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Graph Rewritings with Partial Functions

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Abstract

In 1984, Raoult proposed a formalization of graph rewritings using a pushout in the category of graphs and partial functions. We modified the matter more generally and leaded another proof of the conditions to exist pushouts using the relational calculus. This paper's aim is not only to correct the Raoult's proposition but also to introduce a more general framework of graph rewritings and to give very simple proofs using the relational calculus.

1 Introduction

There are many researches about graphs using the category theory. For an edge of a directed graph, we can decide a source vertex and a destination vertex. We consider a directed graph structure as a function from the set E of edges to the product set $V \times V$ of the source vertices set and destination vertices set. Ehrig[4][3] characterize the graph grammar and rewriting rules using two pushout squares in the category. The category which object is a pair of a set E of edges and a set V of vertices together with those graph structure is described as a functor category over the category of set and functions. So it becomes a topos and have various useful properties. For a rewriting rule, whether we can apply a rewriting rule to a graph is depend on an existence of pushout complement in the category of graphs. The existing theorem of pushout complement in elementary topos is generally proved by Kawahara[7].

Courcelle[1][2] introduced a method to denote a graph using expressions. He characterize a graph rewriting by rewriting the expressions like term rewriting systems. He showed the rewriting power of the method is equivalence to the Ehrigh's rewritings.

Raoult[10] proposed a formalization of graph rewritings with different way of Ehrigh's rewritings. For a vertex, we can decide vertices which are destinations of edges from the vertex. Drawing line up these vertices as a string, we can consider a function from a set V of the vertices to V^* the set of strings of destinations. He considers the function as a graph structure. Further he proposed a formalization of graph rewritings by one pushout square using partial functions.

In this paper we developed the Raoult's method. For a functor over the category **Pfn** of sets and partial functions we consider a graph as a function $V \to TV$ from a vertex set V to TV, where T is a functor over **Pfn**. If we set a functor T to $TV = V^*$ then the theory is same as Raoult's one. We proved the existing condition of pushouts in

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our general category of graphs. By using the relational calculus, we simply proved the properties avoiding many kind of conditional check and case divisions. The relational calculus is a theory of binary relations which originally applied to the area of topology and homological algebra in pure mathematics. Recently it has been used for representing the notion of nondeterminism in automata theory, the theory of assertion semantics[6] and characterization of pushouts in the theory of graph grammars[7].

Our general result of the condition of existence of pushouts produce a little modification of Raoult's result ([10]Proposition. 5) which lucked some conditions. We give a counter example of his result and corrected conditions. Further we show that if we choose a functor T as the powerset functor P then we are able to make a pushout in any situations. That is, every graph rewriting rule is applicable to matched graph without any conditions. In the category of graphs made by the powerset functor, we show an example of a graph rewriting which does not hold Ehrigh's gluing conditions.

2 Preliminary

In this section, we recall some relational notations, properties and some categorical properties of the category **Set** of set and functions and **Pfn** of set and partial functions.

Let A, B and C be sets. When α is a subset of $A \times B$, we call α is a *relation* from A to B and denote it by $\alpha : A \rightharpoonup B$. For relations $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$, we define a *composite relation* $\alpha \cdot \beta : A \rightharpoonup C$ by $\alpha \cdot \beta := \{(a,c) \in A \times C | (a,b) \in \alpha, (b,c) \in \beta \text{ for some } b \in B\}$. For relation $\alpha : A \rightharpoonup B$, we define the *inverse relation* $\alpha^\sharp : B \rightharpoonup A$ by $\alpha^\sharp = \{(b,a) \in B \times A | (a,b) \in \alpha\}$. We identify a function $f : A \rightharpoonup B$ with a relation $\{(a, f(a)) \in A \times B | a \in A\}$ (the graph of f). A function from a set X to one point set $1 = \{*\}$ is denoted by $\Omega_X : X \rightarrow 1$. We define a subset $\mathrm{dom}(\alpha)$ of A for a relation $\alpha : A \rightharpoonup B$ by $\mathrm{dom}(\alpha) = \{a \in A | (a,*) \in \alpha\Omega_A\}$ and a relation $d(\alpha) : A \rightharpoonup A$ by $d(f) = \{(a,a) \in A \times A | a \in \mathrm{dom}(\alpha)\}$. For two relations $\alpha, \beta : A \rightharpoonup B$, we define $\alpha \cup \beta$ and $\alpha \cap \beta$ by set union and intersection respectively.

Lemma 2.1 *For a relation $f : A \rightharpoonup B$,*

(1) *f is a partial function if and only if $f^\sharp f \subset id_B$.*

(2) *f is a (total) function if and only if $f^\sharp f \subset id_B$ and $id_A \subset f f^\sharp$.*

(3) *f is a injective function if and only if $f^\sharp f \subset id_B$ and $f f^\sharp = id_A$.*

(4) *f is a surjective function if and only if $f^\sharp f = id_B$ and $f f^\sharp \supset id_A$.*

Lemma 2.2 *For relations $\alpha, \alpha' : A \rightharpoonup B$, and $\beta, \beta' : B \rightharpoonup C$, if $\alpha \subset \alpha'$ and $\beta \subset \beta'$, then $\alpha\beta \subset \alpha'\beta'$, $\alpha^\sharp \subset \beta^\sharp$ and $\alpha(\beta \cup \beta') = (\alpha\beta) \cup (\alpha\beta')$.*

Proposition 2.3 (Law of Puppe-Calenko) *If $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : A \rightharpoonup C$ are relations, then $\alpha\beta \cap \gamma \subset \alpha(\beta \cap \alpha^\sharp\gamma)$.*

Lemma 2.4 *Let $\alpha : A \rightharpoonup B$, $\beta : A \rightharpoonup B$ be relations. Then $d(\alpha) \subset d(\beta)$ if and only if $\alpha\Omega_A \subset \beta\Omega_B$.*

Corollary 2.5 *Let $f : A \rightarrow B$ is a partial function. Then f is a total function if and only if $\Omega_A = f\Omega_B$.*

Fact 2.6 *The category* Set *has coequalizers.*
For two functions $f : A \rightarrow B$ and $g : A \rightarrow B$, there exist a function $e : B \rightarrow Q$ such that $fe = ge$. For any function $x : B \rightarrow X$ satisfying $fx = gx$, there exist a unique function $\hat{x} : Q \rightarrow X$ such that $e\hat{x} = x$ holds.

Fact 2.7 *The category* **Set** *has pushouts.*
For two functions $f : A \to B$ and $g : A \to C$, there exists a set D and functions $h : B \to D$ and $k : C \to D$. For any functions $x : B \to S$, $y : C \to S$ satisfying $fx = gy$, there exists a unique function $t : D \to S$ such that $ht = x$ and $ht = y$ hold. The unique function t is expressed by $h^{\sharp}x \cup k^{\sharp}y$.

Fact 2.8 *The category* **Pfn** *has products and equalizers.*

Fact 2.9 *The category* **Pfn** *have coproducts.*
For two objects A and B, the coproduct $A + B$ with inclusion functions $i_A : A \to A + B$ and $i_B : B \to A + B$ in **Set** *is also the coproduct in* **Pfn**.

Fact 2.10 *The category* **Pfn** *has coequalizers.*
For two partial functions $f : A \to B$ and $g : A \to B$, let $i : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to A$ be an inclusion function. Let $e : B \to Q$ be the coequalizer $coeq(if, ig)$ in **Set** *and e_0 is an inclusion function for the subset*

$$E = Q - e(f(\mathrm{dom}(f) - \mathrm{dom}(g)) \cup g(\mathrm{dom}(g) - \mathrm{dom}(f)))$$

of Q. Then $ee_0^{\sharp} : B \to E$ is a coequalizer of f and g in **Pfn**.

Fact 2.11 *The category* **Pfn** *have pushouts.*

For a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

in **Pfn**, the domain of partial function h is

$$\mathrm{dom}(h) = (B - f(A)) \cup (f(A) - i_B^{-1}e^{-1}e(i_B f(\mathrm{dom}(f) - \mathrm{dom}(g)) \cup i_C g(\mathrm{dom}(g) - \mathrm{dom}(f)))).$$

Where $i_B : B \to B + C$ and $i_C : C \to B + C$ are inclusion functions of coproduct $B + C$, and $e : B + C \to D$ is a coequalizer of fi_B and gi_C.

3 Category of graphs over Pfn

In this section, we introduce an abstract definition of a category which represent graphs and graph homomorphisms. Graph rewritings are defined by using a single pushout in the category. We prove a necessary and sufficient condition to exist pushouts. Some concrete categories of graphs including the Raoult's[10] definitions are shown. We prove and correct his proposition 5 using our general framework.

Lemma 3.1 *Let $T :$* **Pfn** $\to$ **Pfn** *be a functor, $a : A \to TA$, $b : B \to TB$ and $c : C \to TC$ be functions and $f : A \to B$ and $g : B \to C$ be partial functions. If $fb = d(f) \cdot a \cdot Tf$ and $gc = d(g) \cdot b \cdot Tg$ then $fgc = d(fg) \cdot a \cdot T(fg)$.*

Definition 3.2 *For a functor $T :$* **Pfn** $\to$ **Pfn**, *a* graph *constructed by T is a pair (A, a) of a set A and a function $a : A \to TA$. For graphs (A, a) and (B, b), a partial function $f : A \to B$ is a* graph morphism *related to T if f satisfies $fb = d(f) \cdot a \cdot Tf$.*

Definition 3.3 *For a functor $T :$* **Pfn** $\to$ **Pfn**, *a graph category $G(T)$ is the category of graphs constructed by T and graph morphisms related to T.*

Example 3.4 (Kleene functor) *For a set A, we define $TA = A^*$ the set of strings over A. For a function $f : A \to B$, we define $Tf : A^* \to B^*$ as follows:*

$$
\begin{aligned}
Tf(w) &= f(a_1)f(a_2)\cdots f(a_n) \qquad (\text{where } w = a_1a_2\cdots a_n), \\
Tf(\varepsilon) &= \varepsilon.
\end{aligned}
$$

We denotes by $G(*)$ the category of graphs constructed by this functor. The category is equivalent to that one considered by Raoult[10].

Example 3.5 (powerset functor) *For a set A, we define $TA = P(A)$ the set of all subsets of A. For a function $f : A \to B$, we define $Tf : P(A) \to P(B)$ by $Tf(X) = f(X)$, $(X \subset A)$. We denotes by $G(P)$ the category of graphs constructed by the functor P.*

Example 3.6 *We define a set N^A of functions from A to the set $N = \{0, 1, \ldots\}$ of natural numbers by $N^A = \{f : A \to N | \Sigma_{x \in A} f(a) \text{ is finite.}\}$. We define the functor W : **Set** $\to$ **Set** as follows. For an object A in **Set**, we define $W(A) = N^A$ the set of For a function $f : A \to B$, we define $Tf : N^A \to N^B$ by $Tf(\alpha)(y) = \Sigma\{\alpha(x) | f(x) = y, x \in A\}$, $(\alpha \in N^A, y \in B)$. We denote by $G(W)$ the category of graphs constructed by the functor W.*

Example 3.7 (L-labeled Kleene functor) *We fix a set L of labels for edges. For a set A, we define $TA = (L \times A)^*$ the set of strings of pairs of a label and an element of A. Other definition of the functor T is similarly to the Example 3.4. We denote by $G(L \times -)^*)$ the category of graphs constructed by the functor T.*

Example 3.8 (L-labeled powerset functor) *We similarly to define a functor $TA = P(L \times A)$ like Example 3.7. We denote by $G(P(L \times -)$ the category of graphs constructed by the functor T.*

We note that if $T = P$ or $T = W$ then $Tf : TA \to TB$ is a total function for any partial function $f : A \to B$.

Theorem 3.9 *Let $f : (A, a) \to (B, b)$ and $g : (A, a) \to (C, c)$ be morphisms and the square*

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle g}\downarrow & (1) & \downarrow{\scriptstyle h} \\
C & \xrightarrow[k]{} & D
\end{array}
$$

be a pushout in **Pfn**. *There exists a unique partial function*

$$d = (h^\sharp \cdot b \cdot Th) \cup (k^\sharp \cdot c \cdot Tk)$$

such that $hd = d(h) \cdot b \cdot Th$, $kd = d(k) \cdot c \cdot Tk$. Further, the square (2)

$$
\begin{array}{ccc}
(A, a) & \xrightarrow{f} & (B, b) \\
{\scriptstyle g}\downarrow & (2) & \downarrow{\scriptstyle h} \\
(C, c) & \xrightarrow[k]{} & (D, d)
\end{array}
$$

is a pushout if and only if d is a total function.

Theorem 3.10 *Under the situation of Theorem 3.9, the following conditions are equivalent:*

(1) $d = (h^\sharp \cdot b \cdot Th) \cup (k^\sharp \cdot c \cdot Tk)$ *is a total function.*

(2) $b(\mathrm{dom}(h)) \subset \mathrm{dom}(Th)$ *and* $c(\mathrm{dom}(k)) \subset \mathrm{dom}(Tk)$

(3) $\mathrm{dom}(h) \subset \mathrm{dom}(b \cdot Th)$ *and* $\mathrm{dom}(k) \subset \mathrm{dom}(c \cdot Tk)$

Corollary 3.11 *The categories $G(P)$, $G(W)$ have pushouts.*

Definition 3.12 *For two partial functions $f : A \to B$ and $g : A \to C$, we define a relation $\Gamma_{(f,g)} : A \to 1$ by $\Gamma_{(f,g)} = \cup\{\alpha : A \to 1 | ff^\sharp\alpha = \alpha$ and $gg^\sharp\alpha = \alpha\}$. That is $\Gamma_{(f,g)}$ is the maximum relation satisfying $ff^\sharp\Gamma_{(f,g)} = \Gamma_{(f,g)}$ and $gg^\sharp\Gamma_{(f,g)} = \Gamma_{(f,g)}$.*

Lemma 3.13 *Let*

$$\begin{array}{ccc} (A,a) & \xrightarrow{f} & (B,b) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (C,c) & \xrightarrow[k]{} & (D,d) \end{array}$$

be a pushout in $G_1(T)$. Then $\Gamma_{(f,g)} = fh\Omega_D(= gk\Omega_D)$.

We note that $\Gamma_{(f,g)} = fh\Omega_D$ means $f^{-1}(\mathrm{dom}(h)) = \{a \in A | (a,1) \in \Gamma_{(f,g)}\}$.

Lemma 3.14 $\mathrm{dom}(h) = (B - f(A)) \cup f(A')$ *where* $A' = \{a \in A | (a,1) \in \Gamma_{(f,g)}\}$.

Lemma 3.15 *Under the situation of Theorem 3.9, and we consider the functor $T = *$. Then following three conditions are equivalent:*

(1) $b(f(A')) \subset \mathrm{dom}(Th)(= (\mathrm{dom}(h))^*)$,

(2) $Tf(a(A')) \subset (\mathrm{dom}(h))^*$,

(3) $a(A') \subset (A')^*$,

where $A' = \{a \in A | (a,1) \in \Gamma_{(f,g)}\}$.

Proposition 3.16 *A commutative diagram*

$$\begin{array}{ccc} (A,a) & \xrightarrow{f} & (B,b) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (C,c) & \xrightarrow[k]{} & (D,d) \end{array}$$

in $G_1(*)$ is a pushout if and only if following three conditions holds:

(1) $b(B - f(A)) \subset (\mathrm{dom}(h))^*$,

(2) $c(C - g(A)) \subset (\mathrm{dom}(k))^*$, *and*

(3) $a(A') \subset (A')^*$

Where $A' = \{a \in A | (a,1) \in \Gamma_{(f,g)}\}$.

Example 3.17 *Let $A = \{x_1 \to x_2, x_3\}$, $B = \{y_1 \to y_2\}$ and $C = \{z_1 \to z_2\}$ be graphs. Define graph morphisms $f : A \to B$ and $g : A \to C$ by $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, $g(x_1) = z_1$ and $g(x_2) = z_2$. The value of $g(x_3)$ is undefined(cf. Figure 1).*

It is easy to check $A' = \{x_1\}$. and the condition in Proposition 3.16(3) does not hold. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

in the category **Pfn**, *Since D is a one-point set, h and k are not graph morphisms.*

This example is a counter example of Raoult's proposition 5[10].

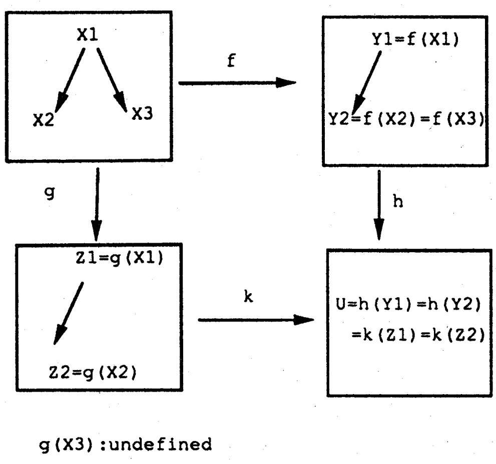

Figure 1:

4 Comment

A graph morphism $f : (A,a) \to (B,b)$ is considered as a rewriting rule. For a graph (C,c), if there exist a graph morphism $g : (A,a) \to (C,c)$ and a graph (C,c) such that the square

$$\begin{array}{ccc} (A,a) & \xrightarrow{f} & (B,b) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (C,c) & \xrightarrow[k]{} & (D,d) \end{array}$$

is a pushout, we say that the rewriting rule f is applicable and (C,c) is rewritten to the graph (D,d).

There is a natural correspondence between the category of **Graph** which is defined by Ehrig[4] and our categories $G(*)$ and $G(P)$, though there are critical differences which we omit to show details in this paper.

Our rewriting ability seems to be less than Ehrigh's one which use two pushout squares in his category **Graph** of graphs (cf. Figure 2). But we show a example which does not satisfies the gluing condition ([4]) but rewritable in our situation(cf. Figure 3).

We do not know completely the essential differences between Ehrigh's rewritings and our rewriting formulation yet.

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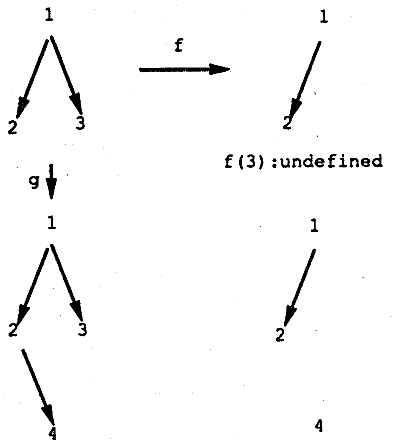

Figure 2:

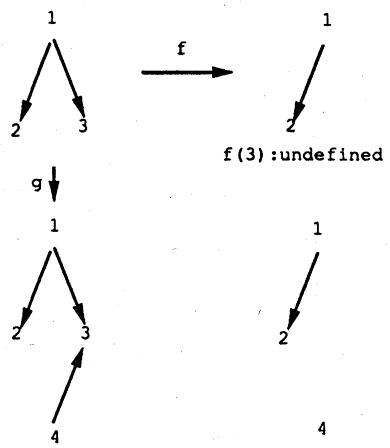

Figure 3:

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