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On some branched surfaces which admit expanding immersions II

by
Eijirou HAYAKAWA
Kyoto University

Abstract. Let K be a branched surface whose branch set S is an embedded circle and such that $K\setminus S$ is connected and oriented. We show that K does not admit expanding immersions. Combined with our previous result [3], this implies that among branched surfaces with branch sets single embedded circles, there are only two which admit expanding immersions.

0. Introduction

In [3], we have studied the existence of expanding immersions for branched surfaces K with branch sets S single embedded circles. But there we restricted ourselves to the case when all connected components of $K\setminus S$ are orientable and their number is two or three. In this paper, we study the remaining case *i.e.* the case when $K\setminus S$ is connected. We still assume $K\setminus S$ is an orientable manifold.

Then this type of branched surfaces is constructed from $\hat{K}^1_g = \Sigma_{g-1}-(\mathring{D}^+ \amalg \mathring{D}^-)$ or $\hat{K}^0_g = \Sigma_{g-1}-(\mathring{D}^+_1 \amalg \mathring{D}^+_2 \amalg \mathring{D}^-)$ as follows, where Σ_{g-1} denotes a Riemann surface of genus $g-1$, and $\mathring{D}^+$, $\mathring{D}^+_1$, $\mathring{D}^+_2$ and $\mathring{D}^-$ denote disjoint embedded open disks. Let S^+ and S^- be connected components of $\partial\hat{K}_g^1$, and let S_1^+, S_2^+ and S^- be connected components of $\partial\hat{K}_g^0$. Let us take a C^1 immersion $\varphi:S^+ \to S^-$, whose mapping degree is +2 or -2 with respect to the orientation on S^+ and S^- induced from an orientation of Σ_{g-1} . We identify $x\in S^+$ with $y\in S^-$ whenever $\varphi(x)=y$ and glue S^+ to S^- , so that we obtain a C^1 branched surface from $\hat{K}_g^1$. By the identification, using C^1 diffeomorphisms $\psi_1:S_1^+ \to S^-$ and $\psi_2:S_2^+ \to S^-$, on S_1^+, S_2^+ and S^- , we can construct a branched surface from $\hat{K}_g^0$ in the same way as above. Ignoring the difference of degrees of attaching maps φ, ψ_1 and ψ_2 , we simply denote the former by K_g^1 and the latter by K_g^0 . In fact, we have two topological types for the former, and three types for the latter. We call the image of S_1^+, S_2^+ and S^- , or S^+ and S^- under the above identification the branch set of K_g^0 or K_g^1 respectively.

Our main result is as follows:

Theorem. *C^1 branched surface K_g^ν , with $\nu= 0$ or 1 and $g\geq 1$, does not admit C^1 expanding immersions.*

For definitions of C^r branched surfaces and C^r expanding immersions, refer to Definition 1 and 2 [3]. Combining this result with the previous one [3], we conclude that in the class of branched surfaces with branch sets single embedded circles, only T^* and T_* (See [3]) admit expanding immersions

In §.1, we define a hyperbolic structure on K_g^ν , that is, a Riemannian metric on the tangent space with negative constant curvature. §.2 is devoted to the proof of the theorem. First we show that there exist isometric immersions $\tilde{j}:D \to \tilde{K}_g^\nu$ and $d:\tilde{K}_g^\nu \to D$, where

D denotes the Poincaré disk and $\tilde{K}_g^\nu$ denotes the universal covering of K_g^ν. Suppose that K_g^ν admits an expanding immersion f. Then the composite $d\circ\tilde{f}\circ\tilde{\jmath}$ is shown to be an expanding quasiconformal map of D, where $\tilde{f}$ denotes a lift of f. In the final lemma, we show that no such maps exist.

The author thanks the referee for pointing out the use of hyperbolic structures. The idea of the proof adopted in this paper is due to him.

1. Hyperbolic structures on K_g^ν

We will define a hyperbolic structure on K_g^ν. For hyperbolic structures on ordinary surfaces, the readers can refer to [1].

First we deal with K_g^0. Using the Poincaré disk model, we can define a hyperbolic structure on $\hat{K}_g^0$ which makes S_1^+, S_2^+ and S^- closed geodesics of the same length. Then choosing ψ_1 and ψ_2 as isometries, we obtain a hyperbolic structure on K_g^0.

Next we consider K_g^1 with $g\geq 2$. In this case, as for K_g^0, we have a hyperbolic structure on $\hat{K}_g^1$ such that S^+ and S^- are closed geodesics and $l(S^+)= 2\cdot l(S^-)$, where $l(\cdot)$ denote the length of arcs. Then we choose φ as an isometric immersion, and we can also define a hyperbolic structure on K_g^1.

Finally we will define a hyperbolic structure on K_1^1. Consider $D\setminus\{0\}$. The hyperbolic metric $-2|dz|/|z|\log|z|$ is easily seen to be invariant under the mapping $\sigma(z)= z^2$. Therefore by pasting two boundaries of $\{z; 1/4\leq|z|\leq 1/2\}$ by σ, one gets a hyperbolic structure on K_1^1

Remark. i) Under usual hyperbolic structures on ordinary surfaces, coordinate changes of local charts are orientation preserving isometries on D . But when $deg\ \varphi = -2$, ours on K_g^1 have orientation reversing isometries. Also when $deg\ \psi_1 = -1$ and/or $deg\ \psi_2 = -1$, K_g^0's admits such ones.

ii) Under the above hyperbolic structures, except K_1^1 , their branch sets are closed geodesics. But in the case of K_1^1 , the branch set is not a geodesic.

iii) We can define a hyperbolic structure on K_g^1 such that there exists a closed geodesic S_0 intersecting S perpendicularly only at one point.

In this place, as Thurston's developing map [5], we will define an isometric immersion from the universal cover $\tilde{K}_g^\nu$ of K_g^ν to the Poincaré disk D .

Let $\pi:\tilde{K}_g^\nu \to K_g^\nu$ be the projection. Take $x_0 \in \tilde{K}_g^\nu$ and a sufficiently small neighborhood U_0 of x_0 . Set $\bar{U}_0 = \pi(U_0)$. Then we have an isometric immersion $d_0:U_0 \to D$ as the composite of a local chart on $\bar{U}_0$ and π . For any $y \in \tilde{K}_g^\nu$, take a path γ from x_0 to y . By analytic continuation of d_0 along γ , we obtain a map $d_y:U_y \to D$, where U_y denotes a neighborhood of y . Since $\tilde{K}_g^\nu$ is simply connected, d_y does not depend on the choice of a path γ . Hence mapping any $y \in \tilde{K}_g^\nu$ to $d_y(y)$, we obtain a desired isometric immersion $d:\tilde{K}_g^\nu \to D$.

2. Proof of the theorem

We will construct an isometric immersion $j:D \to K_g^\nu$.

First we deal with K_g^1 . Recall that there exists a simple closed geodesic S_0 which is orthogonal to the branch set S . Cut K_g^1 first along S and then along S_0 . Prepare infinitely many copies of the resultant surface and glue them together so as to obtain a complete surface L . See Figure 1.

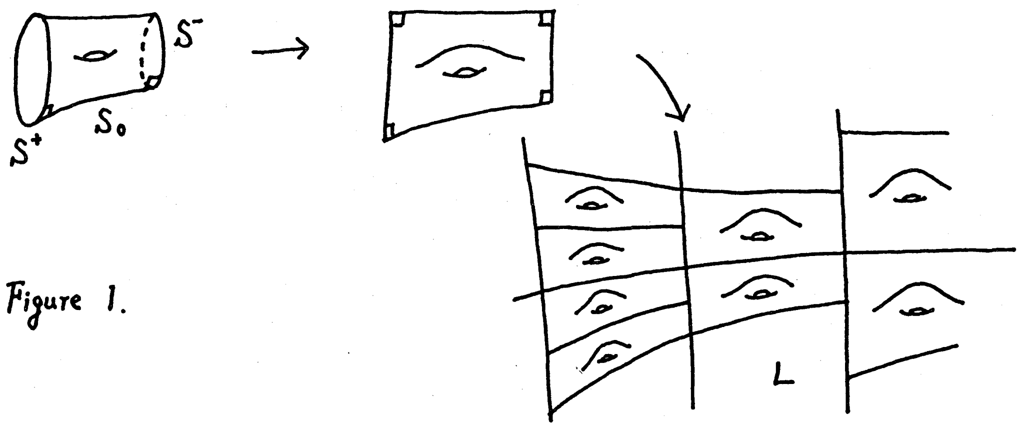

Figure 1.

Clearly there is an isometric immersion of L into K_g^1 . Passing to the universal covering, we get the desired immersion $j:D \to K_g^1$.

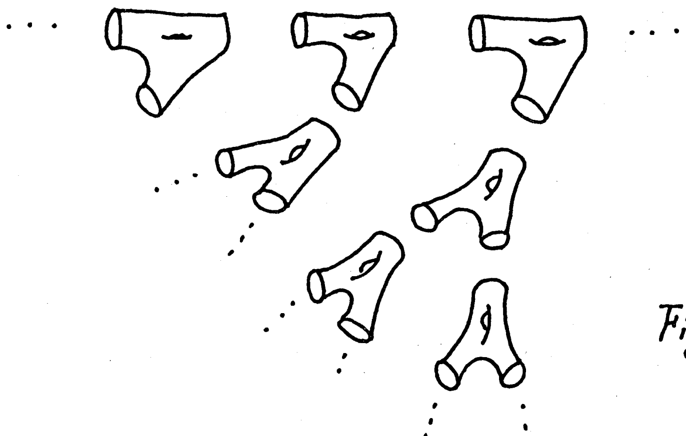

Figure 2.

Next we deal with K_g^0 . We prepare infinitely many copies K_i of the hyperbolic surface $\hat{K}_g^0$, and glue them to obtain a complete

connected hyperbolic surface M as in Figure 2 so that they satisfy:
1) Each copy K_i of $\hat{K}_g^0$ has a natural embedding $\iota_i : K_i \to M$.
2) M has an isometric immersion $p : M \to K_g^0$ such that $p \circ \iota_i : K_i \to K_g^0$ is the same map as the projection $\hat{K}_g^0 \to K_g^0$ up to the identification of K_i with $\hat{K}_g^0$. Let $\pi_M : D \to M$ be the universal covering of M . Set $j = p \circ \pi_M$. This is the desired one.

Let us continue the proof of the theorem. Suppose for contradiction that K_g^ν admits an expanding immersion f . Let us take a lift $\tilde{j} : D \to \tilde{K}_g^\nu$ of j , and take a lift $\tilde{f} : \tilde{K}_g^\nu \to \tilde{K}_g^\nu$ of f .

Set $F = d \circ \tilde{f} \circ \tilde{j}$. We shall show that F is an expanding quasiconformal map. Since f is an immersion, and since K_g^ν is compact, we have

$$k = \sup_{x \in K_g^\nu} \frac{\sup_{v \in T_x K_g^\nu} \|df(v)\| / \|v\|}{\inf_{v \in T_x K_g^\nu} \|df(v)\| / \|v\|} < \infty .$$

As d and $\tilde{j}$ are locally isometric, we have

$$k = \sup_{z \in D} \frac{\sup_{v \in T_z D} \|dF(v)\|_h / \|v\|_h}{\inf_{v \in T_z D} \|dF(v)\|_h / \|v\|_h} < \infty ,$$

where $\|\cdot\|_h$ denotes the hyperbolic metric on D . This shows immediately that F is quasiconformal. (For the definition of quasiconformal maps, refer to [2].) Also we know F is a diffeomorphism of D since F is a quasiisometric immersion.

Passing to f^n for a sufficiently large integer n in the construction of F , if necessary, we may assume that

$$\inf_{v \in TD} \|dF(v)\|_h / \|v\|_h > c > 1 .$$

Then, to complete the proof, we show the following lemma.

Lemma. *The open unit disk* D *does not admit a k-quasiconformal diffeomorphism* φ *with the following properties:*

1) $\inf_{v\in TD} \|d\varphi(v)\|_h/\|v\|_h > c > 1$.

2) $\varphi(0)= 0$.

Proof. Suppose for contradiction that there exists a k-quasiconformal diffeomorphism φ with the properties 1) and 2).

Set $A_r= \{z\in D;\ \rho(0,z)>r\}$, where ρ is the hyperbolic distance. By the properties 1) and 2), we have $\varphi^{-1}(A_r)\supset A_{c^{-1}r}$ for any $r>0$. Since φ is a k-quasiconformal map, we have the following inequality:

$$0 < \frac{1}{k} < \frac{M(A_r)}{M(\varphi^{-1}(A_r))} , \qquad (1)$$

where $M(A)$ denotes the modulus of an annulus A . (See §.6 [4] for the definition of the modulus, and refer to Theorem 7.1 [4] for the inequality (1).)

On the other hand, since $\varphi^{-1}(A_r)\supset A_{c^{-1}r}$, we have:

$$\frac{M(A_r)}{M(\varphi^{-1}(A_r))} \le \frac{M(A_r)}{M(A_{c^{-1}r})} . \qquad (2)$$

As the hyperbolic distance r is equal to the Euclidian distance $(e^r-1)/(e^r+1)$, we have:

$$M(A_r)= \log \frac{e^r+1}{e^r-1} \quad \text{and} \quad M(A_{c^{-1}r})= \log \frac{e^{c^{-1}r}+1}{e^{c^{-1}r}-1} .$$

Hence easily we show the right-hand side of (2) tends to 0 as r tends to infinity. But this contradicts (1). □

Since we can choose the map d such that $d(\tilde{f}\circ\tilde{j}(0))=0$, we may assume $F(0)=0$. Then F has the properties 1) and 2). Hence Lemma implies that f cannot be an expanding immersion. This completes the proof.

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