(cf Table 4 at the end of the paper).

In the following tables we list one half of the set $Q(k)$ (of all the permutations orthogonal to e_k) for $k = 5, 6, 7$. To obtain $Q(k)$ just add the reverse permutations.

3. The asymptotic behavior of $q(k)/k!$

In what follows we consider the ratio $q(k)/k!$ (the proportion of permutations orthogonal to e_k among all permutations). We show that this ratio tends to e^{-2} when k tends to infinity (where $e = 2.7182...$ is the base of natural logarithms). The key is our equality from Theorem 22

$$k! = \sum_{s=2}^{k} (2(-1)^s + q(s))g(k,s).$$

Lemma 25.

$$\sum_{s=2}^{k} c(s)s!g(k,s)/k! = 1.$$

Later we will need the following properties of $q(k)$.

Lemma 26. (1) $q(k) \geq (k-4)q(k-1)$ *for all* $k \geq 5$,
(2) $q(k) \geq (2k-8)q(k-1) - q(k-2)$ *for all* $k \geq 5$,
(3) $q(k) \geq (k-3)q(k-1) + 2k + 4$ *for all* $k \geq 7$.

Proof. 1) Let $\tau \in Q(k-1)$ and let $\tau(i) = k-2$. For $j \in \{1, \dots, k-2\} \setminus \{i, i+1\}$ define $\tau^{(j)} \in S_k$ by $\tau^{(j)}(l) := \tau(l)$ for $l < j$, $\tau^{(j)}(j) := k-1$ and $\tau^{(j)}(l) := \tau(l-1)$ for $l > j$. For example, if $k = 5$ and $\tau = 1302$ we have $\tau^{(3)} = 13042$. Using $\tau \in Q(k-1)$ it is not difficult to see that $\tau^{(j)} \in Q(k)$ and (1) follows.

(2) Let $\tau \in Q(k-1)$ and $\tau(i) = 0$. For $j \in \{1, \dots, k-2\} \setminus \{i, i+1\}$ put $\tau_{(j)}(l) := \tau(l)+1$ for $l < j$, $\tau_{(j)}(j) := 0$ and $\tau_{(j)}(l) := \tau(l-1)+1$ for $l > j$. Again $\tau_{(j)} \in Q(k)$ and so we get $k-4$ elements of $Q(k)$. However, it is possible that $\tau^{(j)} = \sigma_{(j')}$ for some $\tau, \sigma \in Q(k-1)$ and $j, j' \in \{1, \dots, k-2\}$. It may be shown that this happens exactly if there is $\lambda \in Q(k-2)$ such that $\tau = \lambda_{(l)}$ and $\sigma = \lambda^{(m)}$ for some l and m. For example, if $\lambda = 1302$ we have $\tau = \lambda^{(3)} = 13042$ and $\sigma = \lambda_{(1)} = 20413$ and $\tau_{(1)} = \sigma^{(4)} = 204153$. Now it is easy to see that (2) holds.

(3) By (1) we have $q(k-1) \geq (k-5)q(k-2)$ and so

$$(k-5)q(k-1) \geq (k-5)^2 q(k-2). \tag{1}$$

By direct computation the real function $\varphi(x) := (2x+4)/((x-5)^2 - 1)$ is decreasing for $x \geq 7$ and so its maximum on $[7, \infty)$ is $\varphi(7) = 6$. Now by Example 24 we have $q(k-2) \geq q(5) = 6 \geq \varphi(k) = (2k+4)/((k-5)^2 - 1)$. Finally by 2) and (1)

$$\begin{aligned} q(k) &\geq (2k-8)q(k-1) - q(k-2) = (k-3)q(k-1) + (k-5)q(k-1) - q(k-2) \\ &\geq (k-3)q(k-1) + 2k + 4. \quad \square \end{aligned}$$

Put $c(s) := (2(-1)^s + q(s))/s!$. Obviously $c(s) \sim q(s)/s!$ for large s.

Note 27. From Example 24 we have: $c(2) = 1$, $c(3) = -1/3$, $c(4) = 1/6$, $c(5) = 1/30$.

Lemma 28.

$$
\begin{aligned}
&(1)\ c(k) \ \geq\ (k-3)c(k-1)/k \quad \textit{for } k \geq 7,\\
&(2)\ c(k) \ \leq\ 1 \quad \textit{for } k \geq 2.
\end{aligned}
$$

Proof. (1) First we use Lemma 26(3):

$$
\begin{aligned}
c(k) &= (2(-1)^k + q(k))/k! \geq (2(-1)^k + (k-3)q(k-1) + 2k + 4)/k!\\
&\geq (k-3)(2(-1)^{k-1} + q(k-1))/k! \ +\ ((2(-1)^k + 2k + 4 - 2(-1)^{k-1}(k-3))/k!\\
&= (k-3)c(k-1)/k \ +\ (2(-1)^k + 10 + 2(k-3)(1+(-1)^{k-1}))/k!\\
&\geq (k-3)c(k-1)/k.
\end{aligned}
$$

(2) Since the identity permutation e_k and its reverse $((k-1)(k-2)\cdots 1)$ are in $R(k)$ (and hence not in $Q(k)$), we have $q(k) \leq k! - 2$, and therefore

$$c(k) = (2(-1)^k + q(k))/k! \leq (2 + q(k))/k! \leq 1.$$

□

Corollary 29.

(1) $c(s) \geq (k-r)(k-r-1)(k-r-2)c(k-r)/(s(s-1)(s-2))$ *for* $s \geq k-r \geq 6$

(2) $c(s) \leq k(k-1)(k-2)c(k)/(s(s-1)(s-2))$ *for* $k \geq s \geq 6$

Proof. (1) Repeated application of Lemma 28(1) and obvious cancellation. 2) From (1).
□

Now we derive bounds for $g(k,s)$.

Lemma 30.

$$g(k,s) \ <\ 4^{s-1}(k-s+1)! \quad \textit{for } 1 < s \leq k.$$

Proof. We use induction on $s > 1$. First we show the equation for $s = 2$. From definition 20

$$g(k,2) = (k-1)!1! + (k-2)!2! + \cdots, +1!(k-1)! < 2(k-1)! + (k-3)(k-2)!2! < 4(k-1)!.$$

Assume $g(k',s') < 4^{s'-1}(k'-s'+1)!$ holds for all s' and k' such that for $1 < s' \leq k' < k$ and $s' < s$. Now, by definition 20 and applying twice the induction hypothesis

$$g(k,s) = \sum_{n=1}^{k-s+1} n!g(k-n,s-1) < 4^{s-2}\sum_{n=1}^{k-s+1} n!(k-n-s+2)! = 4^{s-1}g(k-s+2,2) < 4^{s}(k-s+1)!.$$

□

Corollary 31. *For all* $k \geq r \geq 5$,

$$1 + 8/k > \sum_{s=r}^{k} c(s)s!g(k,s)/k!.$$

Proof. The values $c(s)$ are all positive except $c(3) = -1/3$. From Lemma 30 we have $g(k,3) < 4^2(k-2)!$ and since $k-1 \geq 4$ so

$$c(3)3!g(k,3)/k! = -2g(k,3)/k! > (-2)4^2/(k(k-1)) > -8/k$$

The statement is now immediate from Lemma 25. □

Denote $\boldsymbol{N}_+$ the number $\{1,2,\ldots\}$ of positive integers and put $A(k,s) := \{(n_1,\ldots,n_s) \in \boldsymbol{N}_+^s : n_1 + \ldots + n_s = k\}$. It is well known that ${}_sH_{k-s} := |A(k,s)| = \binom{k-1}{s-1}$.

Lemma 32. *$s!g(k,s)/k! \geq s2^{k-s}/(k(k-s)!)$ for all $1 \leq s \leq k$.*

Proof. From $n! \geq 2^{n-1}$ we obtain $n_1! \cdot \ldots \cdot n_s! \geq 2^{k-s}$. Therefore

$$s!g(k,s)/k! \geq s!\,{}_sH_{k-s}2^{k-s}/k! = s2^{k-s}/(k(k-s)!).$$

□

Lemma 33. *If $1 \leq s \leq k-5$ then $s!g(k,s)/k! < 1512/(k(k-1)) + 24(-1/(k-s) + 1/(k-s-1))$.*

Proof. Let $n_1, n_2, \ldots, n_s$ be positive integers summing up to k. We divide the summation of the products

$$n_1! \cdot \ldots \cdot n_s! \tag{2}$$

into partial sums, according to the value $N := \max\{n_1, \ldots n_s\}$.

1) Case 1. $N = k-s+1$. There is i such that $n_i = k-s+1$ and $n_j = 1$ for each $j \neq i$. There are s choices for such i and so the sum of the products of the form (2) is $s(k-s+1)!$.

2) Case 2. $N = k-s$. In a similar way we have

$$s(s-1)(k-s)!2! < 2s^2(k-s)!.$$

3) Case 3. $N = k-s-1$. There are only two types of combinations of n_i's:

    1. $n_i = k-s-1$ and $n_j = 3$ for some i and j, and
    2. $n_i = k-s-1$ and $n_j = n_{j'} = 2$ for some i, j and j'.

    The sum of the products for these cases are $6s(s-1)(k-s-1)!$ and $4s(s-1)(s-2)(k-s-1)!/2$, respectively. Summing these two we have

$$2s(s^2-1)(k-s-1)! < 2s^3(k-s-1)!.$$

4) Case 4. $N \leq k-s-2$. Every product (2) is bounded by $(k-s-2)! \cdot 4!$. Indeed, suppose $k-s-2 \geq n_1 \geq \ldots n_s > 0$ and $n_1 + \ldots + n_s = k$ Note that $(x+1)!(y-1)! \geq x!y!$ whenever $x+1 \geq y > 1$. Applying this several times we obtain the required $n_1! \ldots n_s! \leq (k-s-2)!4!1! \ldots 1!$. Since the number of all possible combinations of n_i's is ${}_sH_{k-s}$, the partial sum of the products (2) for $N \leq k-s-2$ is bounded by $(k-s-2)!4!(k-1)!/((k-s)!(s-1)!)$.

Thus $g(k,s)$ is bounded by

$$s(k-s+1)! + 2s^2(k-s)! + 2s^3(k-s-1)! + (k-s-2)!4!(k-1)!/((k-s)!(s-1)!). \tag{3}$$

Now we proceed to evaluate the bound (3) multiplied by $s!/k!$ (as an upper bound for $s!g(k,s)/k!$).

(1) The first term of (3) can be rewritten as

$$\frac{ss!(k-s+1)!}{k!} = (s/(k-2))(\prod_{i=0}^{s-5}(s-i)/(k-3-i))(4!/k(k-1)) < 24/(k(k-1)),$$

since $s < k-2$ and $s-i < k-3-i$ for all i.

(2) The second term.

$$2s^2s!(k-s)!/k! = (s/(k-2))(s/(k-3))\cdot(\prod_{i=0}^{s-5}(s-i)/(k-4-i))\cdot 2\cdot 4!/(k(k-1)) < 48/k(k-1).$$

(3) The third term. In a similar way we have:

$$2s^3s!(k-s+1)!/k! = (s/(k-2))(s/(k-3))(s/(k-4))(\prod_{i=0}^{s-7}(s-i)/(k-5-i))\cdot 2\cdot 6!/(k(k-1)) < 1440/k(k-1).$$

(4) The final term is easier.
$s!(k-s-2)!4!(k-1)!/(k!(k-s)!(s-1)!) = s4!/(k(k-s)(k-s-1)) < 24/((k-s)(k-s-1)) = 24(-1/(k-s)+1/(k-s-1))$.
Summing up the results of (1) – (4) we have the desired result. □

Corollary 34. *If* $5 \leq r \leq k-2$ *then* $\sum_{s=2}^{k-r} s!g(k,s)/k! < 1488/k + 24/(r-1)$.

Proof. $\sum_{s=2}^{k-r} s!g(k,s)/k! < \sum_{s=2}^{k-r} 1512/k(k-1) + 24\sum_{s=2}^{k-r}(-1/(k-s)+1/(k-s-1)) < (k-1)1512/((k(k-1)) + 24/(r-1) - 24/(k-2) < 1512/k + 24/(r-1) - 24/k = 1488/k + 24/(r-1)$. □

Lemma 35. *If* $k > s$ *then* $g(k,s) < \sum_{t=1}^{k-s} 2^{t-1}\binom{s}{t}\binom{k-s-1}{t-1}(k-s-t+2)!$.

Proof. Consider positive integers $n_1, \ldots, n_s$ such that $n_1 + \ldots + n_s = k$. Let $n_{i_j} \geq 2$ for $j = 1, \ldots, t$ and $n_l = 1$ for all $l \in \{1, \ldots, s\} \setminus \{i_1, \ldots, i_t\}$. Note that

$$n_{i_1} + \ldots + n_{i_t} = k - s + t.$$

In particular, $2t \leq n_{i_1} + \ldots + n_{i_t} = k-s+t$ and so $1 \leq t \leq k-s$. For $x \geq y \geq 2$ we have $x!y! \leq 2(x+y-2)!$, because $x!y! \leq (x+1)!(y-1)! \leq (x+2)!(y-2)! \leq \ldots \leq (x+y-2)!2!$. Applying this successively

$$n_{i_1}!\ldots n_{i_t}! \leq 2^{t-1}(n_{i_1} + \ldots + n_{i_t} - 2(t-1))! = 2^{t-1}(k-s-t+2)!.$$

There are $\binom{s}{t}$ choices of $I := \{i_1, \ldots, i_t\}$. Moreover, $n_{i_1}-1, \ldots, n_{i_t}-1$ are positive numbers summing up to $k-s+t-t = k-s$ and so for a fixed I there are ${}_tH_{k-s} = \binom{k-s-1}{t-1}$ choices of $n_{i_1}, \ldots, n_{i_t}$. Together this yields the upper bound. □

Corollary 36. *If* $k \geq s \geq k/2$ *then*

$$s!g(k,s)/k! < 2^{k-s}/(k-s)! + D/k,$$

where $D = 22e^2$.

Proof. If $s = k$ then this inequality is obvious. Suppose that $s < k$. Then by Lemma 35

$$s!g(k,s)/k! < \sum_{t=1}^{k-s} W(t),$$

where $W(t) := 2^{t-1}s!\binom{s}{t}\binom{k-s-1}{t-1}(k-s-t+2)!/k!$. We have

$$W(k-s) = 2^{k-s-1}s!\binom{s}{k-s}\binom{k-s-1}{k-s-1}/k! = \frac{s!s!}{(2s-k)!k!}\frac{2^{k-s}}{(k-s)!},$$

Now from $2s-k < s < k$ and $2s-k+i < s+i$ for $i = 1, \ldots, k-s$ we have

$$\frac{s!s!}{(2s-k)!k!} = \frac{(2s-k+1)\cdots s}{(s+1)\cdots k} < 1$$

and so $W(k-s) < 2^{k-s-1}/(k-s)!$.

Next, if $t \leq k-s-1$ then

$$\begin{aligned}W(t) &= 2^{t-1}\frac{s!}{k!t!}\frac{s!}{(s-t)!}\frac{(k-s-1)!}{(t-1)!(k-s-t)!}(k-s-t+2)! \\ &= \frac{2^{t-1}}{k(t-1)!}\frac{s!}{(s-t)!}\frac{s!}{t!}\frac{(k-s-t+1)(k-s-t+2)}{(k-1)(k-2)\cdots(k-s)}.\end{aligned}$$

From $s \leq k-1$ we have $s-i \leq k-1-i$ $(i=0,\ldots,t-1)$ and from $s \leq k-t-1$ also $s-i \leq k-t-1-i$ $(i=0,\ldots,s-t-3)$. Thus

$$\begin{aligned}s!(s-t)!/(s!t!) &= s(s-1)\cdots(s-t+1)s(s-1)\cdots(t+1) \\ &\leq (k-1)(k-2)\cdots(k-t)(k-t-1)(k-t-2)\cdots(k-s+2)\cdot(t+2)(t+1)\end{aligned}$$

and so

$$W(t) \leq \frac{2^{t-1}}{k(t-1)!}\frac{(k-s-t+1)(k-s-t+2)}{(k-s+1)(k-s)}(t+2)(t+1) \leq 2^{t-1}(t+2)(t+1)/(t-1)!.$$

Thus we have

$$s!g(k,s)/k! < 2^{k-s}/(k-s)! + (1/k)\sum_{t=1}^{k-s-1}(t+2)(t+1)2^{t-1}/(t-1)!.$$

Since the infinite series of positive terms

$$\sum_{t=1}^{\infty}(t+2)(t+1)2^{t-1}/(t-1)!$$

converges to $D = 22e^2$ (differentiate twice the Maclaurin series for $(1/2)x^3e^x$ and evaluate at $x=2$), we have

$$s!k!g(k,s) < 2^{k-s}/(k-s)! + D/k. \quad \square$$

Theorem 37. $\lim_{k\to\infty} q(k)/k! = e^{-2}$.

Proof. We prove the equivalent $\lim_{n\to\infty} c(k) = e^{-2}$. First we show that

$$\overline{\lim}_{n\to\infty} c(k) \leq e^{-2}.$$

By Corollary 31 (replace r by $k-r$)

$$1+8/k > \sum_{s=k-r}^{k} c(s)s!g(k,s)/k!$$

for $k-5 \geq r$. By Corollary 29(1)

$$c(s) > \frac{k-r}{s}\cdot\frac{k-r-1}{s-1}\cdot\frac{k-r-2}{s-2}c(k-r) \geq \frac{k-r}{k}\cdot\frac{k-r-1}{k-1}\cdot\frac{k-r-2}{k-2}c(k-r).$$

On the other hand, by Lemma 32 and summing by $t := k-s$

$$\sum_{s=k-r}^{k}\frac{s!}{k!}g(k,s) \geq \sum_{s=k-r}^{k}\frac{s}{k}\frac{2^{k-s}}{(k-s)!} = \frac{1}{k}\sum_{t=0}^{r}\frac{(k-t)2^t}{t!} = \sum_{t=0}^{r}\frac{2^t}{t!} - \frac{2}{k}\sum_{t=1}^{r}\frac{2^{t-1}}{(t-1)!}.$$

Let $\varepsilon > 0$. When r (and k) is sufficiently large this value is greater than $e^2 - \varepsilon - (2/k)e^2$. Thus we have

$$1+8/k > \frac{k-r}{k}\cdot\frac{k-r-1}{k-1}\cdot\frac{k-r-2}{k-2}c(k-r)(e^2-\varepsilon-(2/k)e^2).$$

If we let k go to infinity while keeping r constant, we have

$$1 \geq \overline{\lim}_{n\to\infty} c(k)(e^2 - \varepsilon).$$

Since ε was arbitrary, we get $\overline{\lim}_{n\to\infty} c(k) \leq e^{-2}$.

Now we show the inequality $\underline{\lim}_{n\to\infty} c(k) \geq e^{-2}$. Let $k/2+1 \geq r \geq 5$. By Lemma 25

$$1 = \sum_{s=2}^{k} c(s)s!g(k,s)/k! = \sum_{s=2}^{k-r} c(s)s!g(k,s)/k! + \sum_{s=k-r+1}^{k} c(s)s!g(k,s)/k!.$$

By Corollary 34 the first sum is less than $1488/k + 24/(r-1)$, while by Corollaries 36 and 29(2) the second one can be bounded as follows

$$\sum_{s=k-r+1}^{k} c(s)s!g(k,s)/k! < \sum_{s=k-r+1}^{k} (2^{k-s}/(k-s)! + D/k)c(s)$$

$$< \sum_{s=k-r+1}^{k} (2^{k-s}/(k-s)! + D/k)k(k-1)(k-2)c(k)/(s(s-1)(s-2))$$

$$< \frac{k}{k-r}\cdot\frac{k-1}{k-r-1}\cdot\frac{k-2}{k-r-2}c(k)\sum_{t=0}^{r-1}\left(\frac{2^t}{t!}+\frac{D}{k}\right) < \frac{k}{k-r}\cdot\frac{k-1}{k-r-1}\cdot\frac{k-2}{k-r-2}c(k)(e^2+Dr/k).$$

If we let k go to infinity for a fixed r we have

$$1 \leq 24/(r-1) + e^2 \underline{\lim}_{n\to\infty} c(k).$$

Since r can be taken arbitrarily large, we have

$$e^{-2} \leq \underline{\lim}_{n\to\infty} c(k).$$

This completes the proof of our theorem. □

Remark. Using formal power series Mark Haiman from M.I.T. independently obtained results [Hai90] which include some of our results.

Put $h(1) := 1$ and $h(s) := -2(-1)^s - q(s)$ for $s \geq 2$ and consider the power series

$$\mu(x) := \sum_{s=1}^{\infty} h(s)x^s, \quad \nu(x) := \sum_{n=1}^{\infty} n!x^n.$$

Then by our Theorem 22

$$\sum_{s=1}^{k} h(s)g(k,s) = \begin{cases} 0 & \text{for } k \geq 2, \\ 1 & \text{for } k = 1. \end{cases}$$

leading to to $\mu(\nu(x)) = x$. This inversion can be directly calculated, for example, using Mathematica. We got the following table from Mark Haiman [Hai90]. The numbers $q(1)$–$q(7)$ coincide with the data we obtained by direct enumeration.

k	1	2	3	4	5	6	7	8	9	10
$h(k)$	1	-2	2	-4	-4	-48	-336	-2,928	-28,144	298,528
$q(k)$	1	0	0	2	6	46	338	2,926	28,146	298,526
$q(k)/k!$			0.0	0.08333	0.05	0.06389	0.06706	0.07256	0.07756	0.08226

The convergence of the ratio $q(k)/k!$ to $e^{-2} = 0.1353\ldots$ can be seen from

k	20	30	40	50	60	70	80	90	100
$q(k)/k!$	0.1086	0.1175	0.1219	0.1246	0.1264	0.1277	0.1286	0.1294	0.1300

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Table 1. $q(5) = 6$: orthogonal permutations to (01234)

(1 3 0 4 2), (1 4 2 0 3), (2 0 4 1 3)

Table 2. $q(6) = 46$: orthogonal permutations to (012345)

(1 3 0 5 2 4) (1 3 5 0 2 4) (1 3 5 0 4 2) (1 3 5 2 0 4) (1 4 0 2 5 3)
(1 4 0 3 5 2) (1 4 2 0 5 3) (1 4 2 5 0 3) (1 5 2 4 0 3) (1 5 3 0 2 4)
(1 5 3 0 4 2) (2 0 3 5 1 4) (2 0 4 1 5 3) (2 0 5 3 1 4) (2 4 0 5 1 3)
(2 4 1 5 0 3) (2 5 0 3 1 4) (2 5 0 4 1 3) (2 5 1 3 0 4) (2 5 1 4 0 3)
(3 0 2 5 1 4) (3 1 5 0 2 4) (3 1 5 2 0 4)

Table 3. $q(7) = 338$: orthogonal permutations to (0123456)

(1 3 0 4 6 2 5) (1 3 0 5 2 6 4) (1 3 0 6 4 2 5) (1 3 5 0 2 6 4) (1 3 5 0 4 6 2)
(1 3 5 0 6 2 4) (1 3 5 0 6 4 2) (1 3 5 2 0 6 4) (1 3 5 2 6 0 4) (1 3 6 0 4 2 5)
(1 3 6 0 5 2 4) (1 3 6 2 4 0 5) (1 3 6 2 5 0 4) (1 3 6 4 0 2 5) (1 3 6 4 0 5 2)
(1 3 6 4 2 0 5) (1 4 0 2 6 3 5) (1 4 0 3 6 2 5) (1 4 0 5 2 6 3) (1 4 0 5 3 6 2)
(1 4 0 6 2 5 3) (1 4 0 6 3 5 2) (1 4 2 0 6 3 5) (1 4 2 5 0 6 3) (1 4 2 6 0 3 5)
(1 4 2 6 0 5 3) (1 4 2 6 3 0 5) (1 4 6 0 2 5 3) (1 4 6 0 3 5 2) (1 4 6 2 0 3 5)
(1 4 6 2 0 5 3) (1 4 6 2 5 0 3) (1 4 6 3 0 2 5) (1 4 6 3 0 5 2) (1 5 0 2 4 6 3)
(1 5 0 3 6 2 4) (1 5 0 3 6 4 2) (1 5 0 4 2 6 3) (1 5 2 0 3 6 4) (1 5 2 0 4 6 3)
(1 5 2 4 0 6 3) (1 5 2 4 6 0 3) (1 5 2 6 3 0 4) (1 5 2 6 4 0 3) (1 5 3 0 2 6 4)
(1 5 3 0 4 6 2) (1 5 3 0 6 2 4) (1 5 3 0 6 4 2) (1 5 3 6 0 2 4) (1 5 3 6 0 4 2)
(1 5 3 6 2 0 4) (1 6 2 4 0 3 5) (1 6 2 4 0 5 3) (1 6 2 5 3 0 4) (1 6 3 0 4 2 5)
(1 6 3 0 5 2 4) (1 6 3 5 0 2 4) (1 6 3 5 0 4 2) (1 6 3 5 2 0 4) (1 6 4 0 2 5 3)
(1 6 4 0 3 5 2) (1 6 4 2 0 3 5) (1 6 4 2 0 5 3) (1 6 4 2 5 0 3) (2 0 3 5 1 6 4)
(2 0 3 6 4 1 5) (2 0 4 1 6 3 5) (2 0 4 6 1 3 5) (2 0 4 6 1 5 3) (2 0 4 6 3 1 5)
(2 0 5 1 3 6 4) (2 0 5 1 4 6 3) (2 0 5 3 1 6 4) (2 0 5 3 6 1 4) (2 0 6 3 5 1 4)
(2 0 6 4 1 3 5) (2 0 6 4 1 5 3) (2 4 0 3 6 1 5) (2 4 0 5 1 6 3) (2 4 0 6 1 3 5)
(2 4 0 6 1 5 3) (2 4 0 6 3 1 5) (2 4 1 5 0 6 3) (2 4 1 6 0 3 5) (2 4 1 6 0 5 3)
(2 4 1 6 3 0 5) (2 4 6 0 3 1 5) (2 4 6 0 5 1 3) (2 4 6 1 3 0 5) (2 4 6 1 5 0 3)
(2 5 0 3 1 6 4) (2 5 0 3 6 1 4) (2 5 0 4 1 6 3) (2 5 0 4 6 1 3) (2 5 0 6 3 1 4)
(2 5 0 6 4 1 3) (2 5 1 3 0 6 4) (2 5 1 3 6 0 4) (2 5 1 4 0 6 3) (2 5 1 4 6 0 3)
(2 5 1 6 3 0 4) (2 5 1 6 4 0 3) (2 5 3 0 6 1 4) (2 5 3 1 6 0 4) (2 6 0 3 5 1 4)
(2 6 0 4 1 3 5) (2 6 0 4 1 5 3) (2 6 0 5 3 1 4) (2 6 1 3 5 0 4) (2 6 1 4 0 3 5)
(2 6 1 4 0 5 3) (2 6 1 5 3 0 4) (2 6 3 0 4 1 5) (2 6 3 0 5 1 4) (2 6 3 1 4 0 5)
(2 6 3 1 5 0 4) (2 6 4 0 3 1 5) (2 6 4 0 5 1 3) (2 6 4 1 3 0 5) (2 6 4 1 5 0 3)
(3 0 2 4 6 1 5) (3 0 2 5 1 6 4) (3 0 2 6 4 1 5) (3 0 4 1 6 2 5) (3 0 4 2 6 1 5)
(3 0 5 1 6 2 4) (3 0 5 2 6 1 4) (3 0 6 1 4 2 5) (3 0 6 1 5 2 4) (3 0 6 2 4 1 5)
(3 0 6 2 5 1 4) (3 1 4 0 6 2 5) (3 1 4 6 0 2 5) (3 1 4 6 2 0 5) (3 1 5 0 2 6 4)
(3 1 5 0 6 2 4) (3 1 5 2 0 6 4) (3 1 5 2 6 0 4) (3 1 6 0 4 2 5) (3 1 6 0 5 2 4)
(3 1 6 2 4 0 5) (3 1 6 2 5 0 4) (3 1 6 4 0 2 5) (3 1 6 4 2 0 5) (3 5 0 2 6 1 4)
(3 5 1 6 0 2 4) (3 5 1 6 2 0 4) (3 5 2 0 6 1 4) (3 6 0 2 4 1 5) (3 6 0 2 5 1 4)
(3 6 1 4 0 2 5) (3 6 1 4 2 0 5) (3 6 1 5 0 2 4) (3 6 1 5 2 0 4) (3 6 2 0 4 1 5)
(3 6 2 0 5 1 4) (4 0 2 6 1 3 5) (4 0 2 6 3 1 5) (4 0 3 1 6 2 5) (4 1 3 0 6 2 5)
(4 1 3 6 0 2 5) (4 1 3 6 2 0 5) (4 1 6 2 0 3 5) (4 1 6 3 0 2 5) (4 2 0 3 6 1 5)
(4 2 0 6 1 3 5) (4 2 0 6 3 1 5) (4 2 6 0 3 1 5) (4 2 6 1 3 0 5)