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INVARIANTS OF 3–MANIFOLDS ASSOCIATED WITH QUANTUM GROUPS AND VERLINDE'S FORMULA

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Introduction

In [14], Witten obtained new topological invariants of closed 3-manifolds and links in 3-manifolds from the quantum field theory. Shortly afterwards, in [11], Reshetikhin and Turaev defined related invariants of closed oriented 3-manifolds and links in such 3-manifolds, by means of representations of quantum groups. More precisely, they use quantized universal enveloping algebra $U_q(sl(2,\mathbb{C}))$, which is a q-deformation of the universal enveloping algebra $sl_2(\mathbb{C})$ discovered independently by Drinfeld [1] and Jimbo ([2],[3]). The algebra $U_q(sl(2,\mathbb{C}))$ has a structure of a Hopf algebra. Reshetikhin and Turaev introduced the additional structure in the case $q = \exp \frac{2m\pi\sqrt{-1}}{r}$ called a 'modular' Hopf algebra to define invariants of 3-manifolds. They obtain invariants of 3-manifolds as a combinational formula using invariants of framed link associated with the algebra $U_q(sl(2,\mathbb{C}))$. This is based on the fact that any closed connected oriented 3-manifold is obtained by Dehn surgery [10] of S^3 along a framed link [7].

As an application of the invariants, we construct a projectively linear representation of $SL(2,\mathbb{Z})$. Let $Z(T^2)$ be an $(r-1)$-dimensional vector spase over $\mathbb{C}$ and $\{e_i\}_{i=0}^{r-2}$ a basis of the vector space $Z(T^2)$ and we associate to a basis e_i a solid torus U_i which has a link in the interior. Gluing such two solid tori U_i and U_j by an element X of the mapping class group of the torus T^2, we obtain a closed 3-manifold M_X with a link. We denote the

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invariant of the resulting manifold by X_{ij}, which is denoted by M_X. We define an action ρ of $SL(2,\mathbb{Z})$ on the vector space $Z(T^2)$ by the formula

$$\rho(X)\, e_j = \sum_{i=0}^{r-2} X_{ij} e_i \quad (j = 0, \cdots, r-2).$$

For generators S and T of $SL(2,\mathbb{Z})$, we obtain the equations

$$S_{ij} = \sqrt{\frac{2}{r}} \sin \frac{m(i+1)(j+1)\pi}{r},$$
$$T_{ij} = q^{\frac{i(i+2)}{4}} \delta_{ij}$$

This matrix (S_{ij}) is the unitary matrix and the representation of $SL(2,\mathbb{Z})$ by means of the matrices above was discovered by Kac and Peterson [4] to discribe the modular property of the character of the affine Lie algeba and was also used by Kohno [5] to defined invariants of 3-manifolds. The above representation

$$\rho : SL(2,\mathbb{Z}) \to GL(Z(T^2))/\langle C \rangle$$

is a projectively linear representation, where $\langle C \rangle$ is the cyclic group generated by a root of unity $C = \exp\sqrt{-1}(-\varphi + \frac{3\pi m}{2r} - \frac{\pi}{2})$. Here φ is determined from the following Gauss sum;

$$\sqrt{2r}\exp(\sqrt{-1}\varphi) = \sum_{k=0}^{2r-1} \exp(\sqrt{-1}\pi k^2 m/2r)$$

As an application, we prove 'Verlinde's Formula' for $SU(2)$ [13]. This is given by the following formula:

$$\frac{S_{ij}\, S_{ik}}{S_{i0}} = \sum_{l=0}^{r-2} S_{il}\, N_{ljk}\,,$$

where

$$N_{ijk} = \begin{cases} 1 & \text{if } |i-j| \le k \le i+j,\ i+j+k \in 2\mathbb{Z},\ i+j+k \le 2(r-2) \\ 0 & \text{otherwise.} \end{cases}$$

We verify it by computing the invariant of $S^2 \times S^1$ with a link in two ways. The proof is similar to that by Witten [14], but our approach is based on representations of $U_q(sl(2,\mathbb{C}))$ with $q = \exp\frac{2m\pi\sqrt{-1}}{r}$.

The paper is organized as follows. In §1, we review some of the results in [11]. We explain a representation of a modular Hopf algebra and define invariants of links and 3-manifolds derived by Reshetikhin and Turaev. In §2, using the invariants derived in §1, we establish a representation of $SL(2,\mathbb{Z})$. The action of generators S and T on the vector space $Z(T^2)$ is represented by matrices and it is shown that they satisfy their relations. In §3, a proof of 'Verlinde's formula' for $SU(2)$ is presented. To compute the invariants, we make use of the idea in §2.

1. Review

1.1 Modular Hopf algebra U_t

In [11], Reshetikhin and Turaev give U_t as an example of 'modular' Hopf algebra. In this paper, we consider the definition of topological invariants of 3-manifolds for this modular Hopf algebra U_t. We explain this modular Hopf algebra U_t. For a non zero $q \in \mathbb{C}$, $U_q(sl_2)$ is the Hopf algebra which is a q-deformation of the universal enveloping algebra of Lie algebra $sl_2(\mathbb{C})$. Let us recall the definition of U_t due to Reshetikhin and Turaev. Let q be a root of unity and $t = \exp(\pi\sqrt{-1}m/2r)$ where m and r are mutually prime integers with odd m, $2r-1 \geq m \geq 1$, $r \geq 2$ and $q = t^4$. We fix an integer r satisfying $r \geq 2$. We define U_t to be the associative algebra with unit over the cyclotomic field $\mathbb{Q}(t)$ with 4 generators K, K^{-1}, X, Y satisfying the following relations:

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}} \tag{1.1.1}$$

$$XK = t^{-2}KX, YK = t^2 KY \tag{1.1.2}$$

$$K^{4r} = 1, X^r = Y^r = 0 \tag{1.1.3}$$

The relations (1.1.1), (1.1.2) define the algebra $U_q(sl_2)$. The structure of Hopf algebra in $U_q(sl_2)$ induces a structure of a Hopf algebra in U_t. The action of comultiplication $\triangle$, counit ε, antipode γ are given on the generators by the following formulas.

$$\Delta(X) = X \otimes K + K^{-1} \otimes X \tag{1.1.4}$$

$$\Delta(X) = Y \otimes K + K^{-1} \otimes Y \tag{1.1.5}$$

$$\Delta(K) = K \otimes K \tag{1.1.6}$$

$$\varepsilon(X) = \varepsilon(Y) = 0, \varepsilon(K) = 1 \tag{1.1.7}$$

$$\gamma(X) = -t^2 X, \gamma(Y) = -t^{-2} Y, \gamma(K) = K^{-1} \tag{1.1.8}$$

The structure of the ribbon Hopf algebra in $U_q(sl_2)$ induces a structure of the ribbon Hopf algebra in U_t. Thus U_t has the universal R-matrix $R \in U_t \otimes U_t$ due to Drinfel'd [1] which satisfies Yang Baxter equation, $u \in U_t$ defined from R, and $v \in U_t$ which is a central element of U_t. If $R = \sum_i \alpha_i \otimes \beta_i$, then $u = \sum_i \gamma(\beta_i)\alpha_i$ and $v = uK^{-2}$. Moreover, U_t satisfies six axioms (see [11, §3]) and has a structure of modular Hopf algebra. We describe the representation of modular Hopf algebra U_t. Let I be a finite set of integers $\{0, 1, \ldots, r-2\}$. For an integer $i \in I$, V_i denotes $(i+1)$-dimensional irreducible representation of U_t. It is an $(i+1)$-dimensional U_t-module. The action ρ of the generator K of U_t on V_i has the following matrix representation:

$$\rho(K) \mapsto \begin{pmatrix} t^i & & & 0 \\ & t^{i-2} & & \\ & & \ddots & \\ 0 & & & t^{-i} \end{pmatrix} \tag{1.1.9}$$

For any U_t-module V_i we provide the dual linear space $V_i^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with the action of U_t:

$$\rho_{V_i^\vee}(a) = (\rho_{V_i}(\gamma(a)))^* \in \mathrm{End}\, V_i^\vee$$

The matrix representation of this action is given by the following matrix:

$$\rho_{V_i^\vee}(K) \mapsto \begin{pmatrix} t^{-i} & & & 0 \\ & t^{-i+2} & & \\ & & \ddots & \\ 0 & & & t^{i} \end{pmatrix} \tag{1.1.10}$$

Let V_i, V_j be U_t-modules and ρ_{V_i} (resp. ρ_{V_j}) the action of U_t on V_i (resp. V_j). Their tensor product is the U_t-module $V_i \otimes V_j$ equipped with the action of U_t defined by the formula for $a \in U_t$:

$$\rho_{V_i \otimes V_j}(a) = (\rho_{V_i} \otimes \rho_{V_j})(\Delta(a))$$

Here $\triangle$ is the comultiplication of U_t. One may consider the category $Rep\,U_t$ of finite dimensional linear representations of U_t. The objects of $Rep\,U_t$ are left U_t-modules

$$V_{i_1}^{\varepsilon_1} \otimes \cdots \otimes V_{i_k}^{\varepsilon_k}$$

where $i_l \in I, \varepsilon_l \in \{\pm 1\}, V_{i_l}^{+1} = V_{i_l}, V_{i_l}^{\vee} = V_{i_l}^{-1}, 1 \leq l \leq k$. The morphisms of $Rep\,U_t$ are U_t-linear homomorphisms.

Definition 1.1. *Let V be an object of $Rep\,U_t$. For any linear operator $f : V \to V$, we define its quantum trace $tr_q f$ to be the ordinary trace over $\mathbb{C}$ of linear operator*

$$f' : V \to V,\ f'(x) = \rho(u^{-1}v)f(x).$$

In particular, if f is the identity map id_V, then we denote $\mathrm{tr}_q id_V$ by $\dim_q V$ and call it the quantum dimension of V. Note that if $V = V_j$, for $j \in I$, then using $v = u^{-1}K^2$ and (1.1.9), we get

$$\begin{aligned} \dim_q V_j &= \mathrm{tr}_q(id_{V_j}) = \mathrm{Tr}\,(\rho_{V_j}(K^2)id_{V_j}) \\ &= \sum_{n=0}^{j} t^{j-2n} = \frac{t^{2j+2} - t^{-2j-2}}{t^2 - t^{-2}} = [j+1] \end{aligned} \tag{1.1.11}$$

where $[n] = \frac{t^{2n}-t^{-2n}}{t^2-t^{-2}} = \frac{\sin(\pi m n/r)}{\sin(\pi m/r)}$

In [11], Reshetikhin and Turaev proved the following theorem.

Theorem 1.2 (Reshetikhin-Turaev). *Let $V_i\,(i \in I)$ be an irreducible representation of U_t. There exists a decomposition*

$$V_i \otimes V_j = (\oplus_k V_k) \oplus Z_{ij} \tag{1.1.12}$$

as a U_t-module, where k satisfies the following conditions

$$|i-j| \leq k \leq i+j, i+j+k \in 2\mathbb{Z}, \tag{1.1.13}$$

$$i+j+k \leq 2\,(r-2). \tag{1.1.14}$$

Moreover Z_{ij} is certain U_t-module and has the next property. For any integers $i, j \in I$ and any U_t-linear homomorphism $f : Z_{ij} \to Z_{ij}$, the quantum trace of f is equal to zero.

$$tr_q f = 0 \tag{1.1.15}$$

1.2 Ribbon graph

An oriented, directed, homogeneous ribbon tangle is a collection of ribbons and annuli as illustrated in Fig.1 ([11],[12]).

Fig.1

A ribbon (annulus) is oriented if it has an orientation as a surface in $\mathbb{R}^3$. By the shaded regions, we express that the tangle is oriented (Fig.1). A tangle is homogeneous if each twist of all ribbons and annuli in the tangle is a full twist. A ribbon tangle is directed if the cores of its ribbons and annuli are provided with directions. For each ribbon tangle we assign a finite dimensional irreducible representation V_i of U_t to each component, where i is called its colour. The procedure is called colouring and we denote it by λ. In Fig.2, elementary coloured ribbon tangles is sketched. We consider ribbons which are called coupons. A small neighborhood of each coupon Q is depicted in Fig.3, where the rectangle illustrates the coupon. A colour of each coupon is a $\mathbb{C}$-linear homomorphism defined from the colours and directions of the ribbons gluing to it. We add coupons to the tangle.

Fig.2 Fig.3

Let us introduce the category $\mathcal{H}$ of ribbon graphs. The objects of $\mathcal{H}$ are sequences

$$\eta = ((i_1, \varepsilon_1), \cdots, (i_k, \varepsilon_k)) \quad (i_1, \cdots, i_k \in I, \varepsilon_1, \cdots, \varepsilon_k \in \{1, -1\}),$$

where $i_1, \cdots, i_k \in I$ and $\varepsilon_1, \cdots, \varepsilon_k \in \{1, -1\}$. We denote the set of such sequences by N. If $\eta, \eta' \in N$, then a morphism $\eta \to \eta'$ is a coloured ribbon graph (considered up to isotopy) such that the sequence of colours and directions of the bottom (resp. top) ribbons is equal to η (resp. η'). The composition $\Gamma' \circ \Gamma$ of such two morphisms $\Gamma : \eta \to \eta'$, $\Gamma' : \eta' \to \eta''$ is the ribbon graph obtained by gluing the bottom ends of Γ' with the corresponding top ends of Γ. The tensor product of objects η, η' is their juxtaposition η, η' (see Fig.4).

Fig.4

1.3 Invariants of closed 3-manifolds

For two categories $Rep\, U_t$ and $\mathcal{H}$, Reshetikhin and Turaev show that there exists a unique covariant functor with five properties (see §2.5 in [11]). They define U_t−linear homomorphisms corresponding to elementary coloured ribbon graphs pictured in Fig.2 and graphspictured in Fig.5.

Fig.5

Since the graphs $J_i^+, J_i^-, X_{ij}^+, X_{ij}^-, a_i, b_i, c_i, d_i$ generate the category $\mathcal{H}$, the compositions and tensor products of the corresponding homomorphisms determine $F(\Gamma)$ for a coloured ribbon tangle Γ. In particular, a coloured (0,0)-ribbon tangle Γ defines $\mathbb{C}$-linear homomorphism $\mathbb{C} \to \mathbb{C}$, i.e. a multiplication by a certain element of $\mathbb{C}$. The element is a regular isotopy invariant of Γ. It is also denoted by $F(\Gamma)$.

Example 1.3 Let Γ be a coloured (0,0)-ribbon tangle in Fig.6.
Then $F(\Gamma) = F(b_i) \circ F(c_i)$ and an easy computation shows $F(\Gamma) = \dim_q V_i$.

Fig.6

Let us recall that $\dim_q V_i$ is equal to the quantum trace of identity homomorphism. The following lemma generalizes this computation.

Lemma 1.4. *Let Γ be a coloured (k,k)-ribbon graph which corresponds to an endomorphism of a certain sequence $\eta \in N$. Let L be the coloured (0,0)-ribbon tangle obtained by closing Γ (see Fig.7). Then $F(L) = tr_q F(\Gamma)$.*

Fig.7

We introduce the presentation of closed 3-manifolds via framed links. A framed link in the 3-sphere is a finite collection L of disjoint smoothly embedded circles $L_1, \cdots, L_l$ in S^3, each component L_k of L is provided with a framing which is an integer n_k. Let ω be an orientation of L. We may regard each component L_k of the annulus with n_k full twists. This identification gives us a (0,0)-ribbon tangle $\Gamma(L, \omega)$. The notation ω may be thought of as the directions of the annuli. Let λ be a colouring of $\Gamma(L,\omega)$. Then $F(\Gamma(L,\omega,\lambda))$ is a regular isotopy invariant of coloured (0,0)-ribbon tangle $\Gamma(L,\omega,\lambda)$. By means of the above results, we define invariants of closed 3-manifolds. The idea of their construction is reduced to the following theorem which relates framed links to closed 3-manifolds.

Theorem 1.5 (Lickorish [7]). *Each closed connected oriented 3-manifold can be obtained by Dehn surgery on S^3 along a certain framed link.*

Let M be a closed connected oriented 3-manifold and L a framed link in S^3 with components $L_1, \cdots, L_l$ and framing $n_1, \cdots, n_l$ which can be related to M by the above theorem. Dehn surgery is the following process. We remove an open tubular neighborhood of each L_k on the resulting toral boundary and glue l solid tori such that their meridians are identified with the curves on the boundaries. We consider such a pair (M, L). Let ω be an orientation of the framed link L. By col(L) we denote the set of colourings of the (0,0)-ribbon tangle $\Gamma(L,\omega)$. Put

$$F(M,L) = C^{\sigma(L)} \sum_{\lambda \in col(L)} \prod_{k=1}^{l} d_{\lambda(L_k)}\, F(\Gamma(L,\omega,\lambda)) \in \mathbb{C}. \tag{1.3.1}$$

Here $C, d_k\, (k = 0, \cdots, r-2)$ are constants contained in the data of the modular Hopf algebra U_t and given by the following formulas:

$$C = \exp(-\sqrt{-1}d), \tag{1.3.2}$$

$$d_k = \sqrt{\frac{2}{r}} \sin \frac{m(k+1)\pi}{r}, \tag{1.3.3}$$

where

$$d = \varphi - \frac{3\pi m}{2r} + \frac{\pi}{2}, \tag{1.3.4}$$

the number φ being determined from the following Gauss sum

$$\sqrt{2r}\exp(\sqrt{-1}\varphi) = \sum_{k=0}^{2r-1} \exp(\sqrt{-1}\pi k^2 m/2r). \tag{1.3.5}$$

The notation $\sigma(L)$ stands for the signature of the linking matrix of the framed link L. We remark that the normalization coincides with that in [6].

Theorem 1.6 (Reshetikhin-Turaev). *For a closed connected oriented 3-manifold M, $F(M,L)$ is a topological invariant of M.*

We may denote $F(M,L)$ by $F(M)$. The invariant is multiplicative with respect to a connected sum:

$$F(M_1 \sharp M_2) = F(M_1)F(M_2). \tag{1.3.6}$$

We have the following relations between invariants with opposite orientations

$$F(M) = \overline{F(-M)},$$

where the bar is the complex conjugation.

Example 1.7 The formula (1.3.6) implies that $F(S^3) = 1$.

Since $S^2 \times S^1$ is obtained by Dehn surgery on S^3 along an unknotted circle with framing 0, we have

$$\begin{aligned} F(S^2 \times S^1) &= \sum_{i=1}^{r-2} d_i \dim_q V_i \\ &= \sqrt{\frac{r}{2}} \left(\sin \frac{m\pi}{r} \right)^{-1} \end{aligned} \tag{1.3.7}$$

Here we used the equation $\dim_q V_i = \sin \frac{m(i+1)\pi}{r} / \sin \frac{m\pi}{r}$. In the case $m = 1$, $F(S^2 \times S^1)$ is equal to Kohno's invariant $\phi_K(S^2 \times S^1)$ with $K = r + 2$.

Let M be a closed connected oriented 3-manifold and T be a coloured (0,0)-ribbon tangle in M. As above, let us present M as the result of surgery on S^3 along a framed link L with components $L_1, \cdots, L_l$. The ribbon tangle $T \cup \Gamma(L, \omega, \lambda)$ may be thought of as a coloured (0,0)-ribbon tangle in S^3. We put

$$F(M,T;L,\omega) = C^{\sigma(L)} \sum_{\lambda \in col(L)} \prod_{k=1}^{l} d_{\lambda(L_k)} F(T \cup \Gamma(L,\omega,\lambda)). \tag{1.3.8}$$

Then $F(M,T;L,\omega)$ is a topological invariant of the pair (M,T). We put $F(M,T) = F(M,T;L,\omega)$. In particular, we have $F(S^3,T) = F(T)$.

2. A representation of $SL(2,\mathbb{Z})$

Using the invariants defined in §1, we establish a projectively linear representation of $SL(2,\mathbb{Z})$. Let M_1 be the mapping class group of torus T^2. We fix a basis a, b in $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ as depicted in Fig.8.

Fig.8

The group M_1 may be canonically identified with $SL(2,\mathbb{Z})$. A presentation of $SL(2,\mathbb{Z})$ is given by

$$SL(2,\mathbb{Z}) = \langle S, T : S^4 = I, (ST)^3 = S^2 \rangle, \tag{2.1}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $Z(T^2)$ be an $(r-1)$-dimensional vector space over $\mathbb{C}$ and $\{e_0, e_1, \cdots, e_{r-2}\}$ a basis of the vector space. We associate to each e_i a solid torus U_i with an annulus T_i in the interior, depicted in Fig.9.

Fig.9

We suppose that the colour of annulus T_i is $i \in \{0, \cdots, r-2\}$ and the direction as in Fig.9. We construct a projectively linear representation

$$\rho : SL(2,\mathbb{Z}) \to GL(Z(T^2))/\langle C \rangle,$$

where C is given by (1.3.2) and $\langle C \rangle$ means the cyclic group generated by $C \cdot I$, when I denotes the identity matrix.

For any element X of $SL(2,\mathbb{Z})$, put

$$\rho(X)e_j = \sum_{i=0}^{r-2} X_{ij} e_i. \tag{2.2}$$

Let $[h]$ be an isotopy class in M_1 corresponding to X. The map h is a degree 1 homeomorphism $T^2 \to T^2$. We identify ∂U_i and ∂U_j using h. The resulting closed connected 3-manifold with the (0,0)-ribbon tangle consisting of two annuli T_i, T_j is denoted by M_X. Then X_{ij} in (2.2) is defined by the following formula:

$$X_{ij} = F(M_X, T_i \cup T_j)/F(S^2 \times S^1) \tag{2.3}$$

Clearly, it follows from the definition that X_{ij} does not depend on the choice of the representative element of the isotopy class.

Theorem 2.1. *The following homomorphism constructed above is a projectively linear representation.*

$$\rho : SL(2,\mathbb{Z}) \to GL(Z(T^2))/\langle C\rangle,$$

where $\langle C\rangle$ means the cyclic group generated by $C\cdot I$ in $GL(Z(T^2))$ with C given by (1.3.2). The values of S_{ij}, I_{ij} and T_{ij} are given by the following formulas:

$$S_{ij} = \sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r},$$

$$I_{ij} = \delta_{ij},$$

$$T_{ij} = t^{i(i+2)}\delta_{ij}.$$

proof. Firstly, let us compute S_{ij}, I_{ij}, and T_{ij}.

(1) the case $X = S$

M_S is the 3-sphere S^3. Two annuli T_i, T_j are linked in M_S and make up the Hopf link (see Fig.10).

Fig.10

Therefore we get $F(M_S, T_i \cup T_j) = F(T_i \cup T_j)$. One computes

$$F(T_i \cup T_j) = \sin\frac{m(i+1)(j+1)\pi}{r} \Big/ \sin\frac{m\pi}{r}. \tag{2.4}$$

Applying (2.3) with (1.3.7) and (2.4), we get

$$S_{ij} = \sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r}. \tag{2.5}$$

(2) the case $X = I$

M_I is $S^2 \times S^1$. In M_I, T_i and T_j are unlinked unknotted annuli with no twists (see Fig.11). Let us consider S^3 with the above annuli and the unknotted circle L that links a pair of the annuli and that has the zero framing as illustrated in Fig.12a.

Fig.11

The Dehn surgery on S^3 along L produces $S^2 \times S^1$ with T_i and T_j depicted in Fig.11. To calculate $F(T_i \cup T_j \cup \Gamma(L, \omega, \lambda))$, we can use the formula (1.1.2)

$$V_i \otimes V_j = (\oplus_k V_k) \oplus Z_{ij}.$$

Let us replace T_i and T_j with a unknotted annulus T_k which runs parallel to T_i and T_j (Fig.12b). We assume that T_k has a colour k and the same direction as two annuli. Then $T_k \cup \Gamma(L,\omega,\lambda)$ is a (0,0)-ribbon tangle in S^3.

Fig.12a Fig.12b

The property (1.1.15) of the U_t-module Z_{ij} ensures the equation

$$F(T_i \cup T_j \cup \Gamma(L,\omega,\lambda)) = \sum_k F(T_k \cup \Gamma(L,\omega,\lambda)), \tag{2.6}$$

where the summation runs over k satisfying (1.1.13) and (1.1.14). As $T_k \cup \Gamma(L,\omega,\lambda)$ is the Hopf link, we casn apply (2.4) to the computation of $F(T_k \cup \Gamma(L,\omega,\lambda)$. If $\lambda(L) = l$, then we obtain

$$F(T_k \cup \Gamma(L,\omega,\lambda)) = F(S^2 \times S^1)\sqrt{\frac{2}{r}} \sin \frac{m(k+1)(l+1)\pi}{r}. \tag{2.7}$$

Thus, we get

$$I_{ij} = \frac{1}{F(S^2 \times S^1)} \sum_{l=0}^{r-2} d_l \left(\sum_k F(S^2 \times S^1)\sqrt{\frac{2}{r}} \sin \frac{m(k+1)(l+1)\pi}{r} \right),$$

where k satisfies the conditions (1.1.13) and (1.1.14). We have the following formula:

$$\sum_{l=0}^{r-2} \sin \frac{m(i+1)(l+1)\pi}{r} \sin \frac{m(l+1)(j+1)\pi}{r} = \frac{r}{2}\delta_{ij}. \tag{2.8}$$

Using (2.8), we show the formula:

$$I_{ij} = \frac{2}{r} \sum_k \frac{r}{2} \delta_{0k}.$$

The condition (1.1.13) of k asserts that k is equal to zero if and only if $i = j$. Therefore we get

$$I_{ij} = \delta_{ij}. \tag{2.9}$$

(3) the case $X = T$

M_T is also $S^2 \times S^1$. But the unknotted annulus T_i with no twists links the unknotted annulus T_j with one full twist (Fig.13). To obtain $(M_T, T_i \cup T_j)$, we start from S^3 with the two above annuli T_i and T_j and with an unknotted circle L which has the zero framing and which links them (Fig.14a). Carrying out the Dehn surgery on S^3 along the circle L turns S^3 into $M_T \cong S^2 \times S^1$.

Fig.13

One claims that we can make use of the idea of the case $X = I$to calculate $F(T_i \cup T_j \cup \Gamma(L, \omega, \lambda))$. We deform the annulus T_i adding the same twist as the annulus T_j. One denotes the resulting annulus by T_i'. The computaion in [11, the proof of Lemma 7.1] implies

$$F(T_i' \cup T_j \cup \Gamma(L, \omega, \lambda)) = (v_i)^{-1} F(T_i \cup T_j \cup \Gamma(L, \omega, \lambda)),$$

where $v_i = t^{i(i+2)}$. A full twist can be expressed by a curl (Fig.14b). It follows from it that we can turn $T_i' \cup T_j$ into two parallel annuli with no twists (Fig.14c).

Let T_k be an annulus of colour k provided with the same twist and direction as two annuli. We replace two annuli by T_k (Fig.14d).

Fig.14a Fig.14b Fig.14c Fig.14d

Then, applying theorem 1.2, one may get the following equation

$$F(T_i' \cup T_j \cup \Gamma(L, \omega, \lambda)) = \sum_{\substack{k \\ |i-j| \leq k \leq i+j \\ i+j+k \in 2\mathbb{Z} \\ i+j+k \leq 2(r-2)}} F(T_k \cup \Gamma(L, \omega, \lambda)),$$

Thus

$$T_{ij} = \frac{1}{F(S^2 \times S^1)} \sum_{l=0}^{r-2} d_l \, v_i \sum_k F(T_k \cup \Gamma(L, \omega, \lambda))$$

here $\lambda(L) = l$. Substituting $v_i = t^{i(i+2)}$, we obtain

$$T_{ij} = t^{i(i+2)} \delta_{ij}. \qquad (2.10)$$

We put $I_{id} = (I_{ij}), S = (S_{ij})$ and $T = (T_{ij})$. They are $(r-1) \times (r-1)$ matrices.

Let us prove that ρ is a projectively linear representation. To do this, it is sufficient to show the following:

$$S^4 = I_{id} \qquad \text{mod} \quad C \cdot I \qquad (2.11)$$

$$(ST)^3 = S^2 \qquad \text{mod} \quad C \cdot I \qquad (2.12)$$

One easily computes

$$S^2 = I_{id}. \tag{2.13}$$

Note that the equation $(ST)^3 = S^2$ is equivalent to the equation $STS = T^{-1}ST^{-1}$. It is easy to compute that an (i,j)-entry of $T^{-1}ST^{-1}$ is

$$\sqrt{\frac{2}{r}}\, t^{i(i+2)+j(j+2)} \sin \frac{m(i+1)(j+1)\pi}{r}. \tag{2.14}$$

Using $t = \exp(\pi\sqrt{-1}m/2r)$ and Gauss sum (1.3.5), an (i,j)-entry of STS is

$$C\sqrt{\frac{2}{r}}\, t^{i(i+2)+j(j+2)} \sin \frac{m(i+1)(j+1)\pi}{r}. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$STS = T^{-1}ST^{-1} \cdot CI_{id}. \tag{2.16}$$

(2.13) implies (2.11) and (2.16) implies (2.12). □

3.Proof of Verlinde's formula

As another application of the invariants given in §1, we prove 'Verlinde's formula' (see [13]). It is given by the following formula.

$$\frac{S_{ij}\, S_{ik}}{S_{i0}} = \sum_{l=0}^{r-2} S_{il}\, N_{ljk} \tag{3.1}$$

where m and r are mutually prime integers with odd m, $1 \le m \le 2r-1, r \le 2$, and

$$S_{ij} = \sqrt{\frac{2}{r}} \sin \frac{m(i+1)(j+1)\pi}{r}, \tag{3.2}$$
$$N_{ijk} = \begin{cases} 1 & \text{if } |i-j| \le k \le i+j, i+j+k \in 2\mathbb{Z}, i+j+k \le 2(r-2) \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Verlinde's formula. Let us consider $S^2 \times S^1$ with three parallel non-twisted annuli T_l, T_j, T_k in the interior (see Fig.15). The directions of them is as in Fig.15 and the colour of T_l (resp. T_j, T_k) is l (resp. j, k).

Fig.15

We call this configuration of three annuli $\widetilde{L_{ljk}}$. The idea of the proof is to evaluate $F(S^2 \times S^1, \widetilde{L_{ljk}})$ in two ways.

Let us begin with the surgery representation of $(S^2 \times S^1, \widetilde{L_{ljk}})$. Let L be an unknotted circle with the zero framing which links $\widetilde{L_{ljk}}$ in S^3 (Fig.16a). The Dehn surgery on S^3 along the circle L produces $(S^2 \times S^1, \widetilde{L_{ljk}})$.

In the first evaluation, we use an analogue of the computation of I_{ij} and T_{ij} in §2. We replace T_j and T_k by an unknotted non-twisted annulus T_p with colour p and the same direction as them (Fig.16b). Then applying Theorem 1.2 with i replaced by l, we obtain the following equation:

$$F(\widetilde{L_{ljk}} \cup \Gamma(L, \omega, \lambda)) = \sum_p F(T_l \cup T_p \cup \Gamma(L, \omega, \lambda)).$$

Here p satisfies the conditions (1.1.13) and (1.1.14) replaced i by p.

Fig.16a Fig.16b

Then we can apply the formula (2.9) to the computation. Thus we get

$$\begin{aligned} F(S^2 \times S^1, \widetilde{L_{ljk}}) &= \sum_{l=0}^{r-2} d_t \left(\sum_p F(T_l \cup T_p \cup \Gamma(L, \omega, \lambda)) \right) \\ &= F(S^2 \times S^1) \sum_{\substack{p \\ |i-j| \le p \le j+k \\ p+j+k \in 2\mathbb{Z} \\ p+j+k \le 2(r-2)}} \delta_{l,p} \end{aligned}$$

It follows from the condition of p that

$$F(S^2 \times S^1, \widetilde{L_{ljk}}) = F(S^2 \times S^1)\, N_{ljk} \tag{3.3}$$

To evaluate $F(S^2 \times S^1, \widetilde{L_{ljk}})$ in the second way, we rotate the (0,0)-ribbon tangle $\widetilde{L_{ljk}} \cup \Gamma(L)$ in S^3 (Fig.17a). The result may be thought of as the closure of the (1,1)-ribbon tangle B^t_{ljk} illustrated in Fig.17b. $F(B^t_{ljk})$ is the homomorphism $V_t \to V_t$. Moreover, it may be thought of as the composition of three homomorphisms determined by (1,1)-ribbon tangles $\tau^t_l, \tau^t_j, \tau^t_k$ illustrated in Fig.17c.

Fig.17a Fig.17b Fig.17c

The map $F(\tau_l^t)$ is a $\mathbb{C}$-linear homomorphism $\mathbb{C} \to \mathbb{C}$, i.e. a multiplication by an element of $\mathbb{C}$. We denote this element by b_l^t. Similarly, $F(\tau_j^t)$ (resp. $F(\tau_k^t)$) is a multiplication by an element b_j^t (resp. b_k^t) of $\mathbb{C}$. The closure of the (1,1)-ribbon tangle τ_l^t makes up the Hopf link. We denote this invariant by s_{tl}. Analogously, the invariant which corresponds to τ_j^t (resp. τ_k^t) is denoted by s_{tj} (resp. s_{tk}). Using (2,4), we derive

$$s_{t\mu} = \sin \frac{m(t+1)(\mu+1)\pi}{r} \Big/ \sin \frac{m\pi}{r},$$

where $\mu \in \{l, j, k\}$. Note that $s_{t0} = \dim_q V_t$. Then Lemma 1.5 shows that

$$s_{t\mu} = b_\mu^t \dim_q V_t = b_\mu^t s_{t0}. \tag{3.4}$$

The above discussion and (3.6) imply that

$$\begin{aligned} F(B_{ljk}^t) &= tr_q(F(\tau_l^t) \circ F(\tau_j^t) \circ F(\tau_k^t)) \\ &= b_l^t\, b_j^t\, b_k^t \dim_q V_t \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5),

$$\begin{aligned} F(S^2 \times S^1, \widetilde{L_{ljk}}) &= \sum_{t=0}^{r-2} d_t\, F(B_{ljk}^t) \dim_q V_t \\ &= \sum_{t=0}^{r-2} d_t\, \frac{s_{tl} s_{tj} s_{tk}}{(s_{t0})^2} \end{aligned} \tag{3.6}$$

Multiplying (3.3) and (3.6) by s_{il} and summing up over $l = 0, \cdots, r-2$, we get

$$\sum_{l=0}^{r-2} F(S^2 \times S^1)\, N_{ljk} = d_i \left(\sin \frac{m\pi}{r}\right)^{-2} \frac{r}{2} \frac{s_{ij} s_{ik}}{(s_{i0})^2}. \tag{3.7}$$

We remark that

$$\begin{aligned} d_i &= \sqrt{\frac{2}{r}} \sin \frac{m(i+1)\pi}{r} \\ &= \sqrt{\frac{2}{r}}\, s_{i0} \sin \frac{m\pi}{r}. \end{aligned} \tag{3.8}$$

Substituting (3.8) in (3.7), we obtain

$$\sum_{l=0}^{r-2} s_{il}\, F(S^2 \times S^1)\, N_{ljk} = F(S^2 \times S^1)\, \frac{s_{ij} s_{ik}}{s_{i0}}. \tag{3.9}$$

The value S_{ij} is related to s_{ij} by the formula

$$s_{ij} = \sqrt{\frac{r}{2}} \left(\sin \frac{m\pi}{r} \right)^{-1} S_{ij}.$$

Thus (3.9) implies (3.1). □

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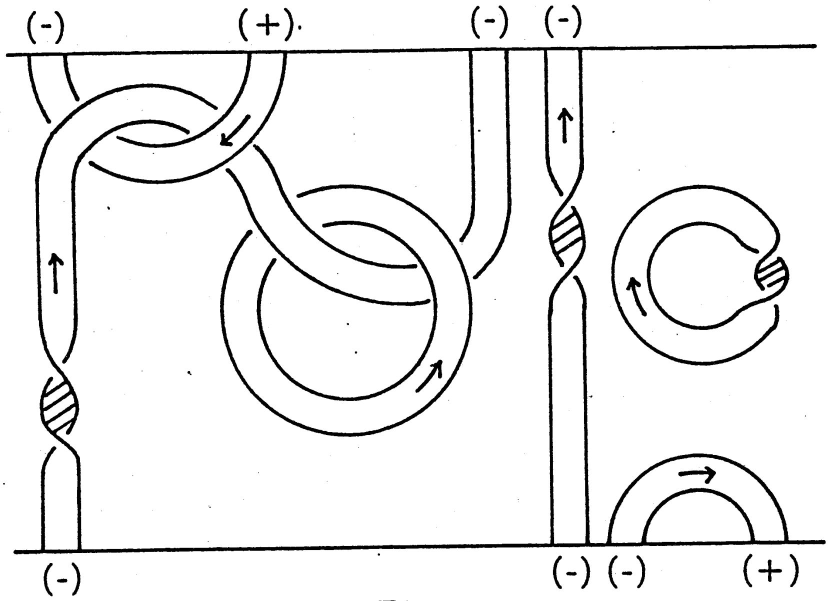
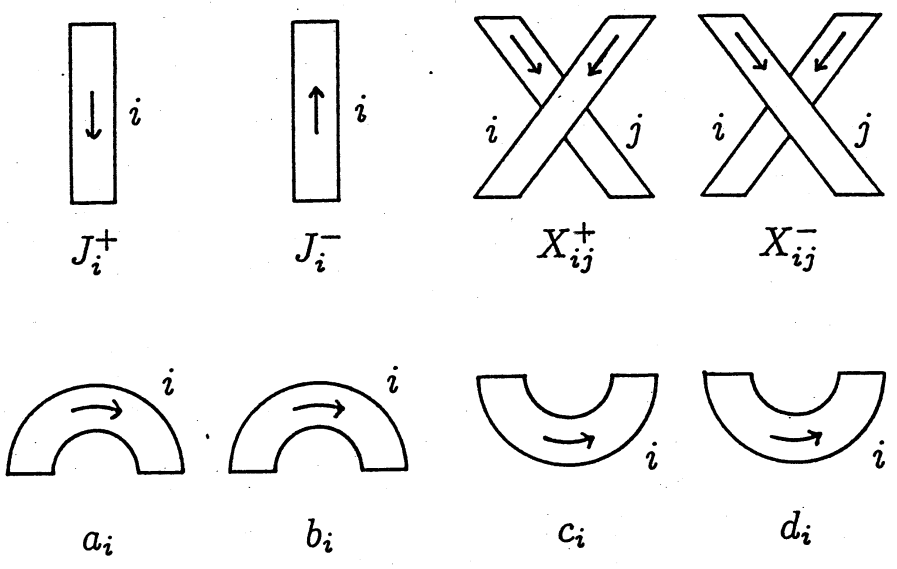


Fig.1



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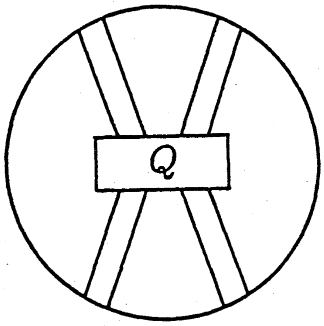

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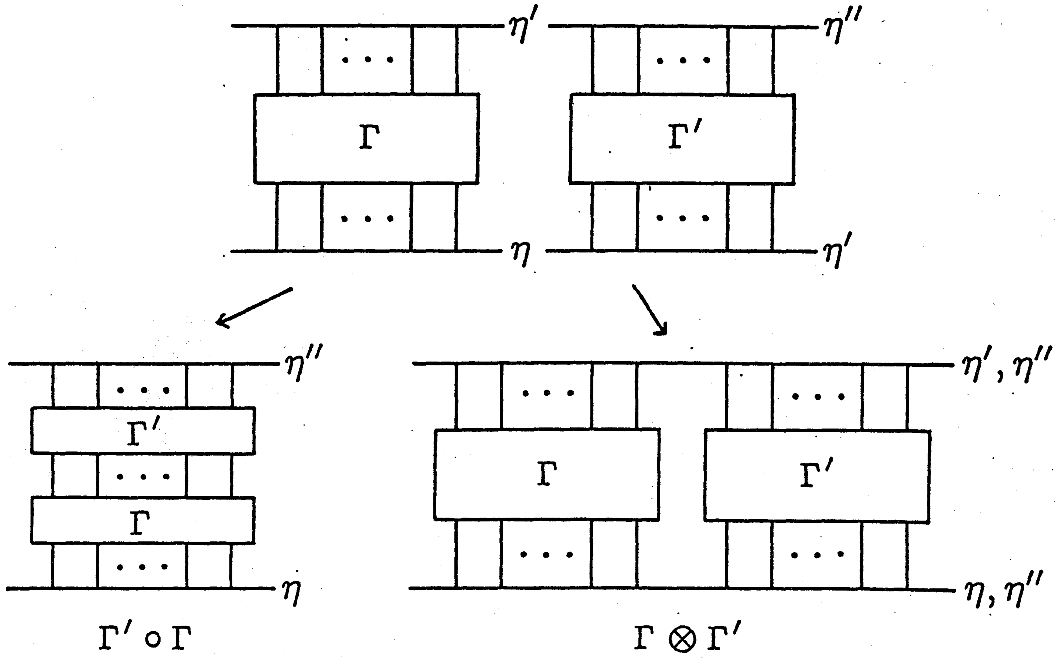

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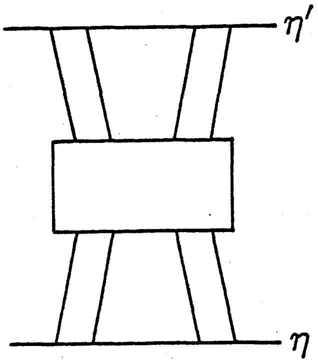


$\Gamma(f : \eta, \eta')$

Fig.5

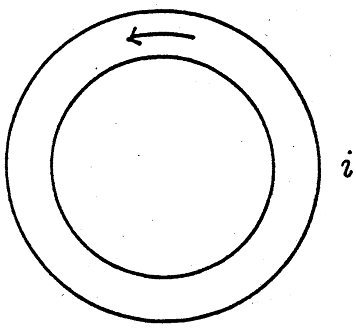
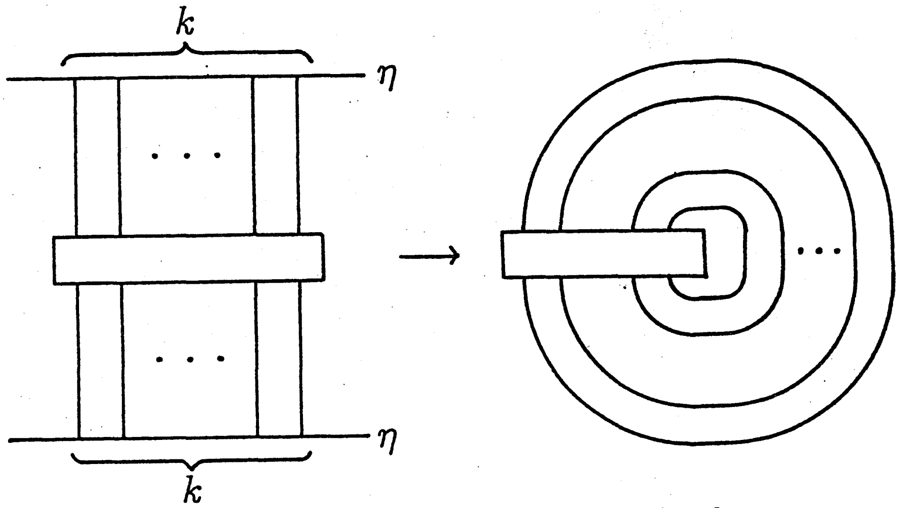


Fig.6



Γ L

Fig.7

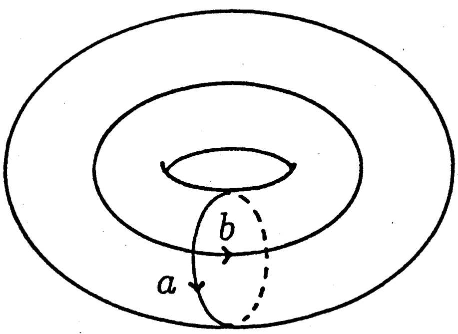


Fig.8

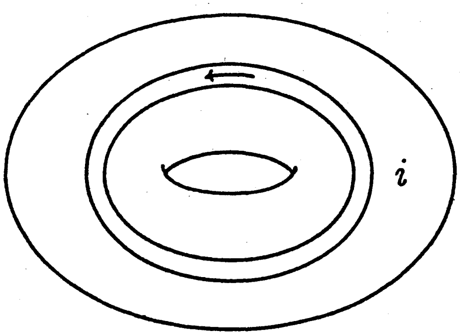


Fig.9

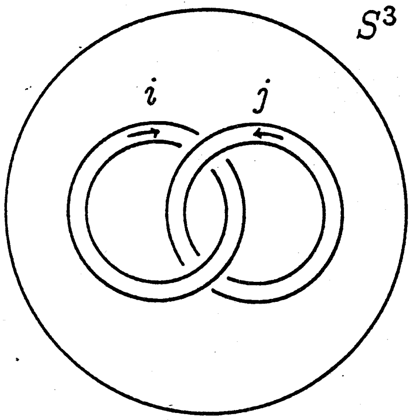


Fig.10

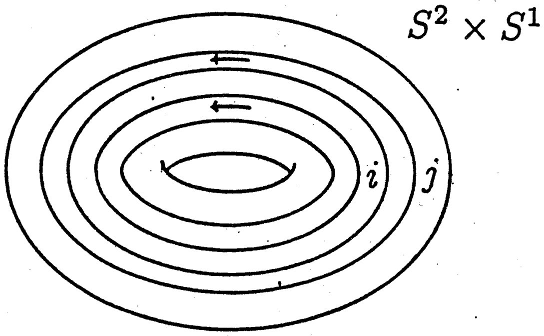


Fig.11

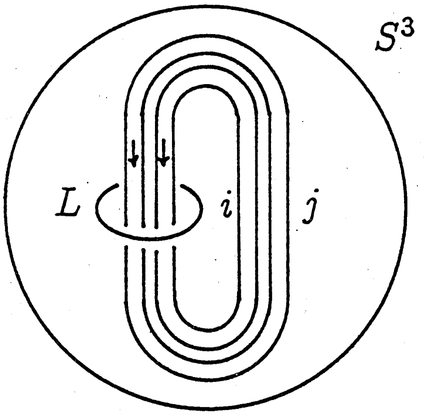
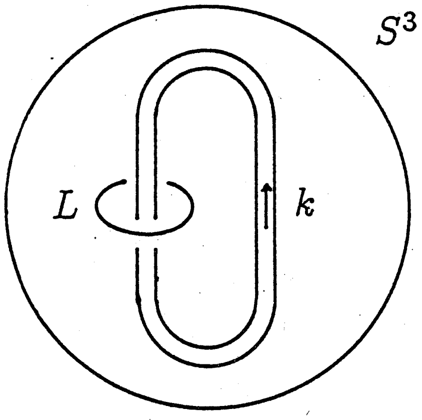


Fig.12a



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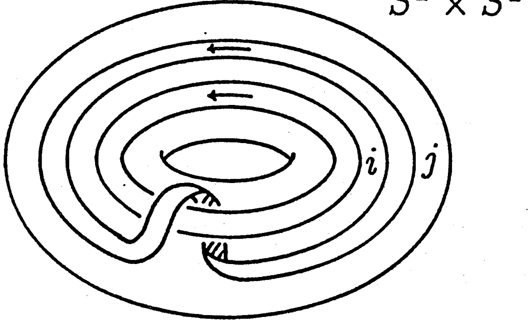


Fig.13

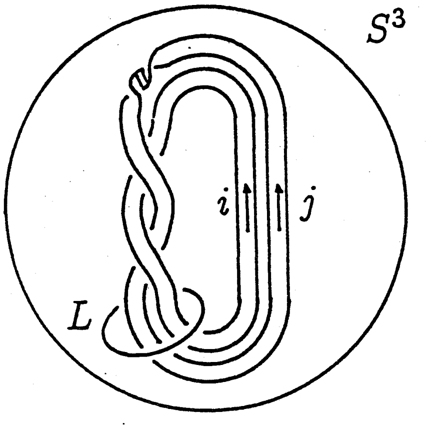


Fig.14a

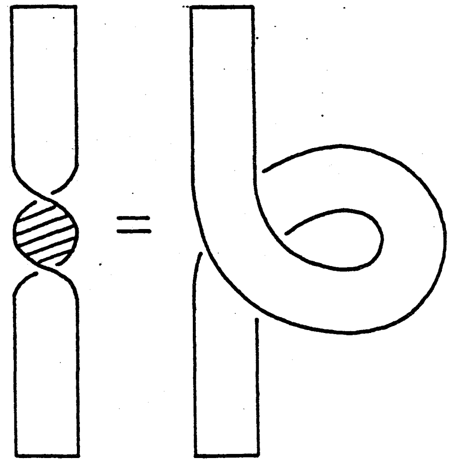


Fig.14b

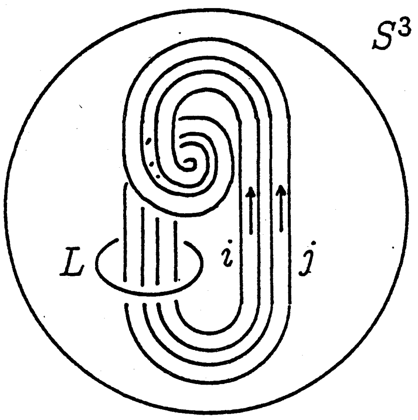


Fig.14c

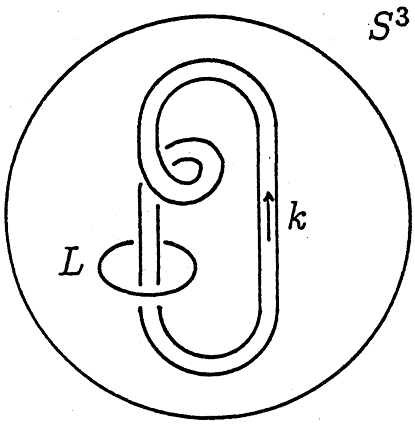


Fig.14d

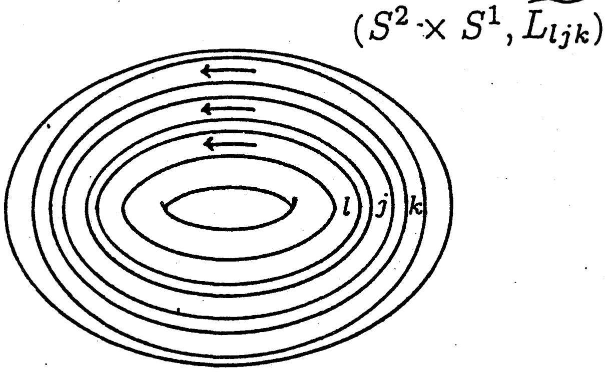


Fig.15

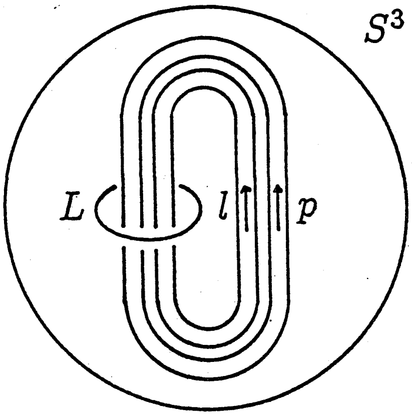


Fig.16b

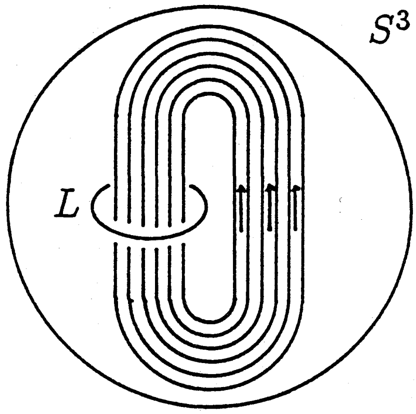


Fig.16a

Fig.17a

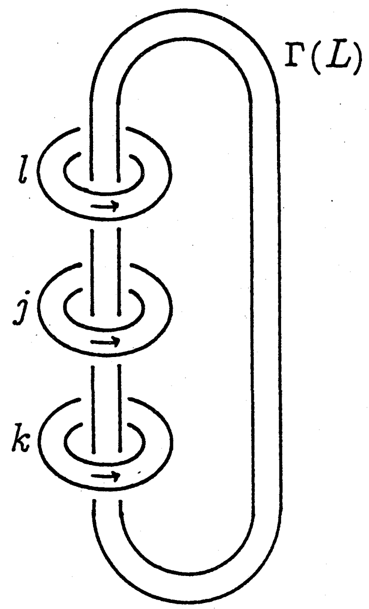
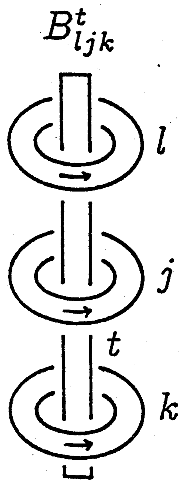


Fig.17b



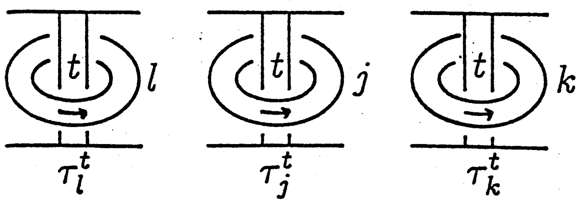


Fig.17c