数理解析研究所講究録
第 755 巻 1991 年 134-155

Semigroups of locally Lipschitzian operators

局所リップシッツ作用素の半群について

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Let $(X, |\cdot|)$ be a Banach space and D a subset of X. A one-parameter family $S = \{S(t) : t \geq 0\}$ of possibly nonlinear operators from D into itself is called a (*nonlinear*) *semigroups* on D if it has the two properties below:

(S1) For $s, t \geq 0$ and $x \in D$, $S(0)x = x$ and $S(s+t)x = S(s)S(t)x$.

(S2) For $x \in D$, $u(\cdot) \equiv S(\cdot)x$ is continuous on $[0, \infty)$ with respect to t.

In order to advance a general theory of nonlinear semigroups, it is necessary to restrict the continuity of the operators $S(t)$. Semigroup $S = \{S(t) : t \geq 0\}$ on D is said to be quasi-contractive if there is a constant ω such that

$$|S(t)x_1 - S(t)x_2| \leq e^{\omega t}|x_1 - x_2| \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x_1, x_2 \in D.$$

For the application to partial differential equations, such a continuity of solution operators can not be expected in case that the number ω depends on the values of the "state variable" x or those of the quantity defined as a function of the "state variable" x. In this paper we employ a lower semi-continuous functional φ on X and subdivide the set D into the "level" sets $D_\alpha = \{x \in D : \varphi(x) \leq \alpha\}$, $\alpha \geq 0$ to describe this situation. Let $\varphi : X \to [0, \infty]$ be a proper lower semi-continuous functional.

We consider the following type of Lipschitz condition on a semigroup $S = \{S(t) : t \geq 0\}$ on D in a local sense with respect to the functional φ:

(L) For $\alpha \geq 0$ and $\tau \geq 0$ there exists $\omega \equiv \omega(\alpha, \tau) \in \boldsymbol{R}$ such that

$$|S(t)x_1 - S(t)x_2| \leq e^{\omega t}|x_1 - x_2| \qquad \text{for } x_1, x_2 \in D_\alpha \text{ and } t \in [0, \tau].$$

Condition (L) defines a considerably general class of semigroups on D and this class is of our main interest in this paper. A semigroup S on D satisfying condition (L) for some lower semi-continuous functional φ is said to be locally quasi-contractive on D with respect to φ or belong to the *class* $\mathcal{S}(D,\varphi)$.

Semigroups as introduced above arise as families of solution operators to the initial-value problems for differential inclusions of the form

$$\text{(DI)} \qquad (d/dt)u(t) \in Au(t), \quad t > 0; \quad \text{(IC)} \qquad u(0) = x_0,$$

where x_0 is an initial-value given in D and A is a possibly multi-valued operator in X. The initial-value problem (DI)–(IC) has been studied by many authors. Especially, under the assumption that A is quasi-dissipative in X, various types of sufficient conditions on A ensuring the existence of solutions (perhaps in a generalized sense) have been investigated and some of the basic results in this direction are given in the papers by Kōmura [18], Kato [15, 16], Crandall and Liggett [7, 8], Kenmochi and Oharu [17], Takahashi [38], Kobayashi [20], Pierre [35, 35], Walker [39], Martin [27, 28], Pazy [28, 33], Schechter [37] and Goldstein [14]. We here show that the results mentioned above can be extended to the case where the nonlinear operator A in (DI) is locally quasi-dissipative with respect to the functional φ in the sense that

(LQD) $D(A) \subset D$, and for each $\alpha \geq 0$ there exists $\omega \equiv \omega(\alpha) \in \boldsymbol{R}$ such that

$$[x_1 - x_2, y_1 - y_2]_- \leq \omega|x_1 - x_2|$$

for $x_1, x_2 \in D(A) \cap D_\alpha$, $y_1 \in Ax_1$ and $y_1 \in Ax_2$.

Condition (LQD) is proper for the class $\mathcal{S}(D,\varphi)$ in the sense that the infinitesimal generator (if it exists in a reasonable sense) of a semigroup belonging to the class $\mathcal{S}(D,\varphi)$ satisfies condition (LQD), and conversely, that under conditions (LQD) on A the semigroup consisting of the solution operators of (DI) belong to the class $\mathcal{S}(D,\varphi)$.

In the subsequent discussions, we first discuss the existence of generalized solutions of the initial value problem for (DI) under the condition (LQD) and so-called *range condition.* These conditions together guarantee the existence of the discrete scheme

$$\text{(DS)} \qquad \begin{cases} (t_k - t_{k-1})^{-1}(x_k - x_{k-1}) - z_k \in Ax_k, & k = 1, 2, \cdots, N \\ z_k \in X, \qquad x_0 \in D, & 0 \leq t_0 < t_1 < \cdots < t_k < \cdots, \end{cases}$$

so far as the norm of the partition $\Delta = (t_k)$ and the error terms (z_k) are sufficiently small. Hence a modified version of the standard method of discretization in time can be applied under the localized quasi-dissipativity condition (LQD), and the generalized solution is obtained as the limits of solutions of the discrete problem (DS) as the norm of Δ and the errors (z_k) tend to zero. We call the generalized solution so obtained a *mild solution* of (DI). Under some general range conditions which will be considered later, such a mild solution of the problem (DI)–(IC) will exist only locally in time in general. But, if we assume the locally quasi-dissipative operator A satisfy stronger range conditions, the mild solutions of the problem (DI)–(IC) exist globally in time and a semigroup $S = \{S(t) : t \geq 0\}$ in the class $\mathcal{S}(D, \varphi)$ on D is obtained as a family of solution operators to the initial-value problem (DI)–(IC). The semigroup $S = \{S(t) : t \geq 0\}$ satisfies a *growth condition* of a certain type which corresponds to *a priori* estimates of mild solutions of (DI)–(IC). Our results extend those of Chambers and Oharu [5] and Goldstein [14], and it is expected that the generation results can be applied to a broad class of nonlinear partial differential equations. In this connection we notice that in the recent papers by Oharu and Takahashi [30, 31] nonlinear semigroups associated with semilinear evolution equations are discussed from the same point of view.

Secondly, we investigate the generators and the differentiability of semigroups in the class $\mathcal{S}(D, \varphi)$ under the additional assumption that X is reflexive and the norm $|\cdot|$ is uniformly Gâteaux differentiable. We shall introduce a notion of generalized infinitesimal generator of a semigroup in the class $\mathcal{S}(D, \varphi)$ and show that such generalized infinitesimal generators satisfy condition (LQD). We will see that in smooth reflexive Banach spaces as mentioned above one can assert the existence of the generalized infinitesimal generator for each semigroup $S = \{S(t) : t \geq 0\}$ in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition with respect to φ. We here focus our attention on the study of semigroups in the class $\mathcal{S}(D, \varphi)$ satisfying the *exponential growth condition* and make an attempt to establish a nonlinear analogue of the Hille-Yosida theorem for such semigroups under the above-mentioned assumptions on X. It turns out that we obtain a (self-contained) general theory for semigroups of locally Lipschitzian operators which includes the theory of quasi-contractive semigroups as a special case.

Section 1 introduces a class of nonlinear operators which are quasi-dissipative in a local sense and then the associated class $\mathcal{S}(D, \varphi)$ of semigroups of locally Lipschitzian operators. The notion of mild solution of the initial-value problem for (DI)–(IC) is introduced and their basic properties are investigated. Section 2 deals with the existence of mild solutions for (DI)–(IC) and the generation of semigroups in the class $\mathcal{S}(D, \varphi)$. In Section 3, the notion of generalized infinitesimal generator of a semigroup in the class $\mathcal{S}(D, \varphi)$ is introduced and the question of the differentiability of the semigroups satisfying the exponential growth condition is investigated . Typical examples are presented in Section 4 to illustrate our results.

1 Preliminaries

Let X be a real Banach space with norm $|\cdot|$. An *operator* A in X means a (possibly multi-valued) operator with domain $D(A)$ and range $R(A)$ in X. In this paper A is identified with its graph $\{(x, y) \in X \times X : x \in D(A), y \in Ax\}$. The identity operator on X is denoted by I.

For $x, y \in X$, we define $[x, y]_\lambda = \lambda^{-1}(|x + \lambda y| - |x|)$ for $\lambda \in \boldsymbol{R} - \{0\}$, $[x, y]_+ = \inf_{\lambda > 0}[x, y]_\lambda = \lim_{\lambda \downarrow 0}[x, y]_\lambda$ and $[x, y]_- = -[x, -y]_+$. The functional $[\cdot, \cdot]_+ : X \times X \to \boldsymbol{R}$ is upper semi-continuous and has the following properties: For $x, y, z \in X$ and $\alpha \in \boldsymbol{R}$,

$$
(1.1) \qquad \left\{
\begin{array}{rcl}
[x, \alpha x + y]_+ & = & \alpha|x| + [x, y]_+, \\
[x, |\alpha| y]_+ & = & |\alpha|[x, y]_+, \\
{[x, y]_- - [x, z]_+} & \leq & [x, y - z]_- \leq [x, y]_+ - [x, z]_+, \\
{[x, y + z]_+} & \leq & [x, y]_+ + [x, z]_+, \\
|[x, y]_+| & \leq & |y|, \quad [x, x]_\pm = |x|.
\end{array}
\right.
$$

See, [10, 24]. Let D be a subset of X and let $\varphi : X \to [0, \infty]$ be a lower semi-continuous functional on X such that $D \subset D(\varphi) = \{x \in X : \varphi(x) < \infty\}$. For each $\alpha \geq 0$ the level set in D of φ is defined as

$$
(1.2) \qquad D_\alpha = \{x \in D : \varphi(x) \leq \alpha\}.
$$

We then introduce a class of nonlinear operators in X that are locally quasi-dissipative for the functional φ.

Definition 1.1. An operator A in X is said to belong to the *class* $\mathcal{G}(D, \varphi)$, if it satisfies condition (LQD) with respect to the functional φ.

As will be seen in the next section, semigroups generated by operators in the class $\mathcal{G}(D, \varphi)$ satisfy the local Lipschitz condition (L). This leads us to the following

Definition 1.2. A semigroup $S = \{S(t) : t \geq 0\}$ on D is said to belong to the class $\mathcal{S}(D, \varphi)$, if $D \subset D(\varphi)$ and condition (L) is satisfied with respect to the functional φ.

As is easily seen, we may assume without loss of generality that D coincides with the effective domain $D(\varphi)$ and each D_α is the usual level set $\{x \in X : \varphi(x) \leq \alpha\}$ of φ itself.

Let A be an operator in the class $\mathcal{G}(D, \varphi)$ and consider the differential inclusion (DI). We here introduce a notion of generalized solution of the differential inclusion (DI) and investigate their properties in conjunction with the functional φ.

We begin by recalling the notion of strong solution of (DI). Let τ denote an arbitrary but fixed positive number.

Definition 1.3. A function $u : [0, \tau] \to X$ is said to be a *strong solution* of (DI) on $[0, \tau]$, if it is Lipschitz continuous over $[0, \tau]$, differentiable a.e. in $(0, \tau), u(t) \in D(A)$ and the strong derivative $u'(t)$ exists and belongs to the set $Au(t)$ for a.e. $t \in (0, \tau)$.

In case that X is a general Banach space, the differential inclusion (DI) does not necessarily admit strong solutions even though the initial values lie in $D(A)$. We here adopt a notion of solution which refers directly to the approximation method used to establish the existence of solutions, so-called *method of discretization in time.*

Definition 1.4. Let $\varepsilon > 0$. A piecewise constant function $v : [0, \tau] \to X$ is said to be an *ε-approximate solution* of (DI) on $[0, \tau]$, if there exists a partition $\{0 = t_0 < t_1 < \cdots < t_N\}$ of the interval $[0, t_N]$ and a finite sequence $((x_i, z_i) : i = 1, \cdots, N)$ with the three properties below:

$$(\varepsilon.1) \qquad v(0) = x_0, v(t) = x_i \text{ for } t \in (t_{i-1}, t_i] \cap [0, \tau] \text{ and}$$

$$(t_i - t_{i-1})^{-1}(x_i - x_{i-1}) - z_i \in Ax_i, \quad i = 1, \cdots, N,$$

$$(\varepsilon.2) \qquad t_i - t_{i-1} \le \varepsilon, i = 1, \cdots, N, \quad \text{and} \quad \tau \le t_N < \tau + \varepsilon,$$

$$(\varepsilon.3) \qquad \sum_{i=1}^{N} (t_i - t_{i-1})|z_i| \le \varepsilon t_N.$$

DEFINITION 1.5. A continuous function $u : [0, \tau] \to X$ is said to be a *mild solution* of (DI) on $[0, \tau]$, provided that for each $\varepsilon > 0$ there is an ε-approximate solution v^ε of (DI) on $[0, \tau]$ such that $|u(t) - v^\varepsilon(t)| \le \varepsilon$ for $t \in [0, \tau]$. If there is a constant $\alpha \in [0, \infty)$ such that $v^\varepsilon(t) \in D_\alpha$ for $\varepsilon > 0$ and $t \in [0, \tau]$, then we say that the mild solution u is *φ-bounded* or *confined to D_α* on the interval $[0, \tau]$.

Notice that if u is a mild solution on $[0, \tau]$ confined to D_α then $u(t) \in D_\alpha$ for $t \in [0, \tau]$ since D_α is closed in X. A mild solutions confined to some D_α is therefore a uniform limit of approximate solutions confined to D_α. A strong solution $u(t)$ confined to some D_α is a mild solution confined to D_α, but the proof is not entirely obvious. The following result is essentially proved in the papers [3] and [24].

PROPOSITION 1.1. *If $u : [0, \tau] \to X$ is a strong solution of* (DI) *on $[0, \tau]$, then it is a mild solution of* (DI) *on $[0, \tau]$. If in addition $u(t) \in D_\alpha$ for $t \in [0, \tau]$ and some $\alpha > 0$, then the mild solution u is confined to D_α.*

We next introduce the notion of integral solution which plays an important role in not only giving a framework of the theory of semigroups of locally Lipschitzian operators which are generated by operators in the class $\mathcal{G}(D, \varphi)$, but also in establishing the uniqueness of mild solutions.

DEFINITION 1.6. A continuous function $u : [0, \tau] \to X$ is said to be an *integral solution* (with respect to φ) of (DI) on $[0, \tau]$, if for each $\beta \in [0, \infty)$ there is $\omega(\beta) \in [0, \infty)$ such that the integral inequality

$$(1.3) \qquad |u(t) - x| - |u(s) - x| \le \int_s^t ([u(\xi) - x, y]_+ + \omega(\beta)|u(\xi) - x|)\, d\xi$$

holds for $s, t \in [0, \tau]$ with $s \le t$ and $(x, y) \in A$ with $x \in D_\beta$.

The number $\omega(\beta)$ appearing in (1.3) is determined by condition (LQD) and corresponds to the Lipschitz constant stated in condition (L). Notice that (1.3) holds for any number $\omega \in [\omega(\beta), \infty)$. We have the following type of uniqueness theorem for φ-bounded mild solutions.

THEOREM 1.2 (Bénilan [3], Kobayasi-Kobayashi-Oharu [24]). *Let* $\alpha \geq 0$ *and let* $u : [0,\tau] \to X$ *be a mild solution of* (DI) *on* $[0,\tau]$ *confined to* D_α. *Then we have:*

(a) *The mild solution* u *is an integral solution of* (DI) *on* $[0,\tau]$.

(b) *If* v *is an integral solution of* (DI) *on* $[0,\tau]$, *then there is* $\omega \equiv \omega(\alpha) \in [0,\infty)$ *such that*

$$|v(t) - u(t)| \leq e^{\omega t}|v(0) - u(0)| \qquad \textit{for} \quad t \in [0,\tau].$$

(c) *If* v *is a mild solution of* (DI) *on* $[0,\tau]$ *confined to* D_α, *then* $v(t) \equiv u(t)$ *on* $[0,\tau]$ *provided that* $v(0) = u(0)$.

To define a notion of *locally* φ-bounded mild solutions defined on semi-open intervals, we denote by σ an arbitrary but fixed extended number in $(0,\infty]$.

DEFINITION 1.7. Let $u : [0,\sigma) \to X$ be continuous over $[0,\sigma)$. We say that u is a *locally* φ*-bounded* mild solution of (DI) on $[0,\sigma)$, if to each $\tau \in [0,\sigma)$ there corresponds $\alpha \in [0,\infty)$ such that the restriction of u to $[0,\tau]$ gives a mild solution of (DI) on $[0,\tau]$ confined to D_α. Further, u is called an *integral solution* of (DI) on $[0,\sigma)$ if for each $\tau \in [0,\sigma)$ the restriction of u to $[0,\tau]$ is an integral solution of (DI) on $[0,\tau]$ in the sense of Definition 1.6. A locally φ-bounded mild solution and an integral solution of (DI) on $[0,\infty)$ are also called locally φ-bounded *global* mild solution and *global* integral solution of (DI), respectively.

The next result is an immediate consequence of Theorem 1.2.

COROLLARY 1.3. *Let* $u : [0,\sigma) \to X$ *be a mild solution of* (DI) *which is locally* φ*-bounded on* $[0,\sigma)$. *Let* $v : [0,\sigma) \to X$ *be an integral solution of* (DI) *on* $[0,\sigma)$. *Then*

(a) u *is an integral solution of* (DI) *on* $[0,\sigma)$;

(b) *for every* $\tau \in [0,\sigma)$ *there is* $\omega \in [0,\infty)$ *such that*

$$|u(t) - v(t)| \leq e^{\omega t}|u(0) - v(0)| \qquad \textit{for} \quad t \in (0,\tau].$$

2 Generation of Semigroups

Suppose that for each $x \in D$ there is a global mild solution $u(\cdot;x)$ of (DI) which is *locally* φ*-bounded* on $[0,\infty)$ and satisfies $u(0;x) = x$. Then one can define for each $t \in [0,\infty)$ an operator $S(t) : D \to D$ by

$$S(t)x = u(t;x) \qquad \text{for} \quad x \in D. \tag{2.1}$$

To assert that the family $S = \{S(t) : t \geq 0\}$ forms a semigroup belonging to the class $\mathcal{S}(D,\varphi)$, we need condition (C) below:

(C) For each $\alpha \in [0,\infty)$ and each $\tau \in [0,\sigma)$ there is $\beta \in [0,\infty)$ such that for $x \in D_\alpha$ the restriction of the associated global mild solution $u(\cdot;x)$ to $[0,\tau]$ is confined to D_β.

THEOREM 2.1. *Let $S = \{S(t) : t \geq 0\}$ be a family of self maps of D defined by* (2.1). *Then S forms a semigroup on D. Assume further that condition* (C) *holds. Then the semigroup S belongs to the class $\mathcal{S}(D,\varphi)$.*

Given a semigroup $S = \{S(t) : t \geq 0\}$ on D, one can assign to each $x \in D$, a D-valued function $u(\cdot;x)$ by (2.1). However condition (C) does not necessarily hold for the family of functions $\{u(\cdot;x) : x \in D\}$.

In this section we introduce a *growth condition* of a certain type to define a specific but natural class of semigroups on D for which condition (C) holds.

First we give a general existence theorem of mild solutions of initial value problem for the differential inclusion (DI) and next present a generation theorem for semigroups in the class $\mathcal{S}(D,\varphi)$ satisfying the growth condition .

Let g be a continuous function defined on the interval $[0,\infty)$ such that $g(r) \geq 0$ for $t \in [0,\infty)$, which we call a *comparison function.* We write $m(\cdot;\alpha)$ for the non-extendable maximal solution of the initial-value problem

$$r'(t) = g(r(t)), \qquad t > 0; \qquad r(0) = \alpha,$$

where α is a given nonnegative number. The interval of existence of the non-extendable maximal solution $m(\cdot;\alpha)$ is denoted by $[0,\sigma_\infty(\alpha))$, where $\sigma_\infty(\alpha) \in (0,\infty]$ in general.

Let A be an operator in X belonging to the class $\mathcal{G}(D,\varphi)$. We consider the following condition (R) which we call the *range condition* for the operator A.

(R) For $\varepsilon > 0$ and $x \in D$ there exist $\delta \in (0,\varepsilon]$, $x_\delta \in D(A)$ and $z_\delta \in X$ which satisfy $|z_\delta| < \varepsilon$ and the two relations below:

$$\begin{aligned} \delta^{-1}(x_\delta - x) - z_\delta &\in Ax_\delta, \\ \delta^{-1}(\varphi(x_\delta) - \varphi(x)) - \varepsilon &\leq g(\varphi(x_\delta)). \end{aligned}$$

Now the existence theorem of mild solutions is stated as follows:

THEOREM 2.2. *Let $A \in \mathcal{G}(D,\varphi)$. Suppose $D \subset \overline{D(A)}$ and the range condition* (R) *holds. Let $x_0 \in D$, $\alpha_0 = \varphi(x_0)$ and $\sigma_0 = \sigma_\infty(\alpha_0)$. Then, there exists a locally φ-bounded mild solution $u(t)$ of differential equation* (DI) *on $[0,\sigma_0)$ satisfying $u(0) = x_0$ and*

$$\varphi(u(t)) \leq m(t,\alpha_0) \qquad \text{for} \quad t \in [0,\sigma_0).$$

In order to state the generation theorem, we employ the following condition for given a semigroup $S = \{S(t); t \geq 0\}$ in the class $\mathcal{S}(D,\varphi)$.

(G) For $\alpha \in [0,\infty)$, $\sigma_\infty(\alpha) = \infty$, and, for $x \in D$ and $t \in [0,\infty)$,

$$\varphi(S(t)x) \le m(t;\varphi(x))$$

We call condition (G) the *growth condition* for $S = \{S(t); t \ge 0\}$ with respect to φ. The growth condition in which $g(r) = ar + b$ for some nonnegative constants a and b will be called the *exponential growth condition*. In this case, the non-extendable maximal solution $m(\cdot;\alpha)$ can be explicitly represented as

$$m(t;\alpha) = \alpha e^{at} + b\int_0^t e^{a(t-s)}\,ds$$

for $t \in [0,\infty)$ and $\alpha \in [0,\infty)$.

The generation theorem is then stated as follows:

THEOREM 2.3. *Let $A \in \mathcal{G}(D,\varphi)$. Suppose $D \subset \overline{D(A)}$, the range condition* (R) *holds and $\sigma_\infty(\alpha) = \infty$ for any $\alpha \in [0,\infty)$. Then there exists a semigroup $S = \{S(t) : t \ge 0\}$ in the class $\mathcal{S}(D,\varphi)$ such that the growth condition* (G) *holds, for each $x \in D$ the function $u(\cdot) = S(\cdot)x$ gives a unique global mild solution of* (DI) *and $u(\cdot)$ is locally φ-bounded on $[0,\infty)$.*

A semigroup $S = \{S(t); t \ge 0\}$ on D does not necessarily satisfy the growth condition (G), even if it provides mild solutions of some differential inclusion (DI) via the relation (2.1) and the nonlinear operator A in (DI) belongs to the class $\mathcal{G}(D,\varphi)$. In applications to partial differential equations, the use of such functionals φ corresponds to *a priori* estimates or energy estimates which ensure the global existence of the solutions as well as their asymptotic properties. Appropriate functionals φ are often derived in accordance with the nature of the equation under consideration so that the mild solutions may satisfy a growth condition of the type (G). See also the recent papers [30, 31].

In order to give the proofs of the theorems mentioned above, we apply the following result which follows readily from the generation theorems due to Kobayashi [20], Crandall and Evans [9] and Kobayasi, Kobayashi and Oharu [24].

THEOREM 2.4. *Let A be an operator in the class $\mathcal{G}(D,\varphi)$ satisfying $D \subset \overline{D(A)}, \tau > 0, \alpha > 0$ and let $x \in D_\alpha$. Suppose that there exists a positive number ε_0, and that for each $\varepsilon \in (0,\varepsilon_0)$ there is an ε-approximate solution $u^\varepsilon : [0,\tau] \to X$ such that $u^\varepsilon(t) \in D_\alpha$ for $t \in [0,\tau]$. If $\lim_{\varepsilon\downarrow 0} u^\varepsilon(0) = x$, then there exists a unique mild solution u of* (DI) *on $[0,\tau]$ confined to D_α and*

$$\lim_{\varepsilon\downarrow 0}(\sup\{|u^\varepsilon(t) - u(t)| : t \in [0,\tau]\}) = 0.$$

For each $\varepsilon > 0$ we write $m_\varepsilon(t;\alpha)$ for the maximal solution of the initial-value problem

$$r'(t) = g_\varepsilon(r(t)), \qquad t > 0; \qquad r(0) = \alpha,$$

where g_ε is defined by

$$g_\varepsilon(r) = g(r) + \varepsilon, \qquad r \in [0, \infty).$$

The maximal interval of existence of the non-extendable solution $m_\varepsilon(t;\alpha)$ is denoted by $[0, \sigma_\infty^\varepsilon(\alpha))$.

If in particular $g(r) = ar + b$, it is seen that $m_\varepsilon(t;\alpha)$ is represented as

$$m_\varepsilon(t;\alpha) = \alpha e^{at} + (b+\varepsilon)\int_0^t e^{a(t-s)}\,ds.$$

We can prove Theorem 2.2 and then Theorem 2.3 after preparing the following lemma which together with its proof contains fundamental estimates in our generation theory.

LEMMA 2.5. *Let $A \in \mathcal{G}(D,\varphi)$. Suppose that $D \subset \overline{D(A)}$ and the range condition* (R) *holds. Let $x_0 \in D$. Then for each $\varepsilon > 0$ there exists a sequence $(h_n, x_n, y_n)_{n=1}^\infty$ in $(0,\varepsilon] \times D(A) \times X$ with the following properties :*

$$\begin{aligned}
\sigma_\infty^\varepsilon(\varphi(x_0)) &\le \sum_{n=1}^\infty h_n, \\
y_n &\in Ax_n, \qquad n = 1, 2, \cdots, \\
|x_n - x_{n-1} - h_n y_n| &\le \varepsilon h_n, \qquad n = 1, 2, \cdots, \\
\varphi(x_n) &\le m_\varepsilon(h_n; \varphi(x_{n-1})), \qquad n = 1, 2, \cdots.
\end{aligned}$$

3 Differentiability of Semigroups

Let $S = \{S(t) : t \ge 0\}$ be a semigroup which belongs to the class $\mathcal{S}(D,\varphi)$. The most natural way to attempt to associate the initial-value problem (DI)–(IC) involving an operator A in the class $\mathcal{G}(D,\varphi)$ with the semigroup is to compute the operator

$$A_+x = \lim_{h\downarrow 0} h^{-1}(S(h)x - x),$$

whose domain $D(A_+)$ is the set of $x \in D$ such that the limit exists in X, and then hope that "solving" (DI)–(IC) with A replaced by an appropriate extension of A_+ will return $S = \{S(t) : t \ge 0\}$. The operator A_+ is usually called the *infinitesimal generator* of S in the theory of operator semigroups. For an arbitrary semigroup S in the class $\mathcal{S}(D,\varphi)$ in a general Banach space X, the domain $D(A_+)$ may be empty in general as indicated by Crandall and Liggett [8]. Moreover, it is observed by Webb [40] that A_+ need not be large enough to satisfy the range condition and does not necessary determine the semigroup S even though $D(A_+)$ is dense in D. It is interesting to seek an optimal concept of infinitesimal generator and find conditions on S, its domain D, the functional φ and the space X under consideration which together assure the existence of such an infinitesimal generator. This can be accomplished if φ is convex on X and if the Banach space X is reflexive and smooth in the following sense.

DEFINITION 3.1. The Banach space $(X, |\cdot|)$ is said to have a *Gâteaux differentiable norm* whenever

(3.1) $$\lim_{\lambda\downarrow 0}(|x+\lambda y|^2 + |x-\lambda y|^2 - 2|x|^2)/(2\lambda) = 0$$

holds for $x, y \in X$. If the above formula (3.1) holds uniformly for bounded x in the sense that for $M > 0, y \in X$ and $\varepsilon > 0$ one finds $\delta > 0$ such that

$$(|x+\lambda y|^2 + |x-\lambda y|^2 - 2|x|^2)/(2\lambda) \le \varepsilon$$

for $\lambda \in (0, \delta]$ and x with $|x| \le M$, then we say that $(X, |\cdot|)$ has a *uniformly* Gâteaux differentiable norm.

In this section we shall introduce a notion of infinitesimal generator in a generalized sense and discuss the differentiability of semigroups in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition (G).

Let $S = \{S(t) : t \ge 0\}$ belong to the class $\mathcal{S}(D, \varphi)$ and define for each $h > 0$ an operator $A_h : D \to X$ by

(3.2) $$A_h x = h^{-1}(S(h)x - x) \qquad \text{for } \ x \in D.$$

We then introduce two notions of "infinitesimal generators "of S.

DEFINITION 3.2. Given a semigroup $S = \{S(t) : t \ge 0\}$ in the class $\mathcal{S}(D, \varphi)$ the *right infinitesimal generator* A_+ is defined as follows: $v \in D(A_+)$ and $w \in A_+ v$ if and only if $v \in D$ and there exist $t \in [0, \infty)$ and $x \in D$ such that $v = S(t)x$ and w equals the right-hand strong derivative $(d^+/dt)S(t)x$. Likewise, the *left infinitesimal generator* A_- is defined in the following way: $v \in D(A_-)$ and $w \in A_- v$ if and only if $v \in D$ and there exist $t \in (0, \infty)$ and $x \in D$ such that $v = S(t)x$ and w is equal to the left-hand strong derivative $(d^-/dt)S(t)x$.

The domain $D(A_+)$ is the set of all elements $S(t)x$ such that the strong limit as $h \downarrow 0$ of $h^{-1}(S(t+h)x - S(t)x)$ exists, and hence it is the set of elements $x \in D$ such that the strong limit $\lim_{h\downarrow 0} h^{-1}(S(h)x - x)$ exists. The domain $D(A_-)$ is the set of elements $S(t)x$ such that $\lim_{h\downarrow 0} h^{-1}(S(t)x - S(t-h)x)$ exists. The domains $D(A_+)$ and $D(A_-)$ may be empty.

The right infinitesimal generator A_+ is necessarily single-valued and what so called the infinitesimal generator of S in the usual sense, while the left infinitesimal generator A_- is multi-valued in general. Let $v \in D(A_+)$ and let $v = S(t)x = S(s)y$ for some $s, t \in [0, \infty)$ and some $x, y \in D$. Then there exists $\omega \in [0, \infty)$ such that $|S(t+h)x - S(s+h)y| \le e^{\omega h}|S(t)x - S(s)y| = 0$ for $h \in (0, 1]$. Hence $h^{-1}(S(t+h)x - S(t)x) = h^{-1}(S(s+h)y - S(s)y)$ for $h \in (0, 1)$ and $(d^+/d\xi)S(\xi)x|_{\xi=t} = (d^+/d\xi)S(\xi)y|_{\xi=s}$, where $(d^+/d\xi)S(\xi)y|_{\xi=s}$ denotes the value of the right-hand derivative of $S(\xi)y$ at the point s and so on. This shows that A_+ is necessarily single-valued. If $v \in D(A_-)$ and $v = S(t)x = S(s)y$ for some $s, t \in [0, \infty)$ and some $x, y \in D$, it is possible that the left-hand derivative $(d^+/d\xi)S(\xi)x|_{\xi=t}$ differs from the left-hand derivative $(d^+/d\xi)S(\xi)y|_{\xi=s}$. Accordingly, the left infinitesimal generator A_- should be understood as a multi-valued operator in general. This situation may be illustrated by the following example:

EXAMPLE . Let $X = \boldsymbol{R}$ and $D = [0, \infty)$. The space X is regarded as a 1-dimensional Hilbert space. On the closed convex set D we define a semigroup $S = \{S(t) : t \geq 0\}$ by $S(t)x = (x - t) \vee 0$ for $t \geq 0$ and $x \in D$. For each $v \in D$ let $v = S(s)x = S(t)y$ for some $x, y \in D$ and some $s, t \geq 0$. Assume that $0 < x < y$. Then $0 \leq s \leq t$. If $0 \leq s < x$, then $y - t = x - s > 0$ and so $(d^+/d\xi)S(\xi)x|_{\xi=s} = (d^+/d\xi)S(\xi)y|_{\xi=t} = -1$. If $s \geq x$, then $v = 0$ and $t \geq y$. Therefore in this case $(d^+/d\xi)S(\xi)x|_{\xi=s} = (d^+/d\xi)S(\xi)y|_{\xi=t} = 0$. If in particular $x < s < t = y$, then $(d^-/d\xi)S(\xi)y|_{\xi=t} = -1$, while $v = S(\sigma)x = 0$ for $x < \sigma < y$ and $(d^-/d\xi)S(\xi)x|_{\xi=s} = 0$. From this we see that the right and left infinitesimal generators A_+ and A_- of S are the operators defined, respectively, by

$$A_+x = 0 \quad \text{for} \quad x = 0, \quad A_+x = -1 \quad \text{for} \quad x > 0,$$

$$A_-x = \{-1, 0\} \quad \text{for} \quad x = 0 \quad \text{and} \quad A_-x = -1 \quad \text{for} \quad x > 0$$

In this case, $A_+ \subset A_-$ and A_- is a multi-valued dissipative operator in X satisfying the range condition (R). In fact, for $x = 0$ put $x_\lambda = 0$ for $\lambda > 0$. Then $x_\lambda - \lambda A_- x_\lambda = 0 - \lambda\{-1, 0\} \ni 0$. For $x > 0$, let $0 < \lambda < x$ and $x_\lambda = x - \lambda > 0$. Then $x_\lambda - \lambda A_- x_\lambda = x - \lambda + \lambda = x$.

It should be noted that both A_+ and A_- need not be large enough to satisfy the range condition and does not necessarily determine the original semigroup S. We then introduce an extended notion of infinitesimal generator.

DEFINITION 3.3. Let f be a positive nondecreasing function on $(0, \infty)$ such that $f(\alpha) > \alpha$ for $\alpha > 0$. For the function f a family $\{A_{f,\alpha} : \alpha > 0\}$ of possibly multi-valued operators in X is defined as follows: For each $\alpha > 0, v \in D(A_{f,\alpha})$ and $(v, w) \in A_{f,\alpha}$ if and only if $v \in D_\alpha$ and there is a function $v(\cdot) : (0, \infty) \to D_{f(\alpha)}$ satisfying

(i)

$$\lim_{h \downarrow 0} v(h) = v \quad \text{and} \quad \lim_{h \downarrow 0} A_h v(h) = w \quad \text{in } X,$$

(ii)

$$\limsup_{h \downarrow 0} \varphi(v(h)) \leq f(\alpha).$$

REMARK . Let $\{A_{f,\alpha} : \alpha > 0\}$ be a family of operators in X defined for a positive nondecreasing function f on $(0, \infty)$ as mentioned in Definition 3.3. Then one can replace the function f by any positive nondecreasing function g such that $g \geq f$ on $(0, \infty)$. If we take such a function g in Definition 4.2, it may be possible to extend the family $\{A_{f,\alpha}\}$ to a larger family $\{A_{g,\alpha}\}$ such that $A_{f,\alpha} \subset A_{g,\alpha}$ for $\alpha > 0$. Accordingly, in what follows, we assume that the function f is fixed to the family $\{A_{f,\alpha}\}$.

As easily seen, for $0 < \alpha < \beta$, we have the inclusion $A_{f,\alpha} \subset A_{f,\beta}$. This fact leads us to the following

DEFINITION 3.4. By the *generalized infinitesimal generator* A (with respect to f) of a semigroup $S = \{S(t) : t \geq 0\}$ in the class $\mathcal{S}(D, \varphi)$ we mean the operator defined by

$$A = \bigcup_{\alpha>0} A_{f,\alpha},$$

where $\{A_{f,\alpha} : \alpha > 0\}$ is a family of operators defined for a positive nondecreasing function f on $(0, \infty)$ such that $f(\alpha) > \alpha$ for $\alpha > 0$.

The relation between the generalized infinitesimal generators and the right and left infinitesimal generators may be described as follows:

PROPOSITION 3.1. *Let $S = \{S(t) : t \geq 0\}$ be a semigroup in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition* (G). *Then we have :*

(a) *$D(A_+) \subset D(A)$ and $A_+v \in Av$ for $v \in D(A_+)$.*

(b) *For each $v \in D$ the nonnegative function $\varphi(S(\cdot)v)$ is right continuous on $[0, \infty)$. If in addition $\varphi(S(\cdot)v)$ is left-continuous on all of $(0, \infty)$ for $v \in D$, then $A_-v \subset Av$ for $v \in D(A_-)$. Therefore, in this case, $A_+ \cup A_- \subset A$ in the sense of graphs of operators.*

(c) *If in particular φ is the indicator function Ind_D of D, then*

$$A = \liminf_{h \downarrow 0} A_h$$

in the sense of graphs of operators.

We then explain some of basic properties of the generated infinitesimal generators of semigroups in the class $\mathcal{S}(D, \varphi)$.

PROPOSITION 3.2. *Let $S = \{S(t) : t \geq 0\}$ belong to the class $\mathcal{S}(D, \varphi)$. Let A be the generalized infinitesimal generator A of S with respect to t. Then A is an operator in the class $\mathcal{G}(D, \varphi)$.*

Let $S = \{S(t) : t \geq 0\}$ be a semigroup in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition (G) and suppose that the generalized infinitesimal generator A of S in the sense of Definition 3.4 has a "nonempty domain". Then it is expected that S is a family of solution operators (perhaps in a generalized sense) of the differential inclusion (DI) formulated for the A. Indeed, we have the following result:

THEOREM 3.3. *Let $S = \{S(t) : t \geq 0\}$ be a semigroup in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition* (G) *and possessing the generalized infinitesimal generator A. Suppose that $D(A) \neq \emptyset$. Then for each $x \in D$ the function $u(\cdot) = S(\cdot)x$ is a global integral solution of* (DI).

If in Theorem 3.3 the generalized infinitesimal generator A has a sufficiently large domain, then we obtain a result converse to Theorem 2.3.

COROLLARY 3.4. *Let $S = \{S(t) : t \geq 0\}$ be a semigroup in the class $\mathcal{S}(D, \varphi)$ satisfying the growth condition* (G) *and A the generalized infinitesimal generator of S. If $\overline{D(A)} \supset D$ and A satisfies the range condition* (R), *then for each $x \in D$ the function $u(\cdot) = S(\cdot)x$ becomes a locally φ-bounded global mild solution of* (DI) *satisfying* (G).

The very strong conditions imposed on A in Corollary 3.4 are automatically satisfied if we assume that X is reflexive, the norm $|\cdot|$ is uniformly Gâteaux differentiable, φ is convex on X, and that $S = \{S(t) : t \geq 0\}$ satisfies the *exponential* growth condition (G).

This is the main result of this section and the assertion is stated as below. We observe at this point that the one-parameter family $\{m(t; \cdot) : t \geq 0\}$ forms an order-preserving affine semigroup on the real half-line $[0, \infty)$ such that $m(t; \alpha) \vee m(t; \beta) = m(t; \alpha \vee \beta)$ for $t \geq 0$ and $\alpha, \beta \in [0, \infty)$.

THEOREM 3.5. *Let $(X, |\cdot|)$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and suppose that φ is convex on X. Let $S = \{S(t) : t \geq 0\}$ be a semigroup on D satisfying the exponential growth condition* (G). *Let A be the generalized infinitesimal generator of S. Then $\overline{D(A)} \supset D$ and A satisfies the range condition of the following form :*

(R_0) *To each $x \in D$ there corresponds a positive number $\lambda(x)$ such that for each $\lambda \in (0, \lambda(x)]$ there is $x_\lambda \in D(A)$ satisfying*

$$\lambda^{-1}(x_\lambda - x) \in Ax_\lambda \quad \textit{and} \quad \lambda^{-1}(\varphi(x_\lambda) - \varphi(x)) \leq g(\varphi(x_\lambda)),$$

where g is the affine function defined $g(r) \equiv ar + b$.

We notice that for an operator A in the class $\mathcal{G}(D, \varphi)$ the range condition (R_0) is much stronger than (R). In this paper condition (R_0) is called the *strict range condition.* The proof is given after discussing the ranges of the approximate operators A_h which are defined by the formula (3.2) and plays an important role in this section. Combining Theorem 3.5 with Corollary 3.4, we obtain the following result.

THEOREM 3.6. *Let* $(X, |\cdot|)$ *be a reflexive Banach space with a uniformly Gâteaux differentiable norm and suppose that* φ *is convex on* X. *Let* $S = \{S(t) : t \geq 0\}$ *be a semigroup on* D *satisfying the exponential growth condition* (G). *Then the generalized infinitesimal generator* A *of* S *in the sense of Definition* 3.4 *has the domain* $D(A)$ *with* $\overline{D(A)} \supset D$ *and satisfies the strict range condition* (R_0). *Furthermore, for each* $x \in D$ *the function* $u(\cdot) = S(\cdot)x$ *gives a global mild solution of* (DI) *satisfying* (G).

The above result together with Theorem 2.3 implies a nonlinear version of the Hille-Yosida theorem. As shown in Theorem 2.3, an operator A in the class $\mathcal{G}(D, \varphi)$ satisfying $\overline{D(A)} \supset D$ and the range condition (R) generates a semigroup S of class $\mathcal{S}(D, \varphi)$ satisfying (G). It is a delicate but deep problem to investigate the relationship between the operator A and the generalized infinitesimal generator of the semigroups S obtained by theorem 3.6. For earlier results in this direction we refer to for instance [6, 19]. However it is possible to treat the generalized infinitesimal generators from a different point of view, and this problem will be discussed in a subsequent paper.

In what follows, we assume without further mention that φ is convex on X, that $(X, |\cdot|)$ is a reflexive Banach space with uniformly Gâteaux differentiable norm, and that $S = \{S(t) : t \geq 0\}$ satisfies the exponential growth condition (G). Theorem 3.5 can be proved with the aid of the following theorem.

THEOREM 3.7. *Let* $S = \{S(t) : t \geq 0\}$ *be a semigroup in the class* $\mathcal{S}(D, \varphi)$ *satisfying the exponential growth condition* (G). *For each* $h > 0$ *let* $A_h : D \to X$ *be the operator defined by* (3.2) *and let* $g_h : [0, \infty) \to \boldsymbol{R}$ *be defined by*

$$g_h(\alpha) = h^{-1}(m(h; \alpha) - \alpha) \qquad \text{for} \quad \alpha \in [0, \infty).$$

Then for each $x \in D$ *there exist* $\lambda_0 \equiv \lambda_0(x) \in (0, \infty)$ *and* $h_0 = h_0(x) \in (0, \infty)$ *with the two properties below :*

(a) *For each* $\lambda \in (0, \lambda_0)$ *and each* $h \in (0, h_0)$ *there is* $x_{\lambda,h} \in D$ *satisfying*

$$\lambda^{-1}(x_{\lambda,h} - x) = A_h x_{\lambda,h} \quad \text{and} \quad \lambda^{-1}(\varphi(x_{\lambda,h}) - \varphi(x)) \leq g_h(\varphi(x_{\lambda,h})).$$

(b) *The limit* $\lim_{h \downarrow 0} x_{\lambda,h} = x_\lambda$ *exists and* $\lim_{\lambda \downarrow 0} x_\lambda = x$.

See [22, Section 5] for the proof.

PROOF OF THEOREM 3.5. Assume that Theorem 3.7 is already established. Let $x \in D$. Then one finds numbers λ_0 and h_0 in $(0, \infty)$ with the properties (a) and (b) stated in Theorem 3.7. Let f be a positive nondecreasing function satisfying $f(\alpha) > \alpha$ on $(0, \infty)$ and assume that A is the generalized infinitesimal generator of S in the sense of definition 3.4. Fix any $\beta \geq (1-a\lambda_0)^{-1}(\varphi(x)+b\lambda_0)$, $\lambda \in (0,\lambda_0)$, $h \in (0,h_0)$ and let $x_{\lambda,h}$ be the element in D as mentioned in Assertion (a). Then $\varphi(x_{\lambda,h}) \leq \beta_{\lambda,h}$, where

$$\beta_{\lambda,h} = (1-\lambda h^{-1}(e^{ah}-1))^{-1}(\varphi(x)+\lambda b h^{-1}\int_0^h e^{a(h-s)}\,ds).$$

This fact and Assertion (b) together imply the estimates

$$\varphi(x_\lambda) \leq \liminf_{h\downarrow 0}\varphi(x_{\lambda,h}) \leq \limsup_{h\downarrow 0}\varphi(x_{\lambda,h}) \leq (1-a\lambda)^{-1}(\varphi(x)+b\lambda)$$

and

$$\varphi(x) \leq \liminf_{\lambda\downarrow 0}\varphi(x_\lambda) \leq \limsup_{\lambda\downarrow 0}\varphi(x_\lambda) \leq \varphi(x).$$

Therefore $\lim_{\lambda\downarrow 0}\varphi(x_\lambda) = \varphi(x)$ and

$$\begin{aligned}&\limsup_{\lambda\downarrow 0}\left(\limsup_{h\downarrow 0}\varphi(x_{\lambda,h}) - \varphi(x_\lambda)\right)\\ \leq\ &\limsup_{\lambda\downarrow 0}\left(\limsup_{h\downarrow 0}\varphi(x_{\lambda,h})\right) - \varphi(x)\ \leq\ \varphi(x)-\varphi(x) = 0.\end{aligned}$$

This shows that there is a sufficiently small positive number $\lambda(x)$ such that

$$\limsup_{h\downarrow 0}\varphi(x_{\lambda,h}) - \varphi(x_\lambda) \leq f(\beta)-\beta \quad \text{for} \quad \lambda \in (0,\lambda(x)). \tag{3.3}$$

Also, we have $\lim_{h\downarrow 0} x_{\lambda,h} = x_\lambda$ and $\lim_{h\downarrow 0} A_h x_{\lambda,h} = \lim_{h\downarrow 0}\lambda^{-1}(x_{\lambda,h}-x) = \lambda^{-1}(x_\lambda - x)$. Combining these formulae and (3.3), we infer from Definition 3.3 that $x_\lambda \in D(A_{f,\beta})$ and $\lambda^{-1}(x_\lambda - x) \in Ax_\lambda$. Since $\varphi(x_\lambda) \leq (1-a\lambda)^{-1}(\varphi(x)+b\lambda)$, it follows that $\lambda^{-1}(\varphi(x_\lambda)-\varphi(x)) \leq g(\varphi(x_\lambda))$. This shows that A satisfies the strict range condition (R_0). Recalling that $x_\lambda \in D(A)$ and $\lim_{\lambda\downarrow 0} x_\lambda = x$, we see that $x \in \overline{D(A)}$. Since x was arbitrary in D, it is concluded that $\overline{D(A)} \supset D$. This completes the proof of Theorem 3.5. □

REMARK . In the above argument, Assertions (a) and (b) in Theorem 3.7 are essential. That is, Theorem 3.5 is valid without any restrictions on the Banach space $(X, |\cdot|)$ if Theorem 3.7 holds for general Banach spaces. In fact, the first assertion (a) is obtained for any Banach space, although it is not possible to obtain the second assertion (b) via the method employed in the paper [22]. It is known that if the semigroups S is associated with a class of semilinear evolution equations of the form

$$(d/dt)u(t) = Au(t) + Bu(t), \qquad t > 0,$$

then Theorem 3.7 is valid for arbitrary Banach spaces. See the recent works of Oharu and Takahashi [30, 31] for the semilinear Hille-Yosida theory in general Banach spaces.

4 Examples

This section is concerned with the application of the above-mentioned abstract theory to nonlinear partial differential equations. We here treat two simple evolution problems and show how the generation theory for nonlinear semigroups may be applied to such problems. More typical evolution problems will be discussed in the forthcoming paper [23]

First example is nonlinear wave equation and second is nonlinear heat equation.

EXAMPLE 4.1 We here treat the initial value problem for the nonlinear wave equation

$$u_t = v, \quad v_t = u_{xx} - |u|^{q-2}u, \qquad (x,t) \in (-\infty,\infty) \times (0,\infty), \tag{4.1}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \qquad x \in (-\infty,\infty), \tag{4.2}$$

where $q > 2$. For the hyperbolic system (4.1), a natural energy function can be found and *a priori* estimates for the solutions are obtained in terms of the energy function. Therefore it is natural to convert the problem (4.1)–(4.2) to the following abstract Cauchy problem in the product space $X = H^1(-\infty,\infty) \times L^2(-\infty,\infty)$ with the standard norm

$$\|[u,v]\|_X = \left(\int_{-\infty}^{\infty} (|u|^2 + |u_x|^2 + |v|^2)\, dx\right)^{1/2}.$$

Namely, the problem (4.1)–(4.2) is converted to the Cauchy problem

$$(d/dt)[u,v](t) = A[u,v], \quad t > 0; \qquad [u,v](0) = [u_0, v_0], \tag{4.3}$$

where

$$A[u,v] = [v, u_{xx} - |u|^{q-2}u] \quad \text{for} \quad [u,v] \in D(A) = H^2(-\infty,\infty) \times H^1(-\infty,\infty).$$

We take the functional φ defined by

$$\varphi([u,v]) = \int_{-\infty}^{\infty} (\frac{1}{2}|u|^2 + \frac{1}{2}|u_x|^2 + \frac{1}{q}|u|^q + \frac{1}{2}|v|^2)\, dx.$$

Note that $D(\varphi) = X$ and φ is continuous on X.

We define a linear wave operator L in X by $L[u,v] = [v, u_{xx}]$ with domain $D(L) = H^2(-\infty,\infty) \times H^1(-\infty,\infty)$ and a nonlinear continuous operator F in X by $F[u,v] = [0, -|u|^{q-2}u]$ with domain $D(F) = X$. Then $D(A) = D(L)$ and $A = L + F$. It is known that $L - \frac{1}{2}I$ is m-dissipative in X. Since

$$\begin{aligned}\int_{-\infty}^{\infty} ([u,v] - [\hat{u},\hat{v}]) \cdot (F[u,v] - F[\hat{u},\hat{v}])\, dx &= -\int_{-\infty}^{\infty} (v - \hat{v})(|u|^{q-2} - |\hat{u}|^{q-2}\hat{u})\, dx \\ \le\ (q-1)(|u|_{L^\infty(-\infty,\infty)} \vee |\hat{u}|_{L^\infty(-\infty,\infty)})^{q-2} &\int_{-\infty}^{\infty} \frac{1}{2}(|v - \hat{v}|^2 + |u - \hat{u}|^2)\, dx,\end{aligned}$$

it is seen from the Sobolev imbedding theorem that $A = L + F$ is locally quasi-dissipative with respect to the functional φ. In order to check the range condition let $[u_0, v_0] \in X$ and set

$$[u_\delta, v_\delta] = (I - \delta L)^{-1}([u_0, v_0] + \delta F[u_0, v_0]),$$

for $\delta > 0$. We see that $[u_\delta, v_\delta] \to [u_0, v_0]$ in X as $\delta \downarrow 0$, and that

$$\text{(4.4)} \qquad \left\{ \begin{array}{rcl} u_\delta - \delta v_\delta - u_0 & = & 0 \\ v_\delta - \delta(u_{\delta,xx} - |u_\delta|^{q-1} u_\delta) - v_0 & = & -\delta(|u_0|^{q-2}|u_0| - |u_\delta|^{q-2}|u_\delta|). \end{array} \right.$$

These together imply that

$$|\delta^{-1}([u_\delta, v_\delta] - [u_0, v_0]) - A[u_\delta, v_\delta]|_X^2 = \int_{-\infty}^{\infty} ||u_0|^{q-2} u_0 - |u_\delta|^{q-2} u_\delta|^2 \, dx$$

converges to 0 as $\delta \downarrow 0$. The first equality in (4.4) implies

$$\text{(4.5)} \qquad \int_{-\infty}^{\infty} \frac{1}{2}(|u_\delta|^2 - |u_0|^2) \, dx \;\le\; \delta \int_{-\infty}^{\infty} \frac{1}{2}(|u_\delta|^2 + |v_\delta|^2) \, dx,$$

and the second relation together with the first equality implies

$$\text{(4.6)} \qquad \begin{array}{l} \displaystyle\int_{-\infty}^{\infty} \{\frac{1}{2}(|v_\delta|^2 - |v_0|^2) + \frac{1}{2}(|u_{\delta,x}|^2 - |u_{0,x}|^2) + \frac{1}{q}(|u_\delta|^q - |u_0|^q)\} \, dx \\ \displaystyle\le \; \delta \int_{-\infty}^{\infty} v_\delta(|u_\delta|^{q-2} u_\delta - |u_0|^{q-2} u_0) \, dx. \end{array}$$

Combining the estimates (4.5) and (4.6), we see that

$$\limsup_{\delta \downarrow 0} \{\delta^{-1}(\varphi([u_\delta, v_\delta]) - \varphi([u_0, v_0])) - \varphi([u_\delta, v_\delta])\} \le 0.$$

Therefore, the semilinear operator A satisfies the range condition (R) with $g(r) = r$. Consequently, a semigroup $S = \{S(t) : t \ge 0\}$ on $D \equiv X$ in the class $\mathcal{S}(X, \varphi)$ is generated by A and S satisfies the growth condition

$$\varphi(S(t)[u, v]) \le e^t \varphi([u, v]) \qquad \text{for } t \in [0, \infty) \text{ and } [u, v] \in X.$$

Furthermore, the semigroup S consists of solution operators to the problem (4.1)–(4.2).

EXAMPLE 4.2 We next consider the initial-boundary value problem for the nonlinear heat equation

$$\text{(4.7)} \qquad u_t = u_{xx} + |u|^{q-2} u, \qquad (x, t) \in (0, 1) \times (0, \infty),$$

$$\text{(4.8)} \qquad \begin{array}{l} u(0, t) = 0, \quad u(1, t) = 0 \qquad t \in (0, \infty), \\ u(x, 0) = u_0(x), \qquad x \in (0, 1), \end{array}$$

where $q > 2$. We take the space $L^2(0, 1)$ with the standard norm

$$|u|_X = (\int_0^1 |u|^2 \, dx)^{1/2}$$

as X and convert the problem (4.7)–(4.8) to the abstract Cauchy problem

$$(4.9) \qquad (d/dt)u(t) = Au(t), \quad t > 0; \qquad u(0) = u_0,$$

where A is defined as

$$Au = u_{xx} + |u|^{q-2}u, \qquad \text{for} \quad u \in D(A) = H^2(0,1) \cap H_0^1(0,1).$$

We take the functional φ on X defined by

$$\varphi(u) = \begin{cases} \int_0^1 \frac{1}{2}(|u|^2 + |u_x|^2)\,dx \quad, & u \in H_0^1(0,1), \\ +\infty \quad, & \text{otherwise.} \end{cases}$$

It is known that locally φ-bounded global solution of the initial-boundary value (4.7)–(4.8) does not always exits. (See, for example, Fujita [13].) Therefore, the generation Theorem 2.3 can not directly applied to this problem, although Theorem 2.2 can be employed to obtain locally φ-bounded global solutions provided initial data are sufficiently "small" in a certain sense. Set $D = D(\varphi) = H_0^1(0,1)$. We define a linear heat operator L in X by $Lu = u_{xx}$ with $D(L) = H^2(0,1) \cap H_0^1(0,1)$ and a nonlinear operator F in X by $Fu = |u|^{q-2}u$ with $D(F) = H_0^1(0,1)$. Then $D(A) = D(L)$ and $A = L + F$. It is well-known that L is m-dissipative in X. Since

$$\begin{aligned} \int_0^1 (u-\hat{u})(Fu - F\hat{u})\,dx &= \int_0^1 (u-\hat{u})(|u|^{q-2}u - |\hat{u}|^{q-2}\hat{u})\,dx \\ &\le (q-1)(|u|_{L^\infty(0,1)} \vee |\hat{u}|_{L^\infty(0,1)})^{q-2} \int_0^1 |u-\hat{u}|^2\,dx, \end{aligned}$$

we infer from the Sobolev imbedding theorem that $A = L + F$ is locally quasi-dissipative with respect to the functional φ. Let $u_0 \in X$ and set

$$u_\delta = (I - \delta L)^{-1}(u_0 + \delta F u_0),$$

for $\delta > 0$. We see that $u_\delta \to u_0$ in X as $\delta \downarrow 0$ and that

$$(4.10) \qquad u_\delta - \delta(u_{\delta,xx} + |u_\delta|^{q-1}u_\delta) - u_0 = \delta(|u_0|^{q-2}u_0 - |u_\delta|^{q-2}u_\delta).$$

These imply that

$$|\delta^{-1}(u_\delta - u_0) - Au_\delta|_X^2 = \int_0^1 ||u_0|^{q-2}u_0 - |u_\delta|^{q-2}u_\delta|^2\,dx,$$

and that the right-hand side converges to 0 as $\delta \downarrow 0$.

The Equation (4.10) implies

$$\begin{aligned} (4.11) \qquad & \int_0^1 u_\delta(u_\delta - u_0)\,dx + \delta\int_0^1 |u_{\delta,x}|^2\,dx \\ &= \delta \int_0^1 u_\delta |u_0|^{q-2}u_0\,dx \\ &\le \delta \int_0^1 \frac{1}{2}|u_\delta|^2 + \int_0^1 \frac{1}{2}|u_0|^{2(q-1)}\,dx \\ &\le \delta \int_0^1 \frac{1}{2}|u_\delta|^2\,dx + \frac{C_1}{2}\varphi(u_0)^{q-1} \end{aligned}$$

and

$$\int_0^1 u_{\delta,x}(u_{\delta,x} - u_{0,x})\,dx + \delta \int_0^1 |u_{\delta,xx}|^2\,dx \tag{4.12}$$
$$= \delta \int_0^1 u_{\delta,x}(q-1)|u_0|^{q-2}u_{0,x}\,dx$$
$$\leq \delta \int_0^1 \frac{1}{2}|u_{\delta,x}|^2 + \int_0^1 \frac{q-1}{2}|u_0|^{2(q-2)}|u_{0,x}|^2\,dx$$
$$\leq \delta \int_0^1 \frac{1}{2}|u_{\delta,x}|^2\,dx + \frac{C_1}{2}\varphi(u_0)^{q-1},$$

where C_1 is a positive constant. Since $|u|^2_{L^2(0,1)} \leq |u_x|^2_{L^2(0,1)}/2$, these estimates (4.11) and (4.12) together imply

$$\delta^{-1}(\varphi(u_\delta) - \varphi(u_0)) + C_2\varphi(u_\delta) \leq C_1(\varphi(u_0))^{q-1}$$

for some positive constant C_2. From this we obtain

$$\limsup_{\delta\downarrow 0}\{\delta^{-1}(\varphi(u_\delta) - \varphi(u_0)) - g(\varphi(u_\delta))\} \leq 0,$$

where $g(r) = (C_1 r^{q-1} - C_2 r) \vee 0$. It turns that the operator A satisfies the range condition (R) for this comparison function $g(r)$. In consequence, for any $u_0 \in D = H_0^1(0,1)$, there exits a unique locally φ-bounded local solution of the Cauchy problem (4.9). If in particular $\varphi(u_0) \leq (C_2/C_1)^{1/(q-2)}$, then there exits a unique locally φ-bounded *global* solution of the Cauchy problem (4.9).

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