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Schrödinger evolution equations and associated smoothing effect

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Abstract. We shall review auther's recent results on the existence, uniqueness, regularity and the smoothing property of a unitary propagator $U(t,s): u(s) \to u(t)$ in $\mathsf{H} = L^2(\mathsf{R}^n)$ for time dependent Schrödinger equations in an external electromagnetic field. Potentials are assumed to be real and are possibly unbounded at infinity $|x| \to \infty$ and scalar potential can have local singularities.

1. Introduction, Assumptions and Theorems.

In this lecture we report author's recent works **[29]**,**[30]** and **[31]**, improving some results in **[31]**, on the Cauchy problem for time dependent Schrödinger equations

$$i\frac{\partial u}{\partial t} = H(t)u = (\sum_{j=1}^{n}\frac{1}{2}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j(t,x))^2 + V(t,x))u, \tag{1.1}$$

where the real potentials $V(t,x)$ and $A(t,x) = (A_1(t,x),\cdots,A_n(t,x))$ are possibly unbounded at infinity $|x| \to \infty$ and $V(t,x)$ can have local singularities. We shall be concerned with the existence, uniqueness, regularity and the smoothing property of a unitary propagator $U(t,s): u(s) \to u(t)$ in $\mathsf{H} = L^2(\mathsf{R}^n)$ solving (1.1).

The existence of a unique dynamics associated with (1.1) is of fundamental importance in quantum mechanics. When $H(t) = H$ is t-independent, this is equivalent to the selfadjointness of H, via celebrated Stone's theorem, and various satisfactory criteria have been known since Kato's fundamental paper **[14]** (see e.g. **[5]**, **[23]**). When $H(t)$ is genuinely time dependent, on the other hand, the situation has been by far less satisfactory. We recall three common methods, among others, for solving (1.1). One of them is to apply to (1.1) the abstract theory of evolution equations (cf. e.g. **[15]**). The method has been widely and successfully employed by many authors. The theory, however, has been originally designed for solving parabolic or hyperbolic equations and, when applied to Schrödinger equations, it often imposes rather severe restrictions on the potentials, which exclude many concrete and important physical examples out of its scope (cf. [**30**]). Another one, which we call the method of semi-classical approximation, is to construct the propagator $U(t,s)$ directly in the form of oscillatory integral operator using, so called, semi-classical approximations. This method was introduced by Fujiwara([**9**] and [**10**]) for Schrödinger equations and extended by Kitada-Kumanogo[**22**], which, however, requires the potentials be C^∞ in the spatial variables. The third one is to study (1.1) through its equivalent integral equation using the perturbation technique.

Synthesizing the last two methods mentioned above, we first establish the following existence and uniqueness theorem. For an interval I, $L^{\ell,\theta}(I) = L^\theta(I, L^\ell(\mathbf{R}^n))$ and $L^{\ell,\theta}_{loc}(I) = L^\theta_{loc}(I, L^\ell(\mathbf{R}^n))$. $|\cdot|$ denotes the norm of matrices as operators on $\mathbf{C}^n$ as well as the Euclidean norm of vectors. $\partial_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We sometimes write $\partial/\partial x_j = \partial_{x_j}$.

ASSUMPTION (A): (1) For $j = 1, \cdots, n$, $A_j(t,x)$ is a real function of $(t,x) \in \mathbf{R}^{n+1}$ such that $\partial_x^\alpha A_j(t,x) \in C^1(\mathbf{R}^{n+1})$ for any α. There exists $\epsilon > 0$ such that

$$(1.2) \qquad |\partial_x^\alpha B(t,x)| \le C_\alpha (1+|x|)^{-1-\epsilon}, \qquad |\alpha| \ge 1,$$

$$(1.3) \qquad |\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \le C_\alpha, \quad |\alpha| \ge 1, \quad (t,x) \in \mathbf{R}^{n+1},$$

where $B(t,x)$ is the strength tensor of the magnetic field, i.e. the skew symmetric matrix with (j,k)-component $B_{jk}(t,x) = (\partial A_k/\partial x_j - \partial A_j/\partial x_k)(t,x)$.
(2) $V(t,x)$ is a real function of $(t,x) \in \mathbf{R}^{n+1}$ such that $V \in L^{p,\alpha}(\mathbf{R}^1) + L^{\infty,1}(\mathbf{R}^1)$ for some $p > n/2$ with $p \ge 1$ and $\alpha = 2p/(2p-n)$.

For $k = 0, 1, \cdots$, $\Sigma(k) = \{u \in L^2(\mathbf{R}^n) : \sum_{|\alpha|+|\beta| \le k} \|x^\beta \partial_x^\alpha u\|^2 = \|u\|^2_{\Sigma(k)} < \infty\}$ and $\Sigma(-k)$ is the dual space of $\Sigma(k)$. For Banach spaces X and Y, $B(X,Y)$ stands for the set of bounded operators from X to Y, $B(X) = B(X,X)$. We use C_* to indicate the strong continuity of operator valued functions, e.g. $C_*(I^2, B(X,Y))$ is the set of strongly continuous $B(X,Y)$-valued functions on I^2.

THEOREM 1. *Let Assumption (A) be satisfied, $q = 2p/(p-1)$ and $\theta = 4p/n$. Then, $H(t) \in B(\mathbf{H} \cap L^q(\mathbf{R}^n), \Sigma(-2))$ and its part in* $\mathbf{H}$ *is selfadjoint for a. e. $t \in \mathbf{R}^1$. There exists a unique propagator $\{U(t,s) : t,s \in \mathbf{R}^2\}$ with the following properties.*
(1) $U(t,s)$ is unitary in $\mathbf{H}$ *with $U(t,s)U(s,r) = U(t,r)$.*
(2) $U(\cdot,\cdot,h) \in C_*(\mathbf{R}^2, B(\mathbf{H}))$.
(3) For any $s \in \mathbf{R}^1$ and $f \in \mathbf{H}$, $U(\cdot,s)f \in L^{q,\theta}_{loc}(\mathbf{R}^1)$ and with constant $C > 0$ independent of s, f and compact intervals I,

$$(1.4) \qquad \|U(\cdot,s)f\|_{L^{q,\theta}(I)} \le C(1+|I|)^{1/\theta}\|f\|.$$

(4) If $f \in \mathbf{H}$, $u(t) = U(t,s)f$ is $\Sigma(-2)$-valued absolutely continuous in t and satisfies the equation (1.1) in $\Sigma(-2)$ at almost every $t \in \mathbf{R}^1$.

In what follows, the exponents $p > n/2$, $q = 2p/(p-1)$ and $\alpha = 2p/(2p-n)$ are reserved to denote those appeared in Assumption (A) and Theorem 1.

Several remarks are in order.

REMARK(A): If $G(t,x)$ is real smooth and T is the gauge transformation:

$$Tu(t,x) = \exp(-iG(t,x))u(t,x), \tag{1.5}$$

then T transforms (1.1) into the one with $A + \partial_x G$ and $V - \partial_t G$, in place of A and V respectively. In particular, by taking $G(t,x) = \int_0^t V(s,x)ds$, smooth scalar potentials may be eliminated from (1.1) by a gauge transformation by changing A to $A + \int_0^t \partial_x V(s,x)ds$. This is the reason why we assumed in Assumption (A) that $V(t,x)$ is bounded at infinity in a certain L^p norm, which allows us to regard (1.1) as a perturbed equation of

$$i\frac{\partial u}{\partial t} = H_0(t)u = \sum_{j=1}^{n} \frac{1}{2}(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j(t,x))^2 u. \tag{1.6}$$

It should be also clear that, if $V_0(t,x)$ is a real function such that $\partial_x^\alpha V_0(t,x)$ is continuous in (t,x) for all $|\alpha| \ge 0$ and $|\partial_x^\alpha V_0(t,x)| \le C_\alpha$, $|\alpha| \ge 2$, then, Theorem 1 remains to hold for (1.1) with $V(t,x) + V_0(t,x)$ in place of $V(t,x)$, the latter being assumed to satisfy Assumption (A). Similar remark will apply to theorems to follow, though we shall not mention it explicitly any more.

REMARK(B): (1.4) implies that for every $f \in \mathbf{H}$ and $s \in \mathbf{R}$, $U(t,s)f \in \mathbf{H} \cap L^q(\mathbf{R}^n)$ for *a.e.* $t \in \mathbf{R}^1$ and manifests the smoothing effect of the propagator $U(t,s)$ in the sense that it improves L^p-smoothness. This is an improvement of Strichartz' inequality

$$(\int\{\int |e^{it\Delta}f(x)|^p dx\}^{\theta/p}dt)^{1/\theta} \le C\|f\|, \quad f \in L^2(\mathbf{R}^n) \tag{1.7}$$

for $0 \le 2/\theta = n(1/2 - 1/p) < 1$ (cf. [**11**]). Note that (1.7) is an estimate global in time and manifests simultaneously the local decay property (as $|t| \to \infty$) of $\exp(it\Delta)f$ (which, incidentally, implies the existence of the restriction of Fourier transforms of $L^{p',\theta'}$-functions on the quadratic surface $\eta = \xi_1^2 + \cdots + \xi_n^2$, cf. [**26**]). This is in contrast to (1.4) where the right hand side increases with the length of the time interval. This is, of course, natural because bound states exist in general which never decay as $t \to \pm\infty$.

REMARK(C): If $V(t,x) = V(x)$ satisfies (2) of Assumption (A) with some $p_0 > n/2$, it does so with any $n/2 < p \le p_0$. Hence $U(\cdot,s)f \in L^{\ell,\theta}_{loc}(\mathbf{R}^1)$ for any $2 \le \ell < 2n/(n-2)$ and $1 \le \theta \le 4\ell/n(\ell-2)$.

REMARK(D): The L^p-smoothing property of the propagator is expected to hold, at least in a weaker form, for Schrödinger equations with faster increasing scalar potentials. In fact, if $H = -d^2/dx^2$ on $[0,1]$ with Dirichlet conditions at $x = 0$, 1, which is an extreme case that $V_0(x) = \infty$ for $x \in (-\infty, 0) \cup (1, \infty)$ in the terminology of Remark (a), it can be easily checked by an explicit computation using Fourier series expansions that $\exp(-itH)u \in L^4_{loc}(\mathbf{R}^1, L^4([0,1]))$, for $u \in L^2([0,1])$ and $\exp(-itH)$ *does* possess certain smoothing effect even in this extreme case. (This is a result of Zygmund [**33**].) In view of Zelditch's work ([**32**]), we note that faster increase of scalar potential implies increasing number of caustics for higher momentum which suggests stronger singularities in the integral kernel of the propagator $U(t,s)$. We suspect that the smoothing effect may be used for measuring the *strength of its singularity.*

REMARK(E): When $H(t) = H$, the L^p-smoothing property implies an interesting result on *change of phases* of Fourier coefficients in (generalized) eigenfunction expansions associated with H. The following is an obvious consequence of (1.4) and the fact that $\{u \in \mathsf{H} : (-i\partial_x - A(x))u \in \mathsf{H}^n\} \subset L^q(\mathbf{R}^n)$. We refer to Edward[**8**] for informations on related results for the ordinary Fourier series.

COROLLARY 2. *Let A and V be independent of t and satisfy Assumption (A) and let $H = H(t)$ be the selfadjoint operator in H defined by the Friedrichs extension:*

$$D(H) = \{u \in \mathsf{H} : (-i\partial/\partial x_j - A_j(x))u \in \mathsf{H},\ j = 1, \cdots, n,\ Hu \in \mathsf{H}\}.$$

Suppose that H has only pure point spectrum $\{\lambda_j\}_{j=1}^{\infty}$ and let $\{\phi_j\}_{j=1}^{\infty}$ be the associated complete set of orthonormal eigenfunctions of H. Then for any $\{c_j\} \in \ell^2$, we have $\sum_{j=1}^{\infty} \exp(-it\lambda_j)c_j\phi_j(x) \in L^{\ell}$, a.e. $t \in \mathbf{R}^1$ for $2 \le \ell < 2n/(n-2)$.

REMARK(F): The L^p-smoothing property of $e^{it\Delta/2}$ has wide applications in the study of *non-linear* Schrödinger equations $i\partial_t u = -(1/2)\Delta u + F(u)$ (cf. e.g. [**11**] and [**17**]). Theorem 1 (and Theorems 7 and 8 to follow) may be used to extend some of their results to more general non-linear Schrödinger equations $i\partial_t u = H(t)u + F(u)$, with $H(t)$ as in (1.1) (cf. [**7**]).

The propagator $U(t,s)$ has not only L^p-smoothing property but also the differentiability improving property, which is the content of the following two theorems. Because of a technical difficulty, we were forced to assume either $A(t,x)$ or $V(t,x)$ vanishes identically. If Theorems 3 and 4 are true under the conditions of Theorem 1 is still an open question.

THEOREM 3. *Let Assumption (A) be satisfied and $\mu > 1/2$. Suppose either $A(t,x) = 0$ or $V(t,x) = 0$. Then, there exists a constant $C_\mu > 0$ such that for $s \in \mathbf{R}^1$ and $T > 0$*

$$\int_{s-T}^{s+T} \|\langle x\rangle^{-\mu}\langle D\rangle^{1/2}U(t,s)f\|^2 dt \le C_\mu(1+T)\|f\|^2, \quad f \in \mathcal{S}(\mathbf{R}^n). \tag{1.7}$$

THEOREM 4. *Let Assumption (A) be satisfied,* $2 \le \ell \le q = 2p/(p-1)$, $0 \le 2\sigma + n(1/2 - 1/\ell) \le 2/\theta < 1$ *and* $\mu > 1/2$. *Suppose either* $A(t,x) = 0$ *or* $V(t,x) = 0$. *Then, there exists a constant* $C_{\ell\theta\sigma} > 0$ *such that for* $s \in \mathbf{R}^1$ *and* $T > 0$

$$(1.8) \quad \int_{s-T}^{s+T} \|\langle x\rangle^{-2\mu\sigma}\langle D\rangle^{\sigma} U(t,s) f(x)\|_{L^\ell}^{\theta} dt \le C_{\ell\theta\sigma}(1+T)\|f\|^{\theta}, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

REMARK (G): As in Remark(c), if $V(t,x) = V(x)$, the condition $2 \le \ell \le q = 2p/(p-1)$ in Theorem 4 may be replaced by $2 \le \ell < 2n/(n-2)$.

It was Kato [**16**] who discovered that a certain unitary group in L^2 can improve the differentiability property: The solutions of the KdV equation

$$\partial_t u + \partial_x^3 u + a(u)\partial_x u = 0, \quad t > 0,\, x \in \mathbf{R}^1$$

with $u(0) = f \in L^2(\mathbf{R}^1)$ satisfy

$$\int_0^T \int_{-R}^{R} |\partial_x u(t,x)|^2 dx dt \le K(\|f\|, T, R)$$

if $\limsup_{|\lambda|\to\infty} |\lambda|^{-4} a(\lambda) \le 0$. Later it was found (cf. [**25**], [**4**] and [**19**]) that such property is common for unitary groups generated by dispersive differential equations with constant coefficients and, in particular, for the free Schrödinger propagator they proved

$$(1.9) \qquad \int_{\mathbf{R}^1} \int_{\mathbf{R}^n} |\phi(t,x)(1-\Delta)^{1/4} e^{it\Delta} u(x)|^2 dx dt \le C\|u\|^2, \quad u \in L^2(\mathbf{R}^n)$$

and for $1 \le q \le 2$, $r \ge 2$ and $\alpha < 1/r - n(1/q - 1/r)$,

$$(1.10) \quad [\int_{\mathbf{R}^1} \int_{\mathbf{R}^n} |\phi(t,x)(1-\Delta)^{\alpha/2} e^{it\Delta} u(x)|^r dx dt]^{1/r} \le C\|u\|_{L^q}, \quad u \in L^q(\mathbf{R}^n),$$

where $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ (see also [**18**] and [**24**] where ϕ in (1.9) is replaced by $\langle x\rangle^{-1/2-\varepsilon}$, $\varepsilon > 0$).

Theorem 3 and Theorem 4 are extensions of (1.9) and (1.10) (with $q = 2$) to differential equations with variable coefficients, however, by the same reason as in (1.4), the right hand sides of (1.7) and (1.8) increase linearly in time length T in contrast to (1.9) and (1.10) which imply also the local decay as $|t| \to \infty$ of $\exp(it\Delta)f$ in the local Sobolev norm. In this connection we should mention that Ben-Artz and Klainermann [**2**] have recently shown, in the case $H(t) = H$ and $A = 0$ and $|V(x)| \le C(1+|x|)^{-2-\epsilon}$, $\epsilon > 0$, that, if zero is not a resonance energy of H, the estimate global in time of type (1.9) is valid for u belonging to the continuous spectral subspace of H (cf. also [**4**]). We should emphasize that

Theorem 4 is an improvement of (1.10) (with $q = 2$) even when $A(t,x) \equiv 0$ and $V(t,x) \equiv 0$ as far as the smoothness is concerned. It uses different L^p-norms for space and time variables in the first place and when $\theta = p$, σ in (1.8) can be as big as $1/p - (n/2)(1/2 - 1/p)$ whereas in (1.10), $\sigma < 1/p - n(1/2 - 1/p)$.

We record two consequences of Theorem 3. One is the *maximal inequality* of Schrödinger type and the other is on the almost everywhere convergence of $U(t,s)f(x)$ as $t \to s$.

THEOREM 5. *Assume the condition of Theorem 3, $\gamma > 1/2$ and $\delta > 2\gamma + 1/2$. Then there exists a constant $C > 0$ such that*

$$\int \sup_{|t-s|<T} |\langle x\rangle^{-\delta} U(t,s)f(x)|^2 dx \le C\|f\|_{H^\gamma}^2, \quad f \in \mathcal{S}(\mathbf{R}^n). \tag{1.12}$$

THEOREM 6. *Let Assumption (A) be satisfied and $f \in H^\gamma(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ with $\gamma > 1/2$. Then for any $s \in \mathbf{R}^1$*

$$\lim_{t\to s} U(t,s)f(x) = f(x), \quad a.e.\, x \in \mathbf{R}^n. \tag{1.13}$$

Note that, when $H(t) = H$, Theorem 6 gives a *summability theorem* for (generalized) eigenfunction expansions associated with H. For $\exp(it\Delta)$ Carleson [**3**] proved the summability theorem with $\gamma = 1/4$ when $n = 1$. This result is shown to be sharp by Dahlberg and Kenig [**6**]. Again for $n = 1$, Kenig and Ruiz[**20**] showed that the maximal inequality (1.9) is satisfied with $\gamma = 1/4$ and that the estimate is sharp. These one-dimensional results were subsequently extended by Sjölin [**25**] and Kenig-Ponce-Vega [**19**] to higher dimensions. It seems still open, however, if these estimates are sharp or not in general.

We close this section with three theorems on the regularity of the solutions of (1.1). The solution $u(t) = U(t,s)f$, $f \in \mathbf{H}$, obtained in Theorem 1 is a *weak solution* in the sense that it satisfies (1.1) only in $\Sigma(-2)$ for *a.e.*$t \in \mathbf{R}^1$, and it is natural to ask the following question: When data f is nice, say $f \in \Sigma(2)$, is $u(t) = U(t,s)f$ a *strong solution*, that is, does $u \in C^1(\mathbf{R}^1, \mathbf{H})$, $u(t) \in D(H(t))$ for every $t \in \mathbf{R}^1$ and satisfy (1.1) in $\mathbf{H}$? This is equivalent to *finding an invariant subspace* $\mathcal{D} \subset D(H(t))$ for $U(t,s)$: $U(t,s)\mathcal{D} = \mathcal{D}$ such that $U(\cdot,\cdot)f \in C^1(\mathbf{R}^2, \mathbf{H})$ for $f \in \mathcal{D}$. This turns out to be a rather subtle and difficult question in general and we shall study this problem only under certain stronger conditions on $A(t,x)$ and $V(t,x)$.

When $V(t,x)$ is smooth in the spatial variables, the gauge transformation (1.5) eliminates it from (1.1) reducing it to the form (1.6). For (1.6) we can construct the propagator $U_0(t,s)$ in the form of oscillatory integral operator(OIO) by the method of semi-classical approximation and derive much information out of it. In particular, $\Sigma(2)$ is an invariant subspace for $U_0(t,s)$. We should recall (cf. [**13**]) that $H_0(t)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^n)$ and its closure, which we denote by the same symbol, is identical with its maximal extension. Thus

$$\Sigma(2) \subset D(H_0(t)) = \{u \in \mathbf{H} : H_0(t)u \in \mathbf{H}\}, \qquad t \in \mathbf{R}^1.$$

In virtue of (1.5), the following theorem is an extension of Fujiwara **[9]**,**[10]**, Kitada **[21]** and Kitada-Kumanogo **[22]** to the case with general magnetic potentials.

THEOREM 7. *Let Assumption (A) be satisfied and $H(t) = H_0(t)$. Then the propagator $U(t,s) = U_0(t,s)$ of Theorem 1 satisfies the following additional properties.*
(1) $U_0(\cdot,\cdot) \in C_*(\mathbf{R}^2, B(\Sigma(k))) \cap C_*^1(\mathbf{R}^2, B(\Sigma(k), \Sigma(k-2)))$ for $k = 0, \pm 1, \cdots$.
(2) If $f \in \Sigma(2)$, then $U_0(t,s)f$ satisfies

$$i(\partial/\partial t)U_0(t,s)f = H_0(t)U_0(t,s)f \in C(\mathbf{R}^2, \mathsf{H}),$$

$$i(\partial/\partial s)U_0(t,s)f = -U_0(t,s)H_0(s)f \in C(\mathbf{R}^2, \mathsf{H}).$$

(3) $U_0(t,s)$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbf{R}^n)$.
(4) For a small $T > 0$, $U_0(t,s)$, $0 < |t-s| \le T$, can be written in the form of an OIO:

$$U_0(t,s)f(x) = (2\pi i(t-s))^{-n/2} \int e^{iS(t,s,x,y)} b(t,s,x,y) f(y)dy, \tag{1.14}$$

where $S(t,s,x,y)$ and $b(t,s,x,y)$ satisfy the following properties.
(4.a) $S(t,s,x,y)$ is a real solution of Hamilton-Jacobi equation

$$(\partial_t S)(t,s,x,y) + (1/2)((\partial_x S)(t,s,x,y) - A(t,x))^2 = 0. \tag{1.15}$$

It is C^1 in (t,s,x,y), C^∞ in (x,y) and satisfies the estimate

$$|\partial_x^\alpha \partial_y^\beta \{S(t,s,x,y) - \frac{(x-y)^2}{2(t-s)}\}| \le C_{\alpha\beta}, \qquad |\alpha+\beta| \ge 2. \tag{1.16}$$

(4.b) For any α and β, $\partial_x^\alpha \partial_y^\beta b(t,s,x,y)$ is C^1 in (t,s,x,y) and satisfies

$$|\partial_x^\alpha \partial_y^\beta b(t,s,x,y)| \le C_{\alpha\beta}, \qquad |\alpha+\beta| \ge 0. \tag{1.17}$$

When $V(t,x)$ is singular, we take a perturbation point of view and consider the case $H(t) = H_0(t) + V(t)$ with $H_0(t) = H_0$ first. We give a sufficient condition on $V(t,x)$ so that $D(H_0)$ becomes an invariant subspace. Recall that $p > n/2$ with $p \ge 1$ is fixed as in Assumption (A) and $q = 2p/(p-1)$.

ASSUMPTION (B): *$V(t,x)$ satisfies $V \in C(\mathbf{R}^1, L^{\tilde{p}}(\mathbf{R}^n)) + C(\mathbf{R}^1, L^\infty(\mathbf{R}^n))$ and $\partial_t V \in L^{p_1,\alpha_1}(\mathbf{R}^1) + L^{\infty,1}(\mathbf{R}^1)$ for $\tilde{p} = max(p,2)$, $p_1 = 2np/(n+4p)$ $(n \ge 5)$, $p_1 > 2p/(p+1)$ $(n = 4)$, $p_1 = 2p/(p+1)$ $(n \le 3)$ and $\alpha_1 = 4p/(4p-n)$.*

THEOREM 8. *Let Assumptions (A) and (B) be satisfied and $H_0(t) = H_0$ is independent of t. Then, $H(t)$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^n)$ and its closure, which will be denoted by the same symbol, has constant domain $D(H(t)) = D(H_0)$. The propagator $U(t,s)$ satisfies the following properties:*
(1) $U(\cdot,\cdot) \in C_*(\mathbf{R}^2, B(D(H_0))) \cap C_*^1(\mathbf{R}^2, B(D(H_0), \mathbf{H}))$.
(2) If $f \in D(H_0)$, then

$$i\partial_t U(\cdot, s)f = H(\cdot)U(\cdot, s)f \in C(\mathbf{R}^1, \mathbf{H}) \cap L^{q,\theta}_{loc}(\mathbf{R}^1), \tag{1.18}$$

$$i\partial_s U(\cdot, s)f = -U(\cdot, s)H(s)f \in C(\mathbf{R}^1, \mathbf{H})) \cap L^{q,\theta}_{loc}(\mathbf{R}^1), \tag{1.18}$$

where we have equipped $D(H_0)$ with graph norm and $\theta = \theta(q) = 4p/n$.

When $H_0(t)$ is genuinely t-dependent, $D(H_0(t))$ is in general wildly changing with t, and it is too ambitious to expect $U(t,s)D(H(s)) \subset D(H(t))$ even when $H(t) = H_0(t)$. However, we know by Theorem 7 that $\Sigma(2) \subset D(H_0(t))$ is an invariant subspace for $U_0(t,s)$, and this suggests we try to find conditions on $V(t,x)$ such that $\Sigma(2)$ be also an invariant subspace for $U(t,s)$. The next theorem shows that this is indeed the case, if $V(t,x)$ suitably decays at infinity. We write $\langle x\rangle = (1 + |x|^2)^{1/2}$.

ASSUMPTION (C): $V(t,x)$ satisfies $V \in \langle x\rangle^{-3}C(\mathbf{R}^1, L^{\tilde{p}}(\mathbf{R}^n)) + \langle x\rangle^{-\beta}C(\mathbf{R}^1, L^\infty(\mathbf{R}^n))$ and $\partial_t V \in \langle x\rangle^{-2}L^{p_1,\alpha_1}(\mathbf{R}^1) + \langle x\rangle^{-\gamma}L^{\infty,1}(\mathbf{R}^1)$ for $\tilde{p} = max(p,2)$, $2 < \beta < 3$, $p_1 = 2np/(n+4p)$ $(n \geq 5)$, $p_1 > 2p/(p+1)$ $(n = 4)$, $p_1 = 2p/(p+1)$ $(n \leq 3)$, $\gamma > 0$ and $\alpha_1 = 4p/(4p-n)$.

THEOREM 9. *Let Assumptions (A) and (C) be satisfied. Then,*
(1) $U(\cdot,\cdot) \in C_*(\mathbf{R}^2, B(\Sigma(2))) \cap C_*^1(\mathbf{R}^2, B(\Sigma(2), \mathbf{H}))$.
(2) If $f \in \Sigma(2)$, then $U(t,s)f$ satisfies (1.18) and (1.19) of Theorem 8.

The proofs of Theorems 1 and 7 to 9 can be found in [**30**] and those of Theorem 3 to 6 are given in [**31**] for the case $V(t,x) = 0$. Theorems 3 to 6 for the case $A(t,x) = 0$ with singular potentials are new materials here.

We shall review the outline of the proofs of Theorem 1 and 7 in section 2. In section 3 we shall prove Theorems 3 and 4 for the case $A(t,x) = 0$. Since the proofs of Theorems 5 and 6 for this case are identical with the corresponding ones in [**31**] for the case $V(t,x) = 0$, we omit it here.

§2. Strategy of the proof of Theorems 1 and 7.

Since the proof of Theorem 1 heavily relies on Theorem 7, we begin with a sketch of the proof of the latter theorem.

We denote the classical mechanical Hamiltonian associated with (1.12) by

$$H_0(t,x,\xi) = (1/2)\sum_{j=1}^n (\xi_j - A_j(t,x))^2$$

and by $(x(t,s,y,\eta),\xi(t,s,y,\eta))$ the solutions of Hamilton's equations

$$dx/dt = \partial H_0/\partial\xi, \quad d\xi/dt = -\partial H_0/\partial x \tag{2.1}$$

with the initial conditions $(x(s),\xi(s)) = (y,\eta)$. To make the dimension same, we introduce new variables and set

$$\tilde{x}(t,s,y,\eta) = x(t,s,y,\eta/(t-s)), \quad \tilde{\xi}(t,s,y,\eta) = (t-s)\xi(t,s,y,\eta/(t-s))$$

for $t \neq s$ and $\tilde{x}(t,t,y,\eta) = y+\eta$, $\tilde{\xi}(t,t,y,\eta) = \eta$.

LEMMA 2.1. *$(\tilde{x}(t,s,y,\eta),\tilde{\xi}(t,s,y,\eta))$ exists globally for $t,s \in \mathbf{R}^1$ and, for every α and β, $\partial_y^\alpha\partial_\eta^\beta\tilde{x}$ and $\partial_y^\alpha\partial_\eta^\beta\tilde{\xi}$ are C^1 functions of (t,s,y,η). For $|t-s| \le 1$*

$$\begin{aligned} &|\partial_y^\alpha\partial_\eta^\beta(\partial\tilde{x}_j/\partial y_k - \delta_{jk})| + |\partial_y^\alpha\partial_\eta^\beta(\partial\tilde{x}_j/\partial\eta_k - \delta_{jk})| \\ &\quad + |\partial_y^\alpha\partial_\eta^\beta(\partial\tilde{\xi}_j/\partial y_k)| + |\partial_y^\alpha\partial_\eta^\beta(\partial\tilde{\xi}_j/\partial\eta_k - \delta_{jk})| \le C_{\alpha\beta}|t-s|, \end{aligned} \tag{2.2}$$

where δ_{jk} is Kronecker's delta;

$$|\tilde{\xi}(t,x,y,\eta) - \eta| + |\tilde{x}(t,s,y,\eta) - y - \eta| \le C|t-s|(1+|y|+|\eta|). \tag{2.3}$$

It can be deduced from (2.2) that there exists $T>0$ such that, for every fixed $|t-s|<T$ and $y \in \mathbf{R}^n$, $\mathbf{R}^n \ni \eta \to x = \tilde{x}(t,s,y,\eta) \in \mathbf{R}^n$ is a global diffeomorphism of $\mathbf{R}^n$. Write $\tilde{\eta}(t,s,y,x)$ for its inverse. Then

$$(x(\tau),\xi(\tau)) = (x(\tau,s,y,(t-s)^{-1}\tilde{\eta}(t,s,y,x)),\xi(\tau,s,y,(t-s)^{-1}\tilde{\eta}(t,s,y,x))) \tag{2.4}$$

is a unique solution of (2.1) such that $x(t) = x$ and $x(s) = y$. We denote by $S(t,s,x,y)$ the action integral along this path:

$$S(t,s,x,y) = \int_s^t \{(\partial_\xi H_0)(\tau,x(\tau),\xi(\tau))\cdot\xi(\tau) - H_0(\tau,x(\tau),\xi(\tau))\}d\tau.$$

LEMMA 2.2. *$S(t,s,x,y)$ satisfies the property (a) of Theorem 7, i.e. it satisfies Hamilton-Jacobi equation (1.15) and the estimate (1.16). Moreover,*

$$(t-s)(\partial_x S)(t,s,x,y) = \tilde{\xi}(t,s,y,\tilde{\eta}(t,s,y,x)), \tag{2.5}$$

$$(t-s)(\partial_y S)(t,s,x,y) = -\tilde{\eta}(t,s,y,\dot{x}). \tag{2.6}$$

Note that (1.16) shows that $S(t,s,x,y)$ has the properties of phase functions of OIOs studied in [**1**]. We let $e(t,s,x,y)$ be the solution of the transport equation

$$\partial_t e + (\nabla_x S - A(t,x))\nabla_x e + \frac{1}{2}(\Delta_x S - \frac{n}{t-s} - div_x A(t,x))e = 0$$

associated with $S(t,s,x,y)$ with the initial condition $e(s,s,x,y) = 1$. For $T>0$ small enough (to be specified below) we write $\Delta(T) = \{(t,s) \in \mathbf{R}^2 : |t-s| \le T\}$. $\mathcal{B}(\mathbf{R}^m) = \{f : \|f\|_{\mathcal{B}^k} = sup_{x\in\mathbf{R}^m}\sum_{|\alpha|\le k}|\partial_x^\alpha f(x)| < \infty,\ k = 0,1,2,\ldots\}$ is Fréchet space of bounded C^∞ functions with bounded derivatives.

LEMMA 2.3. *$e(t,s,\cdot,\cdot) \in C(\Delta(T), \mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$ and satisfies*

$$|\partial_x^\alpha \partial_y^\beta (e(t,s,x,y) - 1)| \le C_{\alpha\beta}|t-s|. \tag{2.7}$$

Using S of Lemma 2.2 and e of Lemma 2.3, we define the (first order) semi-classical approximation $E(t,s)$ of the propagator by

$$E(t,s)f(x) = (2\pi i(t-s))^{-n/2} \int e^{iS(t,s,x,y)} e(t,s,x,y) f(y) dy.$$

When $a(t,s,x,y)$ (resp. $p(t,s,x,\xi,y)$) is continuous in $(t,s) \in \Delta(T)$ with values in $\mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$ (resp. $\mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$), we say $a \in Amp$ (resp. $p \in Sym_0$). $I(t,s,a)$ is the OIO with the phase $S(t,s,x,y)$ and amplitude $a(t,s,x,y)$:

$$I(t,s,a)f(x) = (2\pi i(t-s))^{-n/2} \int e^{iS(t,s,x,y)} a(t,s,x,y) f(y) dy \tag{2.8}$$

and $\Psi DO(t,s)$ is the set of all ν-pseudo-differential operators

$$p(t,s,x,\nu D, x) f(x) = (2\pi\nu)^{-n} \int e^{i(x-y)\xi/\nu} p(t,s,x,\xi,y) f(y) dy d\xi$$

with symbols $p \in Sym_0$, $\nu = t - s$.

Operators of the form (2.8) are of class Asada and Fujiwara[**1**] and the following lemma summarizes their properties to be used in what follows.

LEMMA 2.4. *There exists $T > 0$ such that for $(t,s) \in \Delta(T)$ the following statements are satisfied:*
(1) For $k = 0, 1, \cdots$, there exists a constant C_k such that

$$\|I(t,s,a)f\|_{\Sigma(k)} \le C_k \|a(t,s,\cdot,\cdot)\|_{\mathcal{B}^{M(k)}(\mathbf{R}^{2n})} \|f\|_{\Sigma(k)}, \quad a \in Amp, \tag{2.9}$$

where $M(k) = 2n + k + 1$.
(2) $I(t,s,a)$ is a continuous operator on $\mathcal{S}(\mathbf{R}^n)$.
(3) For $a, b \in Amp$ and $p \in Sym_0$, there exist $c, d, f, g \in Amp$ and $q \in Sym_0$ such that

$$I(t,s,a)^* = I(s,t,c), \quad I(t,r,a)I(r,s,b) = I(t,s,d). \tag{2.10}$$

$$I(t,s,a)p(t,s,x,\nu D,x) = I(t,s,f), \quad p(t,s,x,\nu D,x)I(t,s,a) = I(t,s,g). \tag{2.11}$$

$$I(t,s,a)I(s,t,b) = q(t,s,x,\nu D,x). \tag{2.12}$$

Moreover c, d, f, g $\in$ Amp and q $\in$ Sym_0 in (2.10) to (2.12) depend continuously on a, b and p in the topology of $\mathcal{B}(\mathbf{R}^{2n})$ or $\mathcal{B}(\mathbf{R}^{3n})$.
(4) For subdivision Ω : $s = s_0 < s_1 < \cdots < s_L = t$ and $a_1, a_2, \cdots, a_L \in$ Amp there exists $a \in$ Amp such that $I(t, s_{L-1}, a_L) \cdot I(s_{L-1}, s_{L-2}, a_{L-1}) \cdots I(s_1, s, a_1) = I(t, s, a)$. For $m = 0, 1, \cdots,$

$$\|a(t,s,\cdot,\cdot)\|_{\mathcal{B}^m(\mathbf{R}^{2n})} \leq \kappa(m)^L \prod_{j=1}^{L} \|a_j(s_j, s_{j-1}, \cdot, \cdot)\|_{\mathcal{B}^{R(m)}(\mathbf{R}^{2n})}, \tag{2.13}$$

where constants $\kappa(m)$ and $R(m)$ are independent of L, subdivision Ω, functions $a_j \in$ Amp, $1 \leq j \leq L$ and ν.

In what follows $T > 0$ will be always assumed to be sufficiently small so that Lemma 2.4 is satisfied. In virtue of standard semi-group theory, it suffices to prove Theorems 1 and 7 for $|t - s| < T$.

PROOF OF THEOREM 7: A simple computation shows that

$$\begin{aligned} iG(t,s)f(x) &\equiv (-i\partial_t + H_0(t))E(t,s)f(x) \\ &= -(2\pi i\nu)^{-n/2} \int e^{i\tilde{S}(t,s,x,y)/\nu} (1/2)\Delta_x a(t,s,x,y) f(y) dy, \end{aligned} \tag{2.14}$$

where $\tilde{S}(t,s,x,y) = (t-s)S(t,s,x,y)$ and $\nu = t - s$. In virtue of Lemma 2.4 and (2.7) it follows that $E(t,s)$ and $G(t,s)$ satisfy the following properties:
(1) For $t \neq s$, $E(t,s)$ and $G(t,s)$ are continuous operators both in $\Sigma(k)$ and $\mathcal{S}(\mathbf{R}^n)$, and

$$\|E(t,s)f\|_{\Sigma(k)} \leq C\|f\|_{\Sigma(k)}, \quad \|G(t,s)f\|_{\Sigma(k)} \leq C|t-s|\|f\|_{\Sigma(k)}. \tag{2.15}$$

(2) For $f \in \Sigma(k)$, $\|E(t,s)f - f\|_{\Sigma(k)} \to 0$ as $t \to s$ and if we set $E(t,t)f = f$ and $G(t,t)f = 0$, then $E(\cdot,\cdot)$ and $G(\cdot,\cdot) \in C_*(\Delta(T), B(\Sigma(k)))$.
(3) If $f \in \mathcal{S}(\mathbf{R}^n)$, then $E(t,s)f$ and $G(t,s)f$ are continuous in $\mathcal{S}(\mathbf{R}^n)$ for $(t,s) \in \Delta(T)$.

Using properties (1) to (3) above, we see that, if $U_0(t,s)$ is the propagator with the properties of Theorem 7, then (2.14) may be solved for $E(t,s)$ in the form:

$$E(t,s)f = U_0(t,s)f - \int_s^t U_0(t,r)G(r,s)f dr. \tag{2.16}$$

For constructing $U_0(t,s)$ we regard (2.16) as an operator equation for $U_0(t,s)$ and solve it by successive approximation:

$$U_0(t,s) = E(t,s) + (E * G)(t,s) + (E * G * G)(t,s) + \cdots, \tag{2.17}$$

where $(F * G)(t,s) = \int_s^t F(t,r)G(r,s)dr$. (2.15) implies

$$\|E * G * G * \cdots * G\|_{B(\Sigma(\ell))} \leq C_\ell^k |t-s|^{2k} \frac{1}{\Gamma(2(k+1))}$$

and, for every $\ell = 0, 1, \cdots$, the series (2.17) converges in operator norm of $B(\Sigma(\ell))$ uniformly in $(t,s) \in \Delta(T)$ to define $U_0(\cdot,\cdot) \in C_*(\Delta(T), B(\Sigma(\ell)))$, which clearly satisfies (2.16), hence (1) to (3) of Theorem 7.

Moreover, by (2.13), we see that $(E * G * \cdots * G)(t,s) = I(t,s,c_{Nk})$ with $c_{Nk}(t,s,x,y)$ satisfying

$$\|c_k\|_{\mathcal{B}^m} \leq C_m^{k+1} \frac{|t-s|^{2k}}{\Gamma(2k+1)}, \quad m = 0, 1, \cdots. \tag{2.18}$$

Thus, the sequence $b(t,s,x,y,h) = \sum_{k=0}^\infty c_{Nk}(t,s,x,y,h)$ converges in $\mathcal{B}(\mathbf{R}^{2n})$ uniformly in $\Delta(T)$ and $U_0(t,s) = I(t,s,b)$. This implies that $U_0(t,s)$ satisfies (4) of Theorem 7.

PROOF OF THEOREM 1:

We first remark that the multiplication operator $V(t)$ with $V(t,x)$ is $H_0(t)$-form bounded with bound 0 and the Friedrich extension of $H(t) = H_0(t) + V(t)$ coincides with the part of $H(t) \in B(\mathbf{H} \cap L^q(\mathbf{R}^n), \Sigma(-2))$ in $\mathbf{H}$.

We solve (1.1) via the equivalent integral equation:

$$u(t) = U_0(t,s)f - i\int_s^t U_0(t,r)V(r)u(r)dr. \tag{2.19}$$

We set, for the parameter ℓ with $0 \leq 2/\theta = n(1/2 - 1/\ell) < 1$, the function spaces

$$X(I,\ell) = C_b(I,\mathbf{H}) \cap L^{\ell,\theta}(I), \quad X^*(I,\ell) = L^{2,1}(I) + L^{\ell',\theta'}(I),$$

and solve (2.19) in $X(I,q)$, $q = 2p/(p-1)$ with a small interval I, where ℓ' and θ' are indices conjugate to ℓ and θ. We say $u \in X_{loc}(I,\ell)$ if $u \in X(I',\ell)$ for any compact subinterval I' of I. We need the following four lemmas. The first lemma is an immediate consequence of (2.9) and the complex interpolation theory.

LEMMA 2.5. *Let $a \in Amp$. Then:*
(1) There exists a constant C depending only on n such that for any $2 \leq \ell \leq \infty$ and t, $s \in \Delta(T) \setminus \Delta$,

$$\|I(t,s,a)f\|_\ell \leq C\|a(t,s,\cdot,\cdot)\|_{\mathcal{B}^{4n+1}} |t-s|^{-n(1/2-1/\ell)} \cdot \|f\|_{\ell'}. \tag{2.20}$$

(2) For $f \in L^{\ell'}(\mathbf{R}^n)$, $I(t,s,a)f \in C(\Delta(T) \setminus \Delta, L^\ell(\mathbf{R}^n))$.

The next lemma can be proved by using (2.20) and the Hardy-Littlewood-Sobolev inequality.

LEMMA 2.6. *Let $0 \le 2/\theta(\ell) = n(1/2 - 1/\ell) < 1$ and let $Z(t,s)$ be $B(\mathsf{H})$-valued measurable function of $(t,s) \in \mathbf{R}^2$ which satisfies the following conditions:*
(a) For any $f \in \mathsf{H}$ and $t_0 \in \mathbf{R}^1$, $Z(t,s)f$ and $Z(s,t)^*f$ are continuous in H with respect to t at $t = t_0$ for almost all $s \in \mathbf{R}^1$.
(b) For $f \in L^{\ell'}(\mathbf{R}^n) \cap \mathsf{H}$ and $(t,s) \notin \Delta$, $Z(t,s)f \in L^{\ell}(\mathbf{R}^n)$ and

$$\|Z(t,s)f\|_\ell \le C|t-s|^{-n(1/2-1/\ell)}\|f\|_{\ell'}$$

with constant $C > 0$ independent of t, s and f.
(c) For any $r \in \mathbf{R}^1$ and $(t,s) \notin \Delta$, $Z(r,t)^*Z(r,s)$ and $Z(t,r)Z(s,r)^*$ defined on $L^{\ell'}(\mathbf{R}^n) \cap \mathsf{H}$ can be extended to bounded operators from $L^{\ell'}(\mathbf{R}^n)$ to $L^{\ell}(\mathbf{R}^n)$ and

$$\|Z(r,t)^*Z(r,s)f\|_\ell \le C|t-s|^{-n(1/2-1/\ell)}\|f\|_{\ell'},$$
$$\|Z(t,r)Z(s,r)^*f\|_\ell \le C|t-s|^{-n(1/2-1/\ell)}\|f\|_{\ell'}.$$

Then:
(1) For any $f \in \mathsf{H}$ and s, $u(\cdot) = Z(\cdot,s)f \in L^{\ell,\theta}(\mathbf{R}^1)$ and with constant C independent of s and f,

$$\|Z(\cdot,s)f\|_{\ell,\theta} \le C\|f\|.$$

(2) The integral operator $Ju(t) = \int_{-\infty}^{\infty} Z(t,s)u(s)ds$ is bounded from $X^*(\mathbf{R}^1,\ell)$ to $X(\mathbf{R}^1,\ell)$:

$$\|Ju\|_{X(\mathbf{R}^1,\ell)} \le C\|u\|_{X^*(\mathbf{R}^1,\ell)}.$$

Since $U_0(t,s) = I(t,s,b)$ with $b \in Amp$, it obviously satisfies Lemma 2.5, hence the conditions (a) to (c) of Lemma 2.6. Accordingly, Lemma 2.6 implies the following

LEMMA 2.7. *Let $0 \le 2/\theta = n(1/2 - 1/\ell) < 1$. Then there exists a constant $C > 0$ such that the following estimates are satisfied for any $R > 0$ and $a \in \mathbf{R}^1$.*
(1) For $f \in \mathsf{H}$, $U_0(\cdot,a)f \in L^{\ell,\theta}_{loc}(\mathbf{R}^1)$ and

$$\|U_0(\cdot,a)f\|_{L^{\ell,\theta}([a-R,a+R])} \le C(1+R)^{1/\theta}\|f\|, \quad f \in \mathsf{H}.$$

(2) Let G_a be the integral operator defined by $G_a u(t) = \int_a^t U_0(t,s)u(s)ds$. Then G_a is a bounded operator from $X^*([a-R,a+R],\ell)$ to $X([a-R,a+R],\ell)$ and

$$\|G_a u\|_{X([a-R,a+R],\ell)} \le C(1+R)^{2/\theta}\|u\|_{X^*([a-R,a+R],\ell)}.$$

If we define a Banach space for potentials by $M(I) = L^{p,\alpha}(I) + L^{\infty,1}(I)$ with the standard norm:

$$\|u\|_{M(I)} = inf\{\|V_1\|_{L^{p,\alpha}}(I) + \|V_2\|_{L^{\infty,1}}(I) : V = V_1 + V_2\},$$

where p and α are the exponents in Assumption (A), then Hölder's inequality implies that the multiplication operator V satisfies the following properties.

LEMMA 2.8. *Let $q = 2p/(p-1)$. Then:*

$$\|Vu\|_{X^*(I,q)} \leq \|V\|_{M(I)} \cdot \|u\|_{X(I,q)}.$$

Setting $Q_s f(t) = U_0(t,s)f$, we write the integral equation (2.19) in the form

$$u(t) = Q_s f(t) + G_s V u(t). \tag{2.19'}$$

As in the proof of Theorem 3, we only have to consider in a small interval $I = [t_0 - T/2, t_0 + T/2]$. Lemma 2.7 and 2.8 show that $Q_s \in B(\mathbf{H}, X(I,q))$ and $G_s V \in B(X(I,q))$, both depending on s continuously in the strong topology of operator valued functions. Moreover, if we take $T > 0$ sufficiently small, $G_s V$ is a contraction on $X(I,q)$. It follows that (2.19') can be uniquely solved for $u \in X(I,q)$:

$$u = (1 - G_s V)^{-1} Q_s f = \sum_{k=0}^{\infty} (G_s V)^k Q_s f.$$

We define $U(t,s)$, for t, $s \in I$, by $U(t,s)f = u(t)$, that is, $U(t,s) = \Gamma_t(1 - G_s V)^{-1} Q_s$, where $\Gamma_t \in B(X(I,q), \mathbf{H})$ is the evaluation functional defined by $\Gamma_t u = u(t)$. Clearly $U(\cdot,\cdot) \in C_*(I^2, B(\mathbf{H}))$ and $u(\cdot) = U(\cdot,s)f \in L^{q,4p/n}(I)$ for any $f \in \mathbf{H}$, and, as can be easily shown from its construction, this is the desired propagator for (1.1) satisfying the properties on Theorem 1.

§3. Proof of Theorem 3 and 4 for the case $A(t,x) = 0$.

In virtue of the complex interpolation theory for weighted Sobolev spaces (cf. e.g. [**28**]), it is well known that

$$[W^{k_1,p_1}(\mathbf{R}^n, \langle x\rangle^{\mu_1}dx), W^{k_2,p_2}(\mathbf{R}^n, \langle x\rangle^{\mu_2}dx)]_\theta = W^{k_\theta,p_\theta}(\mathbf{R}^n, \langle x\rangle^{\mu_\theta}(x)dx), \tag{3.1}$$

where $k_\theta = (1-\theta)k_1 + \theta k_2$, $1/p_\theta = (1-\theta)/p_1 + \theta/p_2$ and $\mu_\theta = (1-\theta)\mu_1 + \theta\mu_2$, $0 \leq \theta \leq 1$. Hence Theorem 4 is a consequence of Theorem 3 and Theorem 1, (3).

We prove Theorem 3 for the case $A(t,x) = 0$. We assume $s = 0$ and suppress the index s. In this case $U_0(t,s) = \exp(i(t-s)\Delta/2)$ and (2.19) reads

$$u(t) = Qf(t) + GVu(t) = \exp(it\Delta/2)f - i\int_0^t \exp(i(t-s)\Delta/2)V(s)u(s)ds.$$

We let $F_\epsilon(\xi) = \langle\xi\rangle^{1/2}/(1+\epsilon\xi^2)$ and write, with a little abuse of notation, $F_\epsilon(D_j) = (1+D_j^2)^{1/4}/(1+\epsilon D_j^2)$, $\epsilon > 0$, and $\phi(t) = (1+t^2)^{-\mu/2}$, $t \in \mathbf{R}$. The commutators $[F_\epsilon(D), \langle x\rangle^{-\mu}]$ and $F_\epsilon(D_j)[\phi^2(x_j), F_\epsilon(D_j)]$ are uniformly bounded in $\mathbf{H}$ for $\epsilon > 0$, $j = 1, \cdots, n$. Hence we have

$$\|F_\epsilon(D)\langle x\rangle^{-\mu}v\|^2 \leq \sum_{j=1}^n \|F_\epsilon(D_j)\langle x\rangle^{-\mu}v\|^2 \leq \sum_{j=1}^n \|\phi(x_j)F_\epsilon(D_j)v\|^2 + C\|v\|^2;$$

$$\begin{aligned}\|\phi_j(x)F_\epsilon(D_j)v\|^2 &= (F_\epsilon(D_j)\phi^2(x_j)F_\epsilon(D_j)v, v)\\ &= (F_\epsilon(D_j)^2\phi^2(x_j)v, v) + O(\|v\|), \quad v \in \mathsf{H}.\end{aligned}$$

It follows that with a constant $C > 0$ independent of $\epsilon > 0$

(3.2)

$$\int_0^T \|\langle x\rangle^{-\mu}F_\epsilon(D)u(t)\|^2 dt \le \sum_{j=1}^n \int_0^T \|\phi(x_j)F_\epsilon(D_j)u(t)\|^2 dt + C\|f\|^2$$

$$\sum_{j=1}^n \int_0^T (F_\epsilon(D_j)^2\phi^2(x_j)u(t), u(t))dt + O(\|f\|^2).$$

Here we used $\|u(t)\| = \|f\|$. Introducing the notation

$$g(t) = e^{-it\Delta/2}GVu(t) = \int_0^t e^{-is\Delta}V(s)u(s)ds$$

and

$$\Phi_j(T, x, D) = \int_0^T e^{-it\Delta/2}F_\epsilon(D_j)^2\phi^2(x_j)e^{it\Delta/2}dt,$$

we write the integral in the right of (3.2) in the form

$$\begin{aligned}(3.3) \qquad &(\Phi_j(T, x, D)f, f) + \int_0^T (\Phi_j'(t, x, D)f, g(t))dt\\ &+ \int_0^T (\Phi_j'(t, x, D)g(t), f)dt + \int_0^T (\Phi_j'(t, x, D)g(t), g(t)),\end{aligned}$$

where $\Phi_j'(t, x, D) = e^{-it\Delta/2}F_\epsilon(D_j)^2\phi^2(x_j)e^{it\Delta/2}$ is the t-derivative of $\Phi_j(t, x, D)$.
We use the following

LEMMA 3.1. *For every $j = 1, \cdots, n$ and $t \in \mathbf{R}$, the operator $\Phi_j(t, x, D)$, is a bounded operator of $L^\ell(\mathbf{R}^n)$ for every $1 < \ell < \infty$. Its operator norm is bounded uniformly in $|t| \le T$, $T > 0$, and $\epsilon > 0$.*

$$(3.4) \qquad e^{it\Delta/2}\Phi_j(t, x, D)e^{-it\Delta/2} = -\Phi_j(-t, x, D).$$

PROOF: (3.4) is obvious and we prove the boundedness only. Since $F_\epsilon(D_j)^2$ and $e^{-it\Delta/2}$ commutes, we have

$$e^{-it\Delta/2}F_\epsilon(D_j)^2\phi^2(x_j)e^{it\Delta/2} = F_\epsilon(D_j)^2\phi^2(x_j + tD_j)$$

and $\phi^2(x_j + tD_j)$ is Weyl operator with the symbol $\phi^2(x_j + t\xi_j)$. It is esay to see that $(1 + \epsilon D_j^2)^{-1}$ has L^1 convolution integral kernel with uniform L^1-norm in

x_j-variable, hence, is a uniformly bounded operator in every $L^\ell(\mathbf{R}^n)$, $1 \le \ell \le \infty$. Take $\chi(\xi) \in C_0^\infty(\mathbf{R})$ such that $\chi(\xi) = 1$ near zero and split

$$(1+\xi_j^2)^{1/2} = (1+\xi_j^2)^{1/2}\chi(\xi_j) + (1+\xi_j^2)^{1/2}(1-\chi(\xi_j))\xi_j^{-1} \cdot \xi_j.$$

Here $(1+D_j^2)^{1/2}\chi(D_j)$ is obviously L^ℓ-bounded and the standard result of pseudo-differntial operators shows that $(1+D_j^2)^{1/2}(1-\chi(D_j))D_j^{-1}$ is also bounded in $L^\ell(\mathbf{R}^n)$, $1 < \ell < \infty$. (This is essentially the Hilbert-Transform.) Hence it suffices to prove the boundedness of $\int_0^T \phi^2(x_j+tD_j)dt$ and $\int_0^T D_j\phi^2(x_j+tD_j)dt$, the latter being the Weyl operator with the symbol

$$\int_0^T \phi^2(x_j+t\xi_j)\xi_j dt - (i/2)\int_0^T (\phi^2)'(x_j+t\xi_j)dt \tag{3.5}$$

Note that in general

$$\begin{aligned} g(x+tD)f(x) &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi} g((x+y)/2+t\xi)f(y)dyd\xi \\ &= e^{-ix^2/2t}(g(tD)e^{iy^2/2t}f)(x) \end{aligned} \tag{3.6}$$

It follows that $\phi^2(x_j+tD_j)$ and $(\phi^2)'(x_j+tD_j)$, hence the Weyl operator of which the symbol given by the second integral of (3.5) are uniformly bounded in L^ℓ for every $1 \le \ell \le \infty$. Writing the first integral of (3.5) as

$$\int_0^T \phi^2(x_j+t\xi_j)\xi_j dt = \int_{x_j}^{x_j+T\xi_j} \phi^2(t)dt = \Psi(x_j+T\xi_j) - \Psi(x_j)$$

with $\Psi(t) = \int_0^t \phi^2(s)ds$, we see by (3.6) that the Weyl operator with the symbol $\int_0^T \phi(x_j+t\xi_j)^2\xi_j dt$ is given by

$$\Psi(x_j+TD_j)f(x) - \Psi(x_j) = e^{-ix_j^2/2T}(\Psi(TD_j)e^{iy_j^2/2T}f)(x) - \Psi(x_j)f(x). \tag{3.7}$$

Since $\Psi(t)$ is bounded and its derivatives satisfies

$$|d^k\Psi/dt^k(t)| \le C_k\langle t\rangle^{-\mu+1-k}, \quad k = 1, 2, \cdots$$

the classical theorem for L^ℓ-multiplier theorem (cf. e.g. [**27**]) implies that $\Psi(TD_j)$ is L^ℓ-bounded for $1 < \ell < \infty$. This completes the proof of the lemma. ∎

By integration by parts we rewite (3.3) into the following form with $\Phi_j(t) = \Phi_j(t, x, D)$:

$$\begin{aligned} &(\Phi_j(T)f, f) + (\Phi_j(T)f, g(T)) + (\Phi_j(T)g(T), f) + (\Phi_j(T)g(T), g(T)) \\ &+ \int_0^T (\Phi_j(-t)e^{-it\Delta}f, (Vu)(t))dt + \int_0^T (\Phi_j(-t)(Vu)(t), e^{-it\Delta}f)dt \\ &+ \int_0^T (\Phi_j(-t)(Vu)(t), (GVu)(t))dt + \int_0^T (\Phi_j(-t)(GVu)(t), (Vu)(t))dt. \end{aligned}$$

Each term above can be estimated by $C\|f\|^2$ by applying Lemma 3.1 and the estimates

$$\|g(T)\| \leq C\|f\|; \tag{3.8}$$

$$\|e^{-it\Delta}f\|_{L^{4p/n}(I,L^q)} \leq C\|f\|, \quad \|Vu\|_{L^{4p/n}(I,L^q)^*} \leq C\|f\|; \tag{3.9}$$

and

$$\|GVu\|_{L^{4p/n}(I,L^q)} \leq C\|f\| \tag{3.10}$$

which are the results of Lemma 2.7 and 2.8. (3.8) and Lemma 3.1 show that the first four terms are bounded by $C\|f\|^2$; (3.9) and Lemma 3.1 show the following two terms are bounded by $C\|f\|^2$; and (3.10) implies that the last two terms are also bounded by $C\|f\|^2$. This completes the proof of Theorem 3.

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