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Quantized Contact Transformations and Pseudodifferential Operators of Infinite Order

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1. Introduction

In this note we establish a formula which gives quantized contact transformations of pseudodifferential operators in terms of symbols. A quantization of a given contact transformation ϕ is an extension of ϕ to a ring isomorphism ϕ_* between the rings of pseudodifferential operators ([KS], [M], [SKK]). We calculate the symbol of $\phi_*(P)$ in terms of the symbol of an operator P. As an application of the formula, we define the characteristic sets of pseudodifferential operators of infinite order and show that the sets are invariant under quantized contact transformations.

2. Quantized Contact Transformations

Let ϕ be a contact transformation defined by the following relations:

$$\phi(y,\eta)=(x,\xi)$$

with

$$x=y+\frac{\partial S}{\partial \xi}(y,\xi),$$

$$\eta=\xi+\frac{\partial S}{\partial y}(y,\xi),$$

where $x=(x_1,\dots,x_n),\dots$ *etc.*, and S is a holomorphic function homogeneous in ξ of order 1 such that $|S(y,\xi)|/|\xi|$ is very small. Let a be an invertible microdifferential symbol of finite order.

THEOREM 1. *For every formal symbol* $P(x,\xi)$ *there is a formal symbol* $Q(x,\xi)$ *such that*

$$(1) \qquad P(x,\xi)\circ(e^{S(x,\xi)}a(x,\xi))=(e^{S(x,\xi)}a(x,\xi))\circ Q(x,\xi).$$

Here $\circ$ *denotes the composition by the Leibniz-Hörmander rule:*

$$A(x,\xi)\circ B(x,\xi)=\Sigma\frac{1}{\alpha!}\partial_\xi^\alpha A(x,\xi)\cdot\partial_x^\alpha B(x,\xi).$$

In the case of microdifferential operators of finite order, this theorem is given by [M]. The correspondence $P(x,\xi)\mapsto Q(x,\xi)$ induces a ring isomorphism $\phi_*:\mathcal{E}^{\mathbf{R}}\mapsto\mathcal{E}^{\mathbf{R}}$, which we call a quantization of ϕ. Theorem 1 follows from the following theorem (we assume $a=1$ for simplicity):

THEOREM 2. *There are two invertible microdifferential symbols* $A(x,\xi)$, $B(x,\xi,\zeta)$ *of order 0 such that* P *and* Q *satisfy (1) if and only if*

$$Q(x,\xi)$$
$$= A \circ e^{(\partial_x+\partial_z)\cdot\partial_\zeta+\partial_y\cdot\partial_\eta} B\, P(z+\sigma, \xi+\eta+\theta(z+\sigma, z+y+\sigma, \xi))|^{y=0,z=x}_{\eta=0,\zeta=\xi},$$

where σ *is characterized by* $\sigma(x,\xi,\zeta) = -\vartheta(x+\sigma(x,\xi,\zeta),\xi,\zeta)$ *and where* θ, ϑ *are defined by*

$$S(x,\xi) - S(y,\xi) = < x-y, \theta(x,y,\xi) >,$$

$$S(x,\xi) - S(x,\zeta) = < \xi-\zeta, \vartheta(x,\xi,\zeta) > .$$

The symbol A is constructed as follows:

$$(e^{\partial_\xi\cdot\partial_x} e^{-S(x,\xi)}) \circ e^{S(x,\xi)} = A'(x,\xi)$$

is an invertible microdifferential symbol of order 0 (cf. [K], [KW]). A is the inverse symbol of A', that is, a symbol satisfying $A \circ A' = A' \circ A = 1$. We can construct B in a similar way by using ϑ and show that the principal part of B coincides with that of A' modulo $\zeta - \xi$. Anyway, the important fact is the following: both A and B are invertible and of order 0. So they do not affect the "characteristic", which is defined in the following section.

3. Characteristic sets of pseudodifferential operators of infinite order

Let $P(x,\xi)$ be a symbol in the sense of [A].

DEFINITION 3. *An element $x* = (x_0, \xi_0)$ is said to be non-characteristic with respect to $P =: P(x,\xi)$: if there exist a conic neighborhood Ω of $x*$ (in T^*X) and a positive number r such that for every $\varepsilon > 0$, there is $C_\varepsilon > 0$ for which we have*

$$|P(x,\xi)| \geqq C_\varepsilon e^{-\varepsilon|\xi|} \qquad in \qquad \Omega \cap \{|\xi| \geqq r\}.$$

We write $Char(P)$ the compliment of the set of all non-characteristic elements with respect to P.

Of course, if P is of finite order this definition of $Char(P)$ coincides with the usual ones. In general, we have

$$Char(P) \supset Supp(\mathcal{E}^{\mathbf{R}}/\mathcal{E}^{\mathbf{R}}P).$$

If $x*$ does not belong to $Char(P)$, we may assume that $P(x,\xi)$ is written in the form $e^{p(x,\xi)}$ with a symbol $p(x,\xi)$ of order 1-0. By Theorem 2, $Q(x,\xi)$ can be written in the exponential of some symbol of order 1-0 (cf. [A]). Moreover, ϕ is given by

$$\phi : (x + \sigma(x,\xi,\xi), \xi + \frac{\partial S}{\partial x}(x, \sigma(x,\xi,\xi),\xi)) \mapsto (x,\xi)$$

Hence we have

THEOREM 4. $Char(\phi_*(P)) = \phi(Char(P))$.

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