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The Ihara zeta functions of algebraic groups

Takashi ICHIKAWA (市 川 尚 志)
Department of Mathematics, Faculty of Science,
Kyushu University 33, Fukuoka 812, JAPAN

Introduction

Let G be a connected and reductive algebraic group defined over $\mathbf{Q}$ of hermitian type, and X the bounded symmetric domain induced from the identity component $G(\mathbf{R})_+$ of $G(\mathbf{R})$. Let Γ_0 be a congruence subgroup of $G(\mathbf{Z}) \cap G(\mathbf{R})_+$, and M the Shimura model of X/Γ_0. Langlands' program [10] to parametrize the set $M(\overline{\mathbf{F}}_p)$ (p : a prime on which M has good reduction) was partially achieved by Kottwitz [9] for the Siegel modular case. In this note, when G has a similitude-symplectic embedding (for the classification of such groups, see Satake [16] and Deligne [4]), we shall construct, without detailed proofs, a canonical bijection of a certain subset of X/Γ_0 to an algebraically defined subset of $M(\overline{\mathbf{F}}_p)$. This result can be regarded as a generalization of the result of Ihara [8] on zeta functions of Selberg type (Ihara zeta functions) for congruence subgroups of $PSL_2(\mathbf{Z}[1/p])$.

Following Ihara's idea, we take a congruence subgroup Γ of $G(\mathbf{Z}[1/p]) \cap G(\mathbf{R})_+$ such that $\Gamma \cap G(\mathbf{Z}) = \Gamma_0$. We call $x \in X$ is a *p-ordinary point* if there exists a torsion-free stabilizer of x in Γ inducing a p-adic structure on a faithful representation space V of G which is compatible with the Hodge structure on V induced from x (this definition is independent of the choice of V). When G is a similitude-symplectic group, we show that the reduction map induces a canonical bijection of $\{p\text{-ordinary points of } X\}/\Gamma_0$ to the ordinary locus of $M(\overline{\mathbf{F}}_p)$. This is nothing but a reformation of a result of Deligne [2] and the inverse map corresponds to canonical liftings of ordinary abelian varieties. When G has a similitude-symplectic embedding, we show that the image of this bijection is algebraically defined which follows from that canonical liftings of abelian varieties preserve their deformations.

1 Zeta functions

1.1. Let G be a linear algebraic group defined over $\mathbf{Q}$ which is connected and reductive. For any field K containing $\mathbf{Q}$, let $G(K)$ denote the group of K-rational points of G, and put $G_K = G \otimes_{\mathbf{Q}} K$. Let $G(\mathbf{R})_+$ denote the identity component of the Lie group $G(\mathbf{R})$, and put $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$. We assume that there exists an $\mathbf{R}$-homomorphism $h : \mathbf{S} = R_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m/\mathbf{C}}) \to G_{\mathbf{R}}$ such that

$$X = \{\mathbf{R}\text{-homomorphisms } \mathbf{S} \to G_{\mathbf{R}} \text{ conjugate to } h \text{ over } G(\mathbf{R})_+\}$$

is a bounded symmetric domain. Let V be a $\mathbf{Q}$-vector space of finite dimension, and $\phi : G \to GL(V)$ an injective representation defined over $\mathbf{Q}$. Let L be a $\mathbf{Z}$-lattice of V, p a prime number, and put $L[1/p] = L \otimes \mathbf{Z}[1/p]$ which is a $\mathbf{Z}[1/p]$-lattice of V. Let Γ be a congruence subgroup of

$$\phi^{-1}(\mathrm{Aut}(L[1/p]))_+ = \{g \in G(\mathbf{Q})_+ | \phi(g) \in \mathrm{Aut}(L[1/p])\}.$$

One can show that if there exists an integer $n \geq 3$ prime to p such that $\phi(\Gamma) \subset \{g \in \mathrm{Aut}(L[1/p]) | g \equiv 1(n)\}$, then Γ is torsion-free. For each $x \in X$, put $\Gamma_x = \{\gamma \in \Gamma | \gamma(x) = x\}$. Let $h_x : \mathbf{S} \to G_{\mathbf{R}}$ denote the homomorphism corresponding to x. Then $\phi_{\mathbf{R}} \circ h_x$ induces a Hodge decomposition

$$V \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{i,j} V_x^{i,j}$$

such that for any $(z, z') \in \mathbf{S}(\mathbf{C}) = \mathbf{C}^\times \times \mathbf{C}^\times$ and $v \in V_x^{i,j}$, $(\phi_{\mathbf{R}} \circ h_x)((z, z'))(v) = z^i \cdot z'^j \cdot v$. Then for any $\gamma \in \Gamma_x$, $V_x^{i,j}$ is stable under the action of $\phi(\gamma)_{\mathbf{C}} = \phi(\gamma) \otimes_{\mathbf{Q}} \mathbf{C}$. Fix an isomorphism $\iota : \mathbf{C} \tilde{\to} \overline{\mathbf{Q}}_p$, and let Γ'_x be the set which consists of $\gamma \in \Gamma_x$ such that there exists a rational number $d(\gamma)$ satisfying $\mathrm{ord}_p(\iota(e)) = d(\gamma) \cdot i$ for any eigenvalue e of $\phi(\gamma)_{\mathbf{C}}$ on each $V_x^{i,j}$.

1.2. **Proposition.** *For any $x \in X$, Γ'_x is independent of ϕ, and for any $x \in X$ and $\gamma \in \Gamma'_x$, $d(\gamma)$ is independent of ϕ.*

1.3. **Proposition.** *Let Z be the centralizer of $h(\mathbf{S}(\mathbf{R})) = h(\mathbf{C}^\times)$ in $G(\mathbf{R})$, and assume that $Z/h(\mathbf{R}^\times)$ is compact. Then for any $x \in X$ and $\gamma \in \Gamma'_x$, $d(\gamma) \neq 0$ if and only if γ is torsion-free.*

1.4. **Corollary.** *Assume that there exist a positive integer g and an injective $\mathbf{Q}$-homomorphism of G into the similitude-symplectic algebraic group of size $2g$ which induces a map of X into the Siegel upper half space of degree g. Then for any $\gamma \in \Gamma'_x$, $d(\gamma) \neq 0$ if and only if γ is torsion-free.*

1.5. **Proposition.** *Let $X^{\mathrm{ord}}(\Gamma)$ be the set consisting of $x \in X$ such that there exists $\gamma \in \Gamma'_x$ with $d(\gamma) \neq 0$. Then $X^{\mathrm{ord}}(\Gamma)$ depends only on the $\mathbf{Q}$-structure of G, i.e., it is independent of the choice of Γ.*

1.6. *Remark.* Propositions 1.2 and 1.5 follow the fact that any representation $G \to GL(W)$ is a direct summand of

$$G \;\to\; GL(\oplus_l (V^{\otimes m_l} \otimes (V^*)^{\otimes n_l}))$$

for some m_l and n_l ([5], Proposition 3.1). Proposition 1.3 follows from the product formula for eigenvalues of $\phi(\gamma)$.

1.7. By Proposition 1.5, $X^{\mathrm{ord}}(\Gamma)$ is independent of Γ. Then we put $X^{\mathrm{ord}} = X^{\mathrm{ord}}(\Gamma)$, and call it the set of *ordinary points* of X with respect to ι. For any $x \in X^{\mathrm{ord}}$, let $\Gamma'_x(L)$ be the set consisting of $\gamma \in \Gamma'_x$ such that there exists a decomposition $L \otimes_{\mathbf{Z}} \mathbf{Z}_p$ as $\mathbf{Z}_p$-lattices:

$$L \otimes_{\mathbf{Z}} \mathbf{Z}_p \;=\; \oplus_{i,j}\, L^{i,j}$$

which satisfies $\phi(\gamma)_{\mathbf{Q}_p}(L^{i,j}) = \iota(e) \cdot L^{i,j}$ for any eigenvalue e of $\phi(\gamma)_{\mathbf{C}}$ on each $V_x^{i,j}$. Put

$$\deg(x) = \begin{cases} \min\{d(\gamma) | \gamma \in \Gamma'_x(L) \text{ with } d(\gamma) > 0\} & \text{if } \Gamma'_x(L) \neq \emptyset, \\ 0 & \text{if } \Gamma'_x(L) = \emptyset. \end{cases}$$

Let Γ_0 be the subgroup of Γ defined by

$$\Gamma_0 = \{\gamma \in \Gamma | \phi(\gamma) \in \mathrm{Aut}(L)\}.$$

Then $\deg(x)$ depends only on the Γ_0-equivalence class containing x. Hence $\deg : X \to \mathbf{R}$ induces the map of

$$\mathbf{P}(\Gamma) = \{x \in X^{\mathrm{ord}} | \deg(x) : \text{positive integer}\}/\Gamma_0$$

to $\mathbf{N}$, which we denote by the same symbol. Then we define the zeta function $Z(\Gamma, t)$ of Γ as the following formal power series with variable t :

$$\exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}),$$

where N_r is the cardinality of $\{P \in \mathbf{P}(\Gamma) | \deg(P) \le r\}$.

1.8. *Conjecture.* Let x be any ordinary point of X. Then

(1.8.1) x is a special point of X in the sense of [3].

(1.8.2) $\deg(x)$ is a positive integer, and

$$\{d(\gamma) | \gamma \in \Gamma'_x(L)\} = \mathbf{Z} \cdot \deg(x).$$

(1.8.3) If Γ is torsion-free, then $\Gamma'_x(L)$ is a cyclic group generated by an element $\gamma \in \Gamma'_x(L)$ with $d(\gamma) = \deg(x)$.

Assuming this conjecture, $Z(\Gamma, t)$ can be regarded as a generalization of Ihara's zeta function for PSL_2.

1.9. By results of Satake [15] and Baily-Borel [1], the quotient complex manifold $X/\Gamma_\circ$ is algebraizable. By results of Shimura [17], Deligne [4] [5], and Milne [13], there exist canonically a number field $K(\Gamma)$ contained in $\mathbf{C}$ and an integral scheme M_Γ of finite type defined over $K(\Gamma)$, called the canonical model of $X/\Gamma_\circ$, such that $M_\Gamma(\mathbf{C}) = X/\Gamma_\circ$ and the behavior of special point of M_Γ under the action of $\mathrm{Gal}(\overline{K(\Gamma)}/K(\Gamma))$ is described by the theory of complex multiplication.

If $G = GSp(V)$, then M_Γ is the moduli scheme of abelian varieties with polarization and level structure. If G has a similitude-symplectic embedding, then M_Γ is the moduli scheme of these objects with certain absolute Hodge cycles.

1.10. *Conjecture.* Let $k(\Gamma)$ be the residue field of $K(\Gamma)$ with respect to ι, and p^a the order of $k(\Gamma)$. Then there exists a separated scheme F of finite type defined over $k(\Gamma)$ whose zeta function $Z(F,t)$ satisfies

$$Z(\Gamma,t) \; = \; Z(F,t^a).$$

Moreover, if M has good reduction at ι, then F can be given as a locally closed subset of the special fiber of M with respect to ι.

Assuming this Conjecture, by a result of Dwork [6], one can see that $Z(\Gamma,t)$ is a rational function of t.

2 Symplectic case

2.1. Let g be a positive integer, V a $\mathbf{Q}$-vector space with basis $\{v_1, ..., v_{2g}\}$, and $\psi : V \times V \to \mathbf{Q}$ be the alternating $\mathbf{Q}$-bilinear form given by

$$\psi(v_i, v_j) = \delta_{i,j-g} \; (1 \leq i,j \leq 2g).$$

Let G denote the similitude-symplectic algebraic subgroup $GSp(V,\psi)$ of $GL(V)$ defined over $\mathbf{Q}$ with respect to ψ, i.e., $g \in \mathrm{Aut}(V)$ belongs to $G(\mathbf{Q})$ if and only if there exists an element $\nu(g) \in \mathbf{Q}^\times$ such that $\psi(gv, gw) = \nu(g)\cdot\psi(v,w)$ for all $v, w \in V$. Let $h : \mathbf{S} \to G_{\mathbf{R}}$ be the $\mathbf{R}$-homomorphism given by

$$h(a+b\sqrt{-1})(v_i) = \begin{cases} aw + bw' & (1 \leq i \leq g), \\ -bw + aw' & (g+1 \leq i \leq 2g), \end{cases}$$

where $(a,b) \in \mathbf{R}^2 - \{(0,0)\}$ and $w = v_1 + ... + v_g$, $w' = v_{g+1} + ... + v_{2g}$. Then X is the the Siegel upper half space H_g of degree g which is the bounded symmetric domain induced from $G(\mathbf{R})_+ = \{g \in G(\mathbf{R}) | \nu(g) > 0\}$. Let L be a $\mathbf{Z}$-lattice of V such that $\psi(L \times L) = \mathbf{Z}$, and let d_L be the index of L in $\{v \in V | \psi(v,w) \in \mathbf{Z}$ for any $w \in L\}$.

For each $x \in X$, let A_x be the g-dimensional abelian variety defined over $\mathbf{C}$ such that $H^1(A_x, \mathbf{Z}) = L$ and the Hodge decomposition of $H^1(A_x, \mathbf{C}) = V_{\mathbf{C}}$ is given by h_x, and θ_x the polarization of A_x whose Riemann form is given by ψ. Then by the correspondence

$$X \ni x \longmapsto (A_x, \theta_x, i_x = \text{id.} : H^1(A_x, \mathbf{Z}) \tilde{\rightarrow} L),$$

X becomes the moduli space of the isomorphism classes of triples

$$(A, \theta, i : H^1(A, \mathbf{Z}) \tilde{\rightarrow} L),$$

where A is a g-dimensional abelian variety defined over $\mathbf{C}$ and θ is a polarization of A whose Riemann form is given by

$$H^1(A, \mathbf{Z}) \times H^1(A, \mathbf{Z}) \ni (u, v) \longmapsto \psi(i(u), i(v)) \in \mathbf{Z}.$$

Let p be a prime number, and Γ a congruence subgroup of $G(\mathbf{Q})_+ \cap \mathrm{Aut}(L[1/p])$. Then $\Gamma_0 = \Gamma \cap \mathrm{Aut}(L)$ is a subgroup of $G(\mathbf{Q})_+ \cap \mathrm{Aut}(L)$ defined by congruence conditions prime to p. Two triples (A_1, θ_1, i_1) and (A_2, θ_2, i_2) are said to be Γ_0-equivalent if there exists an element $\gamma \in \Gamma_0$ such that $(A_1, \theta_1, \gamma \circ i_1)$ and (A_2, θ_2, i_2) are isomorphic. For each Γ_0-equivalence class (A, θ, σ), σ is called a level Γ_0-structure of A. For each $x \in X$, let $(A_x, \theta_x, \sigma_x)$ denote the Γ_0-equivalence class containing (A_x, θ_x, i_x). Let $M = M_\Gamma$ be the canonical model of X/Γ_0 defined over $K(\Gamma)$. Assume that $(p, d_L) = 1$. Then by a result of Mumford [14], M has good reduction with respect to ι. Let M_0 denote its special fiber with respect to ι. Let U be the ordinary locus of M_0, i.e., the open subscheme of M_0 defined over $k(\Gamma)$ consisting of all points of M_0 corresponding to ordinary abelian varieties.

2.2. Let k be a perfect field of characteristic p, and A_0 an ordinary abelian variety defined oved k of dimension g. Then the p-divisible group $A_0(p)$ associated with A_0 is the product of a multiplicative p-divisible group and an étale p-divisible group. Let $W(k)$ denote the ring of Witt vectors over k, and R a complete discrete valuation ring containing $W(k)$ with residue field k. Then by a result of Lubin-Tate-Serre [11], there exists a unique pair (A, i) up to isomorphim of an abelian scheme A over R and an isomorphism $i : A \otimes_R k \to A_0$ such that $A(p)$ is the product of a multiplicative p-divisible group and an étale p-divisible group. The pair (A, i) is called the canonical lifting of A_0 to R. Moreover, it is known that for all ordinary abelian varieties A_0 and B_0 defined over k, the reduction map induces the isomorphism

$$\text{(2.2.1)} \qquad \mathrm{Hom}_R((A,i),(B,i)) \xrightarrow{\sim} \mathrm{Hom}_k(A_0, B_0),$$

where (A, i) and (B, i) are the canonical liftings of A_0 and B_0 to R respectively ([11]).

Let k be a finite field $\mathbf{F}_q$, and A_0 any ordinary abelian variety defined over k. Then by a result of Messing [12], a lifting (A, i) of A_0 to R is the canonical lifting if and only if there exists an endomorphism f of A such that $f \otimes_R k$ is the q-th power Frobenius endomorphism of A_0. Let (A, i) be the canonical lifting of A_0 to R. Since A_0 has complex multiplication ([18]), by (2.2.1), A has also complex multiplication.

2.3. **Proposition.** *For any $x \in X$, the following two conditions are equivalent.*

(A) *x is an ordinary point of X.*

(B) *There exists an ordinary abelian variety A_0 defined over $\overline{\mathbf{F}}_p$ such that A_x is the canonical lifting of A_0 with respect to ι, i.e.,*

$$A_x \otimes_{\mathbf{C},\iota} \overline{\mathbf{Q}}_p \cong A \otimes_{W(\overline{\mathbf{F}}_p)} \overline{\mathbf{Q}}_p,$$

where A is the canonical lifting of A_0 to $W(\overline{\mathbf{F}}_p)$.

2.4. **Theorem.** *Assume that $(p, d_L) = 1$. Then Conjectures 1.8 and 1.10 hold for any congruence subgroup Γ of $GSp(L[1/p], \psi)_+$, where F is given as the ordinary locus U of M_0.*

2.5. *Remark.* The key point of the proof of Proposition 2.3 and Theorem 2.4 is that any element $\gamma \in \Gamma'_x$ with $d(\gamma) > 0$ is the unique lifting of a Frobenius endomorphism on a certain ordinary abelian variety defined over a finite field to its canonical lifting. To show the existence of such an abelian variety, we use a result of Honda [7].

3 Classical case

3.1. Let $\phi : G \to GL(V)$, X, and Γ be as in 1.1, and let $\psi : V \times V \to \mathbf{Q}$ and L be as in 2.1. In what follows, assume the following:

(3.1.1) The image of ϕ is contained in $GSp(V, \psi)$ and ϕ induces a map $h : X \to H_g$.

(3.1.2) There exists a positive integer $n \geq 3$ prime to p such that

$$\phi(\Gamma) \subset \{g \in \mathrm{Aut}(L[1/p]) | g \equiv 1(n)\}.$$

Then h is known to be a holomorphic embedding, and by Proposition 1.15 of [3], there exists a unique congruence subgroup Γ' of $GSp(L[1/p], \psi)_+$ such that $\Gamma = \Gamma' \cap G(\mathbf{Q})_+$ and the map

$$X/(\Gamma \cap \phi^{-1}(\mathrm{Aut}(L))) \to H_g/(\Gamma' \cap \mathrm{Aut}(L))$$

induced from h is injective. By (3.1.2),

$$\Gamma' \subset \{g \in \mathrm{Aut}(L[1/p]) | g \equiv 1(n)\}.$$

Hence Γ' and Γ are torsion-free.

3.2. Let M' be the canonical model of $H_g/(\Gamma' \cap \mathrm{Aut}(L))$ defined over $K' = K(\Gamma')$. Assume that $(p, d_L) = 1$. Then M' has good reduction with respect to ι. Let k' be the residue field of K' with respect to ι. Let U be the ordinary locus of the reduction of M' with respect to ι. Then U is defined over k'. Let $\alpha : U \to M'$ be the map corresponding to the canonical lifting of ordinary abelian varieties, i.e., if $x \in U$ and $X = \alpha(x)$, then $(A_X, \theta_X, \sigma_X)$ is the canonical lifting of $(A_x, \theta_x, \sigma_x)$ with respect to ι.

3.3. **Proposition.** *Let L be any finite field extention of $\iota(K')$, and $\mathbf{F}_q$ its residue field. Then $\alpha : U \otimes_{k'} \mathbf{F}_q \to M' \otimes_{K',\iota} L$ is continuous map with respect to the Zariski topology, i.e., if $z \in U \otimes_{k'} \mathbf{F}_q$ is a specialization of $y \in U \otimes_{k'} \mathbf{F}_q$, then $\alpha(z)$ is a specialization of $\alpha(y)$ in $M' \otimes_{K',\iota} L$.*

3.4. **Corollary.** *Put $Z = \{x \in U | \alpha(x) \in M\}$. Then Z is a closed subset of U defined over $k(\Gamma)$.*

3.5. **Proposition.** *Under Conditions* (3.1.1) *and* (3.1.2), *for any $x \in X^{\mathrm{ord}}$,*

$$\phi(\Gamma'_x(L)) = \{\gamma \in (\Gamma_1)'_{h(x)}(L) | k(\Gamma) \subset \mathbf{F}_{p^{d(\gamma)}}\}.$$

3.6. **Theorem.** *Assume that $(p, d_L) = 1$. Then under Conditions* (3.1.1) *and* (3.1.2), *Conjectures* 1.8 *and* 1.10 *hold for Γ, where Z is given in Corollary* 3.4.

3.7. *Remark.* To show Proposition 3.3, by using Serre-Tate's q-theory ([11], [12]), we construct an abelian scheme with a polarization and a level structure over a discrete valuation ring whose general and special fibers correspond to $\alpha(y)$ and $\alpha(z)$ respectively. The proof of Proposition 3.5 is straightforward. Theorem 3.6 follows from Theorem 2.4, Corollary 3.4 and Proposition 3.5.

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