数理解析研究所講究録
第 755 巻 1991 年 185-200

Shape Optimization in Multi-Phase Stefan Problem

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1.Formulation of the optimization problem

Let us consider the enthalpy formulation of Stefan problem described as follows:

$$SP(\Omega)\begin{cases} u_t-\Delta\beta(u)=f & \text{in } Q(\Omega):=(0,T)\times\Omega,\\ u(0,\cdot)=u_0 & \text{in } \Omega,\\ \beta(u)=g & \text{on } \Sigma(\Omega):=(0,T)\times\partial\Omega,\end{cases}$$

where $\hat{\Omega}$ is a fixed smooth bounded domain in $R^N (N \geq 2)$, and Ω is a smooth subdomain of $\hat{\Omega}$, $0 < T < \infty, \hat{Q} := (0,T)\times\hat{\Omega}$ and $\hat{\Sigma} := (0,T)\times\partial\hat{\Omega}$; $\beta : R \to R$ is a nondecreasing function on R such that

$$(1.1)\qquad \begin{cases} \beta(0)=0, |\beta(r)| \geq C_0|r|-C_0' & \text{for all } r\in R\,,\\ |\beta(r)-\beta(r')| \leq L_0|r-r'| & \text{for all } r,r'\in R\,,\end{cases}$$

where $C_0 > 0$, $C_0' \geq 0$ and $L_0 > 0$ are constants. Here we suppose that $f \in L^2(\hat{Q})$, $g \in W^{2,2}(0,T;L^2(\hat{\Omega})) \cap L^2(0,T;H^2(\hat{\Omega}))$ and $u_0 \in L^2(\hat{\Omega})$. In this paper, u represents the enthalpy and $\beta(u)$ the temperature.

Now we give the weak formulation of $SP(\Omega)$.

DEFINITION 1.1. A function $u : [0,T] \to L^2(\Omega)$ is a weak solution of $SP(\Omega)$, if the following three conditions (w1) – (w3) are satisfied:

(w1) $u \in C_w([0,T];L^2(\Omega))$, $u(0) = u_0$;

(w2) $\beta(u) \in L^2(0,T;H^1(\Omega))$ and $\beta(u)-g \in L^2(0,T;H_0^1(\Omega))$;

(w3) $-\int_{Q(\Omega)} u\eta_t dxdt + \int_{Q(\Omega)} \nabla\beta(u)\nabla\eta dxdt = \int_{Q(\Omega)} f\eta dxdt$
for all $\eta \in L^2(0,T;H_0^1(\Omega))$ with $\eta_t \in L^2(Q(\Omega))$ and $\eta(0,\cdot) = \eta(T,\cdot) = 0$.

REMARK 1.1. (1) In (w3) of Definition 1.1, it is enough to take as test function η

any smooth function of the form ρz, with $\rho \in \mathcal{D}(0,T)(=\{\rho \in C^\infty(R);\text{supp}\,\rho \subset (0,T)\})$ and $z \in H_0^1(\Omega)$.

(2) We denote by $C_w([0,T];L^2(\Omega))$ the space of all weakly continuous functions from $[0,T]$ to $L^2(\Omega)$ and by $\langle\cdot,\cdot\rangle_\Omega$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now we introduce the notion of convergence of closed convex sets in a Banach space X, which is due to Mosco [13]. Let $\{K_n\}$ be a sequence of closed convex sets in X and K be a closed convex set in X. Then we say "$K_n \to K$ in X as $n \to \infty$ (in the sense of Mosco)" if the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in K_{n_k}$, and $z_k \to z$ weakly in X as $k \to \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset X$ such that $z_n \in K_n, n = 1,2,\ldots$, and $z_n \to z$ in X as $n \to \infty$.

We denote by χ_Ω the characteristic function of Ω in $\hat{\Omega}$ for any subset Ω of $\hat{\Omega}$. We put

$$O := \{\Omega \subset \hat{\Omega};\ \Omega \text{ is a smooth subdomain of } \hat{\Omega}\}$$

and for each $\Omega \in O$ denote by $V(\Omega)$ the set

$$\{z \in H_0^1(\hat{\Omega}); z = 0 \ \text{ a.e. on } \hat{\Omega} - \Omega\}.$$

Clearly $V(\Omega)$ is a closed linear subspace of $H_0^1(\hat{\Omega})$.

We consider the shape optimization problem for any non-empty subset O_c of O which is compact in the following sense:

$$(C)\begin{cases} \text{For any sequence } \{\Omega_n\} \subset O_c \text{ there is a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ with } \Omega \in O_c \\ \text{such that } \chi_{\Omega_{n_k}} \to \chi_\Omega \text{ in } L^1(\hat{\Omega}) \text{ as } k \to \infty \text{ and } V(\Omega_{n_k}) \to V(\Omega) \text{ in } H_0^1(\hat{\Omega}) \\ \text{as } k \to \infty \text{ (in the sense of Mosco).} \end{cases}$$

We give below typical examples of O_c, which are very important in the application of our main results

EXAMPLE 1.1. (1) Let $\hat{\Omega}$ and O be the same as stated before. Let Θ be the class of all C^1-diffeomorphisms from $\overline{\hat{\Omega}}$ onto itself. Here we give Θ the topology induced from $C^1(\overline{\hat{\Omega}})$. Let Ω' be a smooth subdomain of $\hat{\Omega}$ with $\overline{\Omega'} \subset \hat{\Omega}$. For a given a non-empty compact subset Θ_c of Θ, we put

$$(1.2) \qquad O_c = \{\theta(\Omega'); \theta \in \Theta_c\}.$$

Then this subset O_c of O satisfies condition (C).

Let $\{\Omega_n = \theta_n(\Omega')\}$ be any sequence in O_c. Then, by the compactness of Θ_c, there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \to \theta$ in $C^1(\overline{\hat{\Omega}})$ as $k \to \infty$ for some $\theta \in \Theta_c$. We see easily that $\chi_{\Omega_{n_k}} \to \chi_\Omega$, with $\Omega = \theta(\Omega')$, in $L^1(\hat{\Omega})$ as $k \to \infty$. Moreover, $V(\Omega_{n_k}) \to V(\Omega)$ in $H_0^1(\Omega)$ as $k \to \infty$ (in the sense of Mosco). In fact, if $z_{k'} \to z$ weakly in $H_0^1(\hat{\Omega})$ as $k' \to \infty$ for a subsequence $\{n_{k'}\}$ and $z_{k'} \in V(\Omega_{n_{k'}})$, then $\widetilde{z_{k'}}(x) = z_{k'}(\theta_{n_{k'}} \circ \theta^{-1}(x)) \in V(\Omega)$ and $\widetilde{z_{k'}} \to z(\theta \circ \theta^{-1}) = z$ weakly in $H_0^1(\hat{\Omega})$. So we see that $z \in V(\Omega)$. Also, let $z \in V(\Omega)$ and put $z_k(x) := z(\theta \circ \theta_{n_k}^{-1}(x)) \in V(\Omega_{n_k})$. Then, clearly, we have $z_k \to z$ in $H_0^1(\hat{\Omega})$.

EXAMPLE 1.2. Let $\hat{\Omega} := \{x; |x| < 2\} \subset R^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \leq \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Here we put $O_c := \{\Omega_a; 0 < a \leq \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset O_c of O satisfies condition (C).

In fact, by [13; Lemma 1.8], the 2-capacity of any singleton is zero. Then, by [13], we see that $V(\Omega_a) \to V(\Omega)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a \to 0$. In the other hand, by the same argument as in Example 1.1, we obtain that $V(\Omega_{a'}) \to V(\Omega_a)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a' \to a$. Hence O_c satisfies condition (C). ◇

In the case of Example 1.1, problems $SP(\Omega)$ can be reformulated as degenerate parabolic equations on the fixed domain Ω' by using the variable transformation $y = \theta^{-1}(x)$. However, in the case of Example 1.2, the situation is quite different, because there is no C^1-diffeomorphism between domains Ω_a and Ω.

Based on an abstract result of [1] about the solvability of $SP(\Omega)$, we consider a shape optimization problem. For a given non-empty subset O_c of O, our optimization problem, denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \qquad \text{Find } \Omega_* \in O_c \text{ such that } J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

$$(1.3) \qquad J(\Omega) = \frac{1}{2}\int_{Q(\Omega)} |\, \beta(u_\Omega) - \beta_d \,|^2 \, dxdt + \frac{1}{2}\int_{\hat{Q}-Q(\Omega)} |\, g \,|^2 \, dxdt \quad \text{for } \Omega \in O,$$

u_Ω is the weak solution of $SP(\Omega)$, and β_d is a given function in $L^2(\hat{Q})$.

In real problem, the driving variables are f, g and Ω. But, in this paper, we are interested in the effect of the domain Ω for the shape optimization. So, we fix the functions f and g, and take Ω as the driving variable.

The main results are stated in the following theorems. To prove the existence of solutions to $P(O_c)$, an important part is to show the continuous dependence of weak solution $u = u_\Omega$ to $SP(\Omega)$ upon $\Omega \in O$.

THEOREM 1.1. *Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \to V(\Omega)$ in $H_0^1(\hat{\Omega})$ as $n \to \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Also, denote by u_n and u the weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \to \infty$,*

$$(1.4) \qquad (u_n(t), z)_{\Omega_n} \to (u(t), z)_\Omega \quad \textit{for any } z \in L^2(\hat{\Omega})$$

and

$$(1.5) \qquad \tilde{\beta}(u_n) \to \tilde{\beta}(u) \quad \textit{in } L^2(\hat{Q}).$$

Here we denote by $(\cdot,\cdot)_{\Omega'}$ the inner product in $L^2(\Omega')$ and put

$$\tilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \textit{in } Q(\Omega'), \\ g & \textit{in } \hat{Q} - Q(\Omega'), \end{cases}$$

for any $\Omega' \in O$.

The next theorem is concerned with the existence of a solution to $P(O_c)$.

THEOREM 1.2. *Problem $P(O_c)$ has at least one optimal solution Ω_*.*

We shall prove Theorems 1.1 and 1.2 in section 3.

2. Uniform estimates for the weak solutions to $SP(\Omega)$

In this section, we obtain some results from [1] on the existence, uniqueness and uniform estimates for weak solutions to $SP(\Omega)$. We use the following notations.

For simplicity, we denote by H the space $L^2(\hat{\Omega})$ and by X the Sobolev space $H_0^1(\hat{\Omega})$. Moreover, $|\cdot|_H$ stands for the norm in H and $(\cdot,\cdot)$ the inner product in H. For each $\Omega \in O$, we define a bilinear form $a_\Omega(\cdot,\cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u,v) := \int_\Omega \nabla u \nabla v dx \quad \text{for all } u, v \in H^1(\Omega),$$

and denote by F_Ω the duality mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z\rangle := a_\Omega(v,z) \quad \text{for all } v, z \in H_0^1(\Omega),$$

where $\langle\cdot,\cdot\rangle_\Omega$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$. In paticular, we put $a(\cdot,\cdot) := a_{\hat{\Omega}}(\cdot,\cdot)$.

According to the abstract result of [1; Theorem 2.1], problem $SP(\Omega)$ has a unique weak solution u such that $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $\beta(u)-g \in L^2(0,T;H_0^1(\Omega))$ for any $\Omega \in O$. In fact, the weak solution u is obtained as a unique solution of the following evolution problem in $H^{-1}(\Omega)$:

$$(2.1) \qquad \begin{cases} u'(t) + F_\Omega(\beta(u(t)) - g(t)) = f(t) + \Delta g(t) & \text{for a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases}$$

We give some uniform estimates for weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 *There exists a positive constant M_1 independent of Ω such that*

$$(2.2) \qquad |\, u_\Omega \,|_{L^\infty(0,T;L^2(\Omega))} \le M_1, \quad |\, \beta(u_\Omega)\,|_{L^2(0,T;H^1(\Omega))} \le M_1$$

$$(2.3) \qquad |t^{1/2}\frac{d}{dt}\beta(u_\Omega)|_{L^2(0,T;L^2(\Omega))} \le M_1, \quad |\, t^{1/2}\beta(u_\Omega)\,|_{L^\infty(0,T;H^1(\Omega))} \le M_1$$

for all $\Omega \in O$, where u_Ω is the weak solution of $SP(\Omega)$.

Proof. As was seen in [1], problem $SP(\Omega)$ is able to be approximated by non-degenerated problem $SP(\Omega)^\varepsilon$, $\varepsilon \in (0,1]$:

$$SP(\Omega)^\varepsilon \begin{cases} u_t - \Delta\beta^\varepsilon(u) = f & \text{in } Q(\Omega), \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ \beta^\varepsilon(u) = g & \text{on } \Sigma(\Omega), \end{cases}$$

where $\beta^\varepsilon(r) = \beta(r) + \varepsilon r, r \in R$.

In fact, this problem has one and only one weak solution $u^\varepsilon \in C([0,T]; L^2(\Omega))$ such that $t^{1/2}\frac{d}{dt}\beta^\varepsilon(u^\varepsilon) \in L^2(Q(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \in L^2(0,T;H^1(\Omega))$. Moreover, we see that $u^\varepsilon \to u_\Omega$ in C_w $([0,T];\ L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \to \beta(u_\Omega)$ weakly in $L^2(0,T;H^1(\Omega))$ as $\varepsilon \to 0$. After some calculations, we obtain that there is a positive constant C' independent of ε and Ω such that

$$\text{(2.4)} \qquad \sup_{0\le t\le T} | u^\varepsilon(t) |^2_{L^2(\Omega)} + \int_0^T | \nabla(\beta^\varepsilon(u^\varepsilon(t))) |^2_{L^2(\Omega)}\, dt \le C' .$$

Moreover, multiply both sides of $u_t - \Delta\beta^\varepsilon(u^\varepsilon) = f$ by $t\frac{d}{dt}(\beta^\varepsilon(u^\varepsilon) - g)$ and integrate over $Q(\Omega)$. Then, by (2.4), we have

$$\text{(2.5)} \qquad | t^{1/2}\beta^\varepsilon(u^\varepsilon) |_{L^\infty(0,T;H^1(\Omega))} \le C'', \quad |t^{1/2}\frac{d}{dt}\beta^\varepsilon(u^\varepsilon)|_{L^2(0,T;L^2(\Omega))} \le C'', \quad \text{for any } \varepsilon \in (0,1] \text{ and } \Omega \in O,$$

where C'' is a constant independent of $\varepsilon \in (0,1]$ and $\Omega \in O$. Therefore, letting $\varepsilon \to 0$, we see that (2.2) and (2.3) hold. ◇

3.Proofs of Theorems 1.1 and 1.2

First we prove Theorem 1.1 .

Proof of THEOREM 1.1. Let consider the function $u_g \in L^\infty(0,T;H)$ such that $g(t,x) = \beta(u_g(t,x))$ in $\hat{Q}$. Here, we put

$$\tilde{u}_n = \begin{cases} u_n & \text{in } Q_n := Q(\Omega), \\ u_g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Then, we see that $\tilde{u}_n \in L^\infty(0,T;H)$. Moreover, we put $v_n := \beta(\tilde{u}_n)$ in $\hat{Q}$. By using Lemma 2.1, there exist a subsequence $\{n_k\}$ of $\{n\}$, $v \in L^2(0,T;H^1(\hat{\Omega}))$ and $\tilde{u} \in L^\infty(0,T;H)$ such that

$$(3.1) \qquad \tilde{u}_{n_k} \to \tilde{u} \quad \text{weakly* in } L^\infty(0,T;H)$$

and

$$(3.2) \qquad \begin{cases} v_{n_k} \to v & \text{weakly in } L^2(0,T;H^1(\hat{\Omega})), \\ v_{n_k}(t) \to v(t) & \text{weakly in } H^1(\hat{\Omega}) \text{ for all } t \in (0,T]. \end{cases}$$

By using Ascoli-Arzela's theorem and Lemma 2.1, we easily verify that

$$v_{n_k} \to v \text{ in } L^2(0,T;H) \text{ as } k \to \infty.$$

Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in $\hat{Q}$, from (3.1) and (3.2) we show that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in (0,T]$.

Next, let z be any function in $V(\Omega)$ and ρ be any function in $\mathcal{D}(0,T)$. By the assumptions of Theorem 1.1, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \to z$ in X. Then by the definition of solution to $SP(\Omega)$ we have

$$-\int_0^T (u_{n_k}(t), z_{n_k})_{\Omega_{n_k}} \rho'(t)dt + \int_0^T a_{\Omega_{n_k}}(v_{n_k}(t), z_{n_k})\rho(t)dt = \int_0^T (f(t), z_{n_k})_{\Omega_{n_k}} \rho(t)dt.$$

Letting $k \to \infty$, by $z_{n_k} = 0$ a.e. on $\hat{\Omega} - \Omega_{n_k}$ we obtain

$$-\int_0^T (\tilde{u}(t), z)\rho'(t)dt + \int_0^T a(v(t), z)\rho(t)dt = \int_0^T (f(t), z)\rho(t)dt.$$

This shows that $u = \tilde{u}\,|_{Q(\Omega)}$ is the solution of $SP(\Omega)$. $\diamond$

Proof of THEOREM 1.2. Since $J(\Omega) \geq 0$, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J(\Omega_n) \to J_* := \inf\{J(\Omega); \Omega \in O_c\}$$

Then, by the compactness of O_c, there are a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ and $\Omega_* \in O_c$ such that $V(\Omega_{n_k}) \to V(\Omega_*)$ in X (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_{n_k}} \to \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $k \to \infty$. Now, denote by u_k the weak solution of $SP(\Omega_{n_k})$ and by u_* the weak solution of $SP(\Omega_*)$. Then put

$$v_k := \begin{cases} \beta(u_k) & \text{in } Q_k = Q(\Omega_{n_k}), \\ g & \text{in } \hat{Q} - Q_k, \end{cases}$$

and

$$v := \begin{cases} \beta(u_*) & \text{in } Q = Q(\Omega_*), \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_k \to v$ in $L^2(0,T;H)$ as $k \to \infty$. Then we see that

$$J(\Omega_{n_k}) \to J(\Omega_*).$$

Therefore $J(\Omega_*) = J_*$. Hence Ω_* is a solution of $P(O_c)$. $\diamond$

4.Approximations for $SP(\Omega)$ and $P(O_c)$

In this section, from some numerical points of view, we discuss approximations of $SP(\Omega)$ and $P(O_c)$ by smooth problems. At first, we introduce the approximation β^ε and χ_Ω^ν for β and χ_Ω, respectively.

Let $\{\beta^\varepsilon\} = \{\beta^\varepsilon; 0 < \varepsilon \leq 1\}$ be a family of (smooth) functions $\beta^\varepsilon : R \to R$ such that

$$(\beta) \begin{cases} \mid \beta^\varepsilon(r) - \beta(r) \mid \leq \varepsilon(\mid r \mid +1) & \text{for all } r \in R \ ; \\ \beta^\varepsilon(0) = 0, \ \mid \beta^\varepsilon(r) - \beta^\varepsilon(r') \mid \leq \tilde{L}_0 \mid r - r' \mid & \text{for all } r, r' \in R \ , \\ \dfrac{d}{dr}\beta^\varepsilon(r) \geq \varepsilon & \text{for a.e. } r \in R \ , \end{cases}$$

where $\tilde{L}_0 > 0$ is a constant independent of ε .

Next, let $\{\chi_\Omega^\nu\} = \{\chi_\Omega^\nu; 0 < \nu \leq 1, \Omega \in O_c\}$ be a family of smooth functions on $\hat{\Omega}$ and suppose that the following two conditions $(\chi 1)$ and $(\chi 2)$ hold :

$(\chi 1)$ $0 \leq \chi_\Omega \leq \chi_\Omega^\nu \leq 1$ in $\hat{\Omega}$ and supp $(\chi_\Omega^\nu) \subset \{x \in \hat{\Omega}; dist(x, \Omega) \leq \nu\}$
for any $\nu \in (0,1]$ and $\Omega \in O_c$.

$(\chi 2)$ For each $\nu \in (0,1]$, $\{\chi_\Omega^\nu; \Omega \in O_c\}$ is compact in $L^1(\hat{\Omega})$.

We give below typical examples of approximations β^ε and χ_Ω^ν for β and χ_Ω, respectively, which satisfy the conditions mentioned above.

EXAMPLE 4.1. (1) We define $\beta^\varepsilon : R \longrightarrow R$ by $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$ for any $r \in R$. Then, the family of $\{\beta^\varepsilon\}$ satisfies the condition (β) for $\tilde{L}_0 = L_0 + 1$ where L_0 is the constant of (1.1).

(2) Let $\hat{\Omega}$, Ω' and O_c be the same as in Example 1.1. Now, for each $\nu \in (0,1]$ and $\Omega \in O_c$, we denote by $\Omega(\frac{\nu}{2})$ the set $\{x \in \hat{\Omega}; dist(x,\Omega) \le \frac{\nu}{2}\}$. Let χ_Ω^ν be the regularization of $\chi_{\Omega(\frac{\nu}{2})}$ by means of usual mollifiers on $\hat{\Omega}$. Clearly, we see that $(\chi 1)$ holds. Also, we obtain that $(\chi 2)$ holds. Because we can prove that

(4.1) $\quad$ if $\Omega_n = \theta_n(\Omega'), \theta_n \to \theta$ in $C^1(\overline{\hat{\Omega}})$ and $\Omega = \theta(\Omega')$, then $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\hat{\Omega})$.

Now, we define the approximate problem $SP(\Omega)^{\varepsilon\nu\mu}$, $\varepsilon, \nu, \mu \in (0,1]$, by using the penalty method for $SP(\Omega)$:

$$SP(\Omega)^{\varepsilon\nu\mu} \begin{cases} u_t - \Delta\beta^\varepsilon(u) = f - \dfrac{1-\chi_\Omega^\nu}{\mu}(\beta^\varepsilon(u) - g) & \text{in } \hat{Q} \,, \\ u(0,\cdot) = u_0 & \text{in } \hat{\Omega} \,, \\ \beta^\varepsilon(u) = g & \text{on } \hat{\Sigma} \,. \end{cases}$$

Here we give the weak formulation of $SP(\Omega)^{\varepsilon\nu\mu}$.

DEFINITION 4.1. A function $u : [0,T] \to H$ is a solution of $SP(\Omega)^{\varepsilon\nu\mu}$, if the following three conditions $(aw1) - (aw3)$ are satisfied:

$(aw1)$ $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H) \cap L^2(0,T;H^1(\hat{\Omega}))$, $u(0) = u_0$ in $\hat{\Omega}$;

$(aw2)$ $\beta^\varepsilon(u(t)) - g(t) \in X$ for a.e. $t \in [0,T]$;

$(aw3)$ $\langle u'(t), z\rangle_{\hat{\Omega}} + a(\beta^\varepsilon(u(t)), z) = \left(f(t) - \dfrac{1-\chi_\Omega^\nu}{\mu}(\beta^\varepsilon(u(t)) - g(t)), z\right)$
for any $z \in X$, a.e. $t \in [0,T]$.

According to the abstract result in [9; Chapter 2] (or [10]), problem $SP(\Omega)^{\varepsilon\nu\mu}$ has a unique solution u.

Our approximate optimization problem $P(O_c)^{\varepsilon\nu\mu}$, associated with $SP(\Omega)^{\varepsilon\nu\mu}$, is formulated as follows:

$$P(O_c)^{\varepsilon\nu\mu} \quad \text{Find } \Omega_*^{\varepsilon\nu\mu} \in O_c \text{ such that } J^{\varepsilon\nu\mu}(\Omega_*^{\varepsilon\nu\mu}) = \inf_{\Omega\in O_c} J^{\varepsilon\nu\mu}(\Omega),$$

where

$$J^{\varepsilon\nu\mu}(\Omega) = \frac{1}{2}\int_{\hat{Q}} \chi_\Omega^\nu \mid \beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu}) - \beta_d \mid^2 dxdt + \frac{1}{2}\int_{\hat{Q}} (1-\chi_\Omega^\nu) \mid g \mid^2 dxdt,$$

$u_\Omega^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Next, we give the convergence results in the following theorem.

THEOREM 4.1. *We have the following statements (1) and (2):*

(1) For each $\varepsilon, \nu, \mu \in (0,1]$, $P(O_c)^{\varepsilon\nu\mu}$ has at least one solution.

(2) Let $\{\varepsilon_n\}, \{\nu_n\}, \{\mu_n\}$ be null sequences and let $\{\Omega_n\} \subset O_c$ and $\Omega \in O_c$ such that $V(\Omega_n) \to V(\Omega)$ in X as $n \to \infty$ (in the sense of Mosco), $\chi_{\Omega_n}^{\nu_n} \to \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \to \infty$. Denote by u_n the solution of $SP(\Omega_n)^{\varepsilon_n\nu_n\mu_n}$. Then as $n \to \infty$,

$$\begin{cases} \chi_{\Omega_n} u_n \to \chi_\Omega u & \text{weakly* in } L^\infty(0,T;H), \\ \beta^{\varepsilon_n}(u_n) \to v & \text{in } L^2(0,T;H) \text{ and weakly in } L^2(0,T;H^1(\hat{\Omega})), \end{cases}$$

Moreover u is the weak solution of $SP(\Omega)$ and

$$v = \begin{cases} \beta(u) & \text{in } Q = (0,T)\times\Omega, \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

In particular, if Ω_n is a solution of $P(O_c)^{\varepsilon\nu\mu}$ with $\varepsilon = \varepsilon_n, \nu = \nu_n$ and $\mu = \mu_n$ for $n = 1, 2, \ldots$, then Ω is a solution of $P(O_c)$.

In this theorem, $\{\varepsilon_n\}$, $\{\nu_n\}$, and $\{\mu_n\}$ are chosen independently. This is very convenient for numerical computation. Moreover, we show that $P(O_c)^{\varepsilon\nu\mu}$ converges to $P(_c)$ in some sense.

5.Energy estimates for $SP(\Omega)^{\varepsilon\nu\mu}$

For the proof of Theorem 4.1, we prepare some lemmas on energy estimates for solutions of $SP(\Omega)^{\varepsilon\nu\mu}$ with respect to $\varepsilon, \nu, \mu \in (0,1]$ and $\Omega \in O_c$.

LEMMA 5.1. *There is a positive constant M_2 such that*

$$(5.1)\qquad |u_\Omega^{\varepsilon\nu\mu}|_{L^\infty(0,T;H)}\le M_2, |\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})|_{L^2(0,T;H^1(\widehat{\Omega}))}\le M_2$$

and

$$(5.2)\qquad \frac{1}{\mu}\int_{\widehat{Q}}(1-\chi_\Omega^\nu)\,|\,\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})-g\,|^2\,dxdt\le M_2$$

for all $\varepsilon,\nu,\mu\in(0,1]$ and $\Omega\in O_c$, where $u_\Omega^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. For $0<\nu,\mu\le 1, \Omega\in O, 0\le t\le T$, we introduce a proper lower semi-continuous convex function $\varphi_\Omega^{\nu\mu}$ on H as follows:

$$(5.3)\qquad \varphi_\Omega^{\nu\mu}(t,z)=\begin{cases}\frac{1}{2}\,|\,\nabla z\,|_H^2+\frac{1}{2\mu}\int_{\widehat{\Omega}}(1-\chi_\Omega^\nu)\,|\,z-g(t)\,|^2\,dx & \text{for } z-g(t)\in X,\\ +\infty & \text{otherwise.}\end{cases}$$

We easily see that the subdifferential $\partial\varphi_\Omega^{\nu\mu}(t,\cdot)$ in H is singlevalued in H and

$$(5.4)\qquad z^*=\partial\varphi_\Omega^{\nu\mu}(t,z)\Leftrightarrow\begin{cases} z-g(t)\in X,\ z^*\in H,\\ z^*=-\Delta z+\dfrac{1-\chi_\Omega^\nu}{\mu}(z-g(t))\in H.\end{cases}$$

By using (5.4), we can show that $SP(\Omega)^{\varepsilon\nu\mu}$ can be reformulated by the following evolution problem in H:

$$(5.5)\qquad \begin{cases} u'(t)+\partial\varphi_\Omega^{\nu\mu}(t,\beta^\varepsilon(u(t)))=f(t) & \text{in } H \text{ for a.e. } t\in[0,T],\\ u(0)=u_0.\end{cases}$$

For simplicity, we write u for $u_\Omega^{\varepsilon\nu\mu}$,$\chi$ for χ_Ω^ν and $\varphi(t,\cdot)$ for $\varphi_\Omega^{\nu\mu}(t,\cdot)$. Multiplying $u'(t)+\partial\varphi(t,\beta^\varepsilon(u(t)))=f(t)$ by $\beta^\varepsilon(u(t))-g(t)$, by using (5.4), we obtain

$$\begin{aligned}&(u'(t),\beta^\varepsilon(u(t))-g(t))+a(\beta^\varepsilon(u(t)),\beta^\varepsilon(u(t))-g(t))\\&+\frac{1}{\mu}\int_{\widehat{\Omega}}(1-\chi)\,|\,\beta^\varepsilon(u(t))-g(t)\,|^2\,dx\\&=(f(t),\beta^\varepsilon(u(t))-g(t)).\end{aligned}$$

After some calculations, we obtain the following inequality:

$$(5.6)\qquad \begin{aligned}&\frac{d}{dt}\{\int_{\widehat{\Omega}}\widehat{\beta^\varepsilon}(u(t))dx-(g(t),u(t))\}\\&+R_1\{|\,\nabla(\beta^\varepsilon(u(t))-g(t))\,|_H^2+\frac{1}{\mu}\int_{\widehat{\Omega}}(1-\chi)\,|\,\beta^\varepsilon(u(t))-g(t)\,|^2\,dx\}\\&\le R_2\{\int_{\widehat{\Omega}}\widehat{\beta^\varepsilon}(u(t))dt-(g(t),u(t))\}\\&+R_3(1+|\,g(t)\,|_{H^1(\widehat{\Omega})}^2+|\,g'(t)\,|_H^2+|\,f(t)\,|_H^2)\end{aligned}$$

where R_i, $i = 1, 2, 3$, are positive constants independent of ε, ν, μ and Ω. By using Gronwall's inequality and (5.6), we show (5.1) and (5.2) for a positive constant M_2 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. $\diamond$

LEMMA 5.2. *There is a positive constant M_3 such that*

$$(5.7) \qquad | t^{1/2}\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}) |_{L^\infty(0,T;H^1(\widehat{\Omega}))} \leq M_3, |t^{1/2}\frac{d}{dt}\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu})|_{L^2(0,T;H)} \leq M_3,$$

and

$$(5.8) \qquad \sup_{t\in(0,T]} \frac{t}{\mu}\int_{\widehat{\Omega}}(1-\chi_{\Omega}^{\nu}) \mid \beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}(t)) - g(t) \mid^2 dx \leq M_3,$$

for all $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$, where $u_{\Omega}^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. Simply write u for $u_{\Omega}^{\varepsilon\nu\mu}$ and $\widetilde{\beta}$ for $\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu})$. Let us consider the convex function $\psi := \psi_{\Omega}^{\nu\mu}$ on H given by

$$\psi_{\Omega}^{\nu\mu}(z) = \begin{cases} \frac{1}{2} \mid \nabla z \mid_H^2 + \frac{1}{2\mu}\int_{\widehat{\Omega}}(1-\chi_{\Omega}^{\nu}) \mid z \mid^2 dx & \text{for } z \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, it is easy to see that ψ is proper lower semicontinuous and convex on H, and the subdifferential $\partial\psi$ is singlevalued in H. Besides,

$$z^* = \partial\psi(z) \Leftrightarrow \begin{cases} z \in X, \ z^* \in H, \\ z^* = -\Delta z + \dfrac{1-\chi_{\Omega}^{\nu}}{\mu} z \in H. \end{cases}$$

Moreover, by the standard argument of convex analysis, we have

$$(5.9) \qquad \frac{d}{dt}\psi(z(t)) = (\partial\psi(z(t)), z'(t)) \text{ for } z \in W^{1,2}(0,T;H).$$

Then, by using (5.4) and (5.5), we see that

$$\begin{aligned} &(u'(t), \widetilde{\beta}'(t) - g'(t)) + (-\Delta(\widetilde{\beta}(t) - g(t)) + \frac{1-\chi_{\Omega}^{\nu}}{\mu}(\widetilde{\beta}(t) - g(t)), \widetilde{\beta}'(t) - g'(t)) \\ &= (f(t) + \Delta g(t), \widetilde{\beta}'(t) - g'(t)). \end{aligned}$$

Then, by (5.9), we show that

$$
\begin{aligned}
&\frac{t}{2\widetilde{L}_0} \mid \widetilde{\beta}'(t) \mid_H^2 \\
&+\frac{d}{dt}\{\frac{t}{2} \mid \nabla(\widetilde{\beta}(t) - g(t)) \mid_H^2 - t(u(t), g'(t)) + \frac{t}{2\mu}\int_{\widehat{\Omega}}(1-\chi_\Omega^\nu) \mid \widetilde{\beta}(t) - g(t) \mid^2 dx\} \\
&\leq T \mid f(t) + \Delta g(t) \mid_H \{\mid g'(t) \mid_H + \frac{\widetilde{L}_0}{2} \mid f(t) + \Delta g(t) \mid_H\} + T \mid u(t) \mid_H \cdot \mid g''(t) \mid_H \\
&+\frac{1}{2} \mid \nabla(\widetilde{\beta}(t) - g(t)) \mid_H^2 - (u(t), g'(t)) + \frac{1}{2\mu}\int_{\widehat{\Omega}}(1-\chi_\Omega^\nu) \mid \widetilde{\beta}(t) - g(t) \mid^2 dx.
\end{aligned} \tag{5.10}
$$

Here, integrating (5.10) over $[0, t]$ and using Lemma 5.1, we derive the estimates (5.7) and (5.8) for some positive constant M_3 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. ◇

6.Proof of Theorem 4.1.

Now we prove Theorem 4.1.

Proof of (1) of THEOREM 4.1. Fix $\varepsilon, \nu, \mu \in (0, 1]$ and put $I_* = \inf\{J^{\varepsilon\nu\mu}(\Omega); \Omega \in O_c\} \geq 0$. Then, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J^{\varepsilon\nu\mu}(\Omega_n) \to I_* \text{ (as } n \to \infty).$$

By (χ2), there is a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ such that $V(\Omega_{n_k}) \to V(\Omega)$ in X (in the sense of Mosco) and $\chi_k := \chi_{\Omega_{n_k}}^\nu \to \chi_\Omega^\nu =: \chi$ in $L^1(\widehat{\Omega})$ for some $\Omega \in O_c$. In a similar way to that of the proof of Theorem 1.1, we can prove that the solution $u_k := u_{\Omega_{n_k}}^{\varepsilon\nu\mu}$ converges to the weak solution $u := u_\Omega^{\varepsilon\nu\mu}$ of $SP(\Omega)^{\varepsilon\nu\mu}$ in the sense that

$$
\begin{cases}
u_k \to u & \text{in } L^2(0, T; H) \\
\beta^\varepsilon(u_k) \to \beta^\varepsilon(u) & \text{in } L^2(0, T; H)
\end{cases}
$$

Therefore

$$I_* = \lim_{k \to \infty} J^{\varepsilon\nu\mu}(\Omega_k) = J^{\varepsilon\nu\mu}(\Omega),$$

and we see that Ω is a solution of $P(O_c)^{\varepsilon\nu\mu}$. ◇

Proof of (2) of Theorem 4.1. By Lemma 5.1 and Lemma 5.2, we may assume that

$$u_n \to \widetilde{u} \text{ weakly* in } L^\infty([0, T]; H), \tag{6.1}$$

and

$$(6.2)\quad \begin{cases} \widetilde{\beta}_n := \beta^{\varepsilon_n}(u_n) \to \beta(\widetilde{u}) =: \widetilde{\beta} \text{ in } C_{loc}((0,T];H) \text{ and weakly in } L^2(0,T;H^1(\widehat{\Omega})), \\ \widetilde{\beta}_n(t) \to \widetilde{\beta}(t) \text{ weakly in } H^1(\widehat{\Omega}) \text{ for any } t \in (0,T]\ . \end{cases}$$

In fact, (6.1) and (6.2) are obtained in a similar way to the proof of Theorem 1.2. Moreover, by using (5.8) of Lemma 5.2 and (6.2), we have

$$\begin{cases} \chi_{\Omega_n} u_n \to \chi_\Omega u & \text{weakly* in } L^\infty(0,T;H), \\ \widetilde{\beta}_n \to \widetilde{\beta} & \text{in } L^2(0,T;H), \\ \int_{\widehat{\Omega}} (1-\chi_{\Omega_n}^{\nu_n}) \mid \widetilde{\beta}_n(t) - g(t) \mid^2 dx \to 0 = \int_{\widehat{\Omega}} (1-\chi_\Omega) \mid \widetilde{\beta}(t) - g(t) \mid^2 dx & \\ & \text{for any } t \in (0,T], \end{cases}$$

so that

$$(6.3)\qquad \widetilde{\beta}(t) - g(t) \in V(\Omega) \quad \text{for any } t \in (0,T]\ .$$

Next, let ρ be any function in $\mathcal{D}(0,T)$. By assumption, for any $z \in V(\Omega)$, there is a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \to z$ in X. From (5.5) it follows that

$$-\int_0^T (u_n(t), z_n)\rho_t(t)dt + \int_0^T a(\widetilde{\beta}_n(t), z_n)\rho(t)dt + \frac{1}{\mu_n}\int_0^T ((1-\chi_{\Omega_n}^{\nu_n})(\widetilde{\beta}_n - g)(t), z_n)\rho(t)dt$$
$$= \int_0^T (f(t), z_n)\rho(t)dt.$$

Since $(1-\chi_{\Omega_n}^{\nu_n})z_n = 0$ a.e. on $\widehat{\Omega}$, as $n \to \infty$, we get that

$$\int_0^T (\widetilde{u}'(t), z\rho(t))_{\widehat{\Omega}} dt + \int_0^T a(\widetilde{\beta}(t), z)\rho(t)dt = \int_0^T (f(t), z)\rho(t)dt.$$

Therefore $\widetilde{u}$ is the weak solution of $SP(\Omega)$.

In particular, let Ω_n be a solution of $P(O_c)^{\varepsilon_n\nu_n\mu_n}$ for each n. Just as above

$$J^{\varepsilon_n\nu_n\mu_n}(\Omega_n) \to J(\Omega)$$

and

$$J^{\varepsilon_n\nu_n\mu_n}(\Omega') \to J(\Omega') \quad \text{for any } \Omega' \in O_c.$$

Therefore, for any $\Omega' \in O_c$,

$$J(\Omega') = \lim_{n\to\infty} J^{\varepsilon_n\nu_n\mu_n}(\Omega') \geq \lim_{n\to\infty} J^{\varepsilon_n\nu_n\mu_n}(\Omega_n) = J(\Omega).$$

This shows that Ω is a solution of $P(O_c)$. $\diamond$

For the detailed proofs of all results stated in this note, see the forthcoming paper [17].

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