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塩水発振器のモデリングと系の定性的性質について
MODELLING OF A SALTWATER OSCILLATOR AND THE QUALITATIVE BEHAVIOR

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ABSTRUCT

A mathematical model of the saltwater oscillator is proposed. The model is described by the linear ordinary differential equations for the downward flow mode of the saltwater and the upward flow mode of the water. The dynamic behavior of the system is analyzed employing the eigenvalue, and the amplitude of the oscillation and the averaged increase of the saltwater level are estimated. The experimental result is shown and the period of the oscillation is examined. The switching factor which can change the flow direction is considered from the observation.

1. INTRODUCTION

A saltwater oscillator produces a interesting fluid phenomenon which the flows of water and saltwater automatically change the upward flow and downward flow alternately in a tube having the small diameter. Martin[1] observed the phenomena and constructed a mathematical model with regard to the flow in the tube. His model has two nonlinear terms. One is proportional to the square of the fluid velocity in the tube. The term comes from the fluid resistance caused by the divergent flow at the tube end. Another nonlinear term, which is not differentiable at the point of the flow velocity being zero, comes from the density difference between the saltwater and water. The state equations of the upward and downward flow were shown and the period of the oscillation was estimated. Yoshikawa[2] et al. approximated the nonlinear function of Martin's model to the polynomial function of the flow velocity using the Taylor expansion . And they said that the state equation of the saltwater oscillator had the same form to the Reyleigh's equation.

In the paper, the piece-wise linear model of the oscillator based upon Martin's model is proposed. Although all of the oscillations existing in both the proposed model and Martin's model must have the infinite period because of the property of the equilibrium point, the amplitude of the oscillation and the transient of the averaged level of the saltwater surface in the inside container are analytically estimated. Moreover, experimental results are shown. The switching factor that can change the upward and downward flow alternately is considered from experimental results. The comparisons between the experimental results and the analytical results from the model are made.

2. SALTWATER OSCILLATOR

The saltwater oscillator has a very simple structure as illustrated in Fig.1. The outside container is filled with the water, and the inside container is filled with the saltwater. A tube which has a small diameter is attached at the bottom of the inside container as shown in the figure. The water and saltwater can flow upward and downward respectively through the tube. In our experimental apparatus, the dimensions of the oscillator are followings.

cross sectional area of the outside container	: 600 cm^2
cross sectional area of the inside container	: 25 cm^2
length of the tube	: 0.5 - 5 cm
inner diameter of the tube	: 1 - 3 mm
density of the saltwater	: about 1.2 g/cm^3

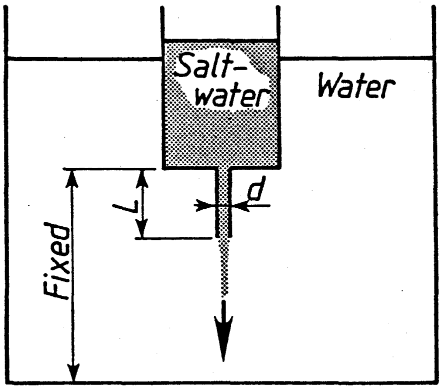

Fig.1 Saltwater Oscillator

In the above specifications of the system, we observe the oscillations of the upward and downward flow change that have the period from 20[sec] to 3[min]. Note that the dimension of the tube affects the period of the oscillation considerably.

In the experiment, we give an adequate height of the saltwater surface in the inside container as an initial condition, for instance as shown in Fig.1, the surface of the saltwater is higher than that of the water in the outside container. Then the saltwater starts to flow downward in the tube and the flow velocity gradually becomes slow. Finally the flow stops when the pressure balance at the tube end is equilibrium state. Then suddenly, the water in the outside container start to flow upward and the flow stops by the pressure balance. In this way, the oscillator continues to change the flow direction alternately.

3. MATHEMATICAL MODEL OF THE SYSTEM

3.1 Fundamental Equation of the Flow in the Narrow Tube

In order to analyze the system, we employ the Navier-Storks equation to the system. The equation is written as the following.

$$\rho\frac{D\mathbf{u}}{Dt} = -\mathrm{grad}\, p + \mu\nabla^2\mathbf{u} + \mathbf{B} \qquad (1)$$

where, ρ: density of fluid, $\mathbf{u}$: flow velocity, p: pressure
μ: coefficient of viscosity, $\mathbf{B}$: body force, t : time

$\frac{D}{Dt}$: materiarl derivative, ∇^2: Laplacian

Here, we must decide where we apply the eq.(1) in the system. By the observation of the phenomena, the main motion of the fluid is limited only in the tube and to the induced flow by the flow in the tube. The other fluid in the containers stagnates. Therefor, we apply the equation to the flow in the tube. In the system behavior, the bi-flow mode occurs under some system conditions as shown in Photo.1. The photograph was taken by the Schlieren method. (See Appendix.) The bi-flow mode is that there exists two stream tubes in the narrow tube and the water and saltwater simultaneously flow upward and downward respectively. Such mode occurs in the case of the ratio d/L being large, where d and L are the inner diameter of the tube and the tube length respectively. In this paper, we pay our attention to the behavior in case of the ratio d/L being small, and assume only one stream tube in the tube and do not consider the bi-flow mode.

In order to analyze the equation to the flow in the tube easily, we assume some conditions for the system simplification when the system model is constructed. The following conditions are assumed to the flow in the tube. The coordinate system is shown in Fig.2 and let (u_r, u_θ, u_z) be the component of the flow velocity $\mathbf{u}$.

(1) The flow is axisymmetric.

(2) The flow directed to coordinate r is zero, that is, $u_r = 0$.

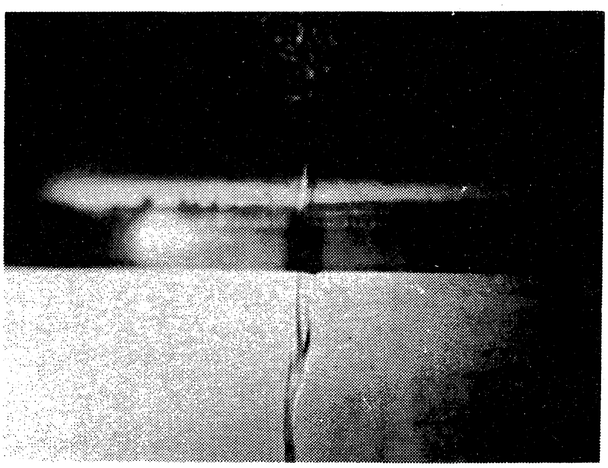

Photo. 1 Bi-flow Mode
d=3mm, L=6mm

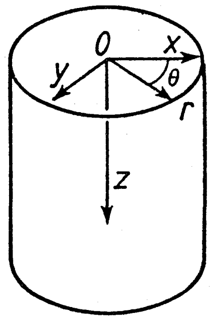

Fig.2 Coordinate System

From the above assumption and the equation of continuity for the incompressible fluid, we easily derive $\partial u_z/\partial z = 0$. Then the eq.(1) becomes the following with regard to u_z.

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \rho g + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \qquad (2)$$

where, g: gravity acceleration

The averaged flow velocity with regard to the cross sectional area of the narrow tube is defined as the following.

$$u \equiv \frac{1}{A}\int_0^{d/2} 2\pi r\, u_z\, dr \qquad (3)$$

where, $A = \pi d^2/4$: cross sectional area of the narrow tube
d: inner diameter of the narrow tube

Then integrating eq.(2) over the volume in the narrow tube, we obtain the following equation.

$$\rho \frac{du}{dt} = -\frac{p(L) - p(0)}{L} + \rho g + \frac{4\mu}{d}\left.\frac{\partial u_z}{\partial r}\right|_{r = d/2} \qquad (4)$$

Since from experimental results the maximum flow velocity in the pipe is about 1 [m/s] and Reynolds number is less than 3000, the Hagen-Poiseuille flow can be assumed to the flow. Then the slope of the velocity distribution at $r=d/2$ is;

$$\left.\frac{\partial u_z}{\partial r}\right|_{r = d/2} = -\frac{8u}{d}\ . \qquad (5)$$

Substituting eq.(5) into eq.(4), we obtain the equation of motion of the fluid in the narrow tube as

$$\rho \frac{du}{dt} = -\frac{p(L) - p(0)}{L} + \rho g - \frac{32\mu}{d^2} u . \tag{6}$$

Martin considered the divergent flow at the tube end and added the pressure loss term which proportional to the square of the flow velocity to eq.(6). Photograph 2 shows the moment of the change of the flow direction. The time interval between each photograph is about 0.35[sec]. Observing the photograph, we obtain that the mushroom shaped flow indicated by the white arrow in the photograph only occurs at the instant of the flow change. Hence we neglect the nonlinear term for the system simplification.

3.2 Modelling of the System

Figure 3 shows the proposed model of the saltwater oscillator. In this model, we assume that the water and saltwater cannot mix each other such as water and oil from the below observations.

Under the condition that both the length L and the diameter d of the tube are adequately small such as d=1[mm] and L=6[mm]. (1) the saltwater going to the outside container downward from the tube gently flows downward without almost spreading and gathers at the bottom of the outside container . (2) the water going to the inside container upward from the tube gently flows upward without almost spreading and gathers at the top of the inside container .

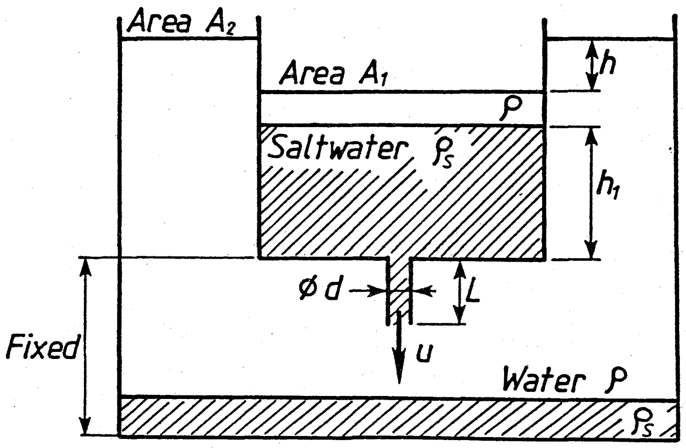

Fig.3 Model of the System

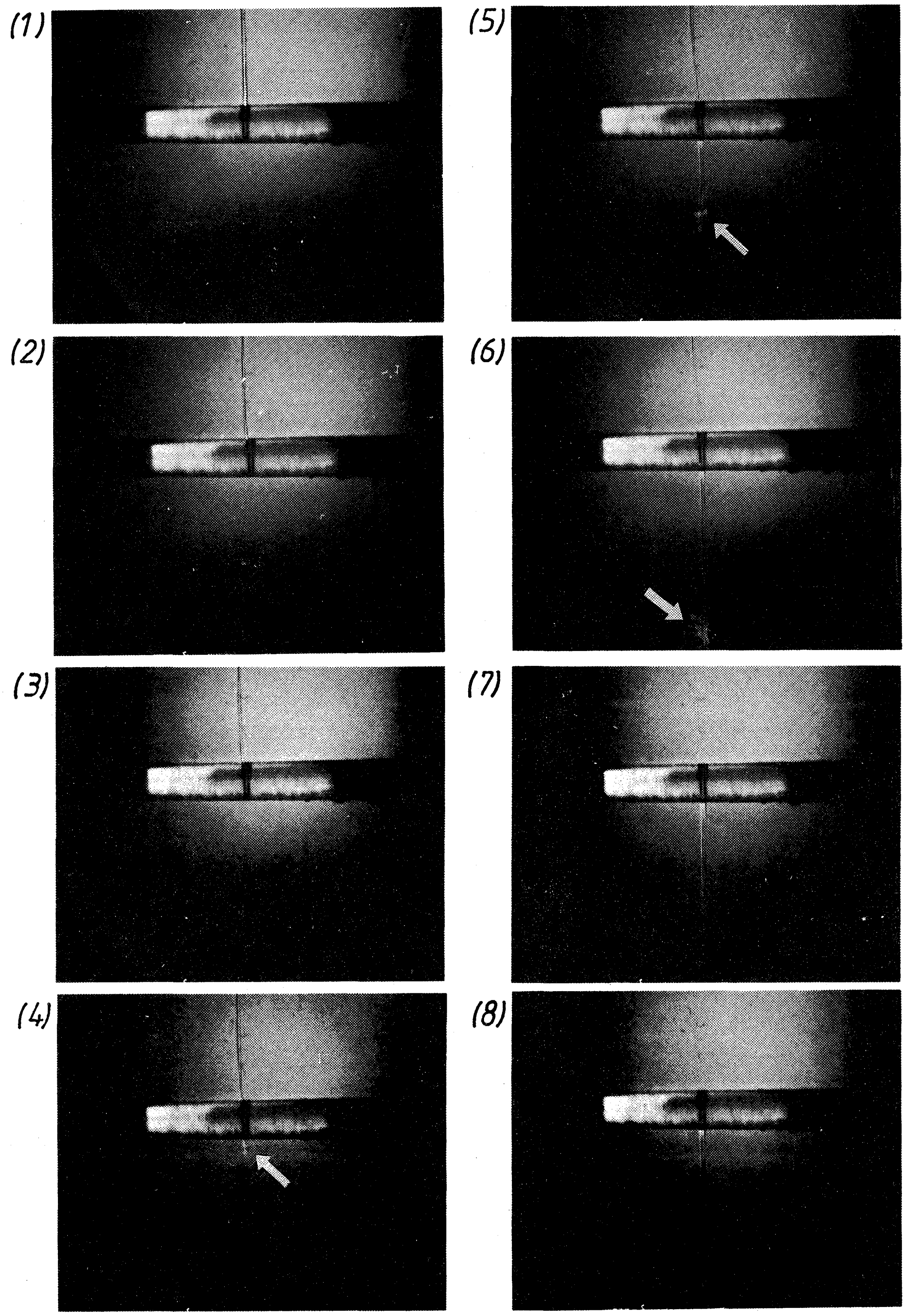

Photo. 2 A Moment of the Flow Direction Change
d=1mm, L=6mm, ρ_s=1.2g/cm^3

Employing the model, the pressure difference p(L)-p(0) in eq.(6) is written as the following.

$$p(L) - p(0) = \rho g(h+h_l+L) - \rho_s g h_l \qquad (7)$$

where, ρ: density of water, ρ_s: density of saltwater

Substituting eq.(7) into eq.(6), and note that the downward and upward flow are respectively the saltwater and water, we obtain the following non-dimensionalized equation.

$$\hat{\rho}_a \frac{d\hat{u}}{d\tau} = (\hat{\rho}_s - \hat{\rho})\hat{h}_l - \hat{h} + (\hat{\rho}_a - \hat{\rho}) - \hat{\rho}_a \hat{u} \qquad (8)$$

where, $\hat{\rho}_a = \hat{\rho}_s$, $(\hat{u} \geq 0)$, $\hat{\rho}_a = \hat{\rho}$, $(\hat{u} < 0)$

$\hat{u} \equiv Ru/g$, $\tau \equiv Rt$, $\hat{\rho}_a \equiv \rho_a/\rho$, $\hat{\rho}_s \equiv \rho_s/\rho$, $\hat{\rho} \equiv \rho/\rho = 1$, $\hat{h}_l \equiv h_l/L$, $\hat{h} \equiv h/L$

$R \equiv 32\mu/\rho_a d^2$

We consider the low of the continuity with regard to the fluid in the containers, we have the following equation.

$$\frac{d\hat{h}}{d\tau} = K\left(\frac{1}{\hat{A}_1} + \frac{1}{\hat{A}_2}\right)\hat{u} \qquad (9)$$

$$\frac{d\hat{h}_l}{d\tau} = \begin{cases} -K(1/\hat{A}_l)\hat{u}, & (\hat{u} \geq 0) \\ 0, & (\hat{u} < 0) \end{cases} \qquad (10)$$

where, $\hat{A}_l \equiv A_l/A$, $\hat{A}_2 \equiv A_2/A$, $K \equiv g/R^2L$

Equations (8), (9) and (10) show the proposed model of the system.

4. ANALYTICAL RESULTS

The proposed model mentioned above is a piece-wise linear in the region of $\hat{u}\geq 0$ and $\hat{u}<0$ respectively. Then we easily obtain the equilibrium set and the eigenvalues of the system, and examine the flow of the solution in the phase space. The equilibrium set and the eigenvalue are the followings.

$$\text{in } \hat{u}\geq 0, \quad \text{equilibrium set (ES1)} \quad \hat{u}=0,\ \hat{h}=(\hat{\rho}_s-1)(\hat{h}_1+1)$$
$$\text{eigenvalues} \quad \lambda=0,\ \frac{-1\pm\sqrt{1-4K(1/\hat{A}_1+1/\hat{\rho}_s\hat{A}_2)}}{2} \tag{11}$$

$$\text{in } \hat{u}\leq 0, \quad \text{equilibrium set (ES2)} \quad \hat{u}=0,\ \hat{h}=(\hat{\rho}_s-1)\hat{h}_1$$
$$\text{eigenvalues} \quad \lambda=0,\ \frac{-1\pm\sqrt{1-4K(1/\hat{A}_1+1/\hat{A}_2)}}{2} \tag{12}$$

In the above, one eigenvalue is always zero. This implies that the state variables of the system are two. Actually, eliminating the unnecessary variable from eq.(8), (9) and (10), we have the following second order ordinary differential equation.

$$\frac{d^2\hat{u}}{d\tau^2}+\frac{d\hat{u}}{d\tau}+K\left(\frac{1}{\hat{A}_1}+\frac{1}{\hat{\rho}_s\hat{A}_2}\right)\hat{u}=0, \qquad (\hat{u}\geq 0)$$
$$\frac{d^2\hat{u}}{d\tau^2}+\frac{d\hat{u}}{d\tau}+K\left(\frac{1}{\hat{A}_1}+\frac{1}{\hat{A}_2}\right)\hat{u}=0, \qquad (\hat{u}<0)$$

In eq.(11) and (12), the diameter of the tube is so small that the ratios $\hat{A}_1$ and $\hat{A}_2$ become more than one thousand. Then the eigenvalues except zero become real and negative, and one is about -1 and the other is very close to zero. Therefore the model is stiff. From this, we have the flow of the solution in the phase space as illustrated in Fig.4. The shaded area in the figure consists of two regions $u\geq 0$ and $u\leq 0$. The initial condition in the shaded area $u\geq 0$ (or $u\leq 0$) contradicts the fact that the upward flow $u<0$ (downward flow $u>0$) exists when the head of the outside container is much greater (less) than that of the inside container. Hence whatever initial conditions we give in the shaded area, the

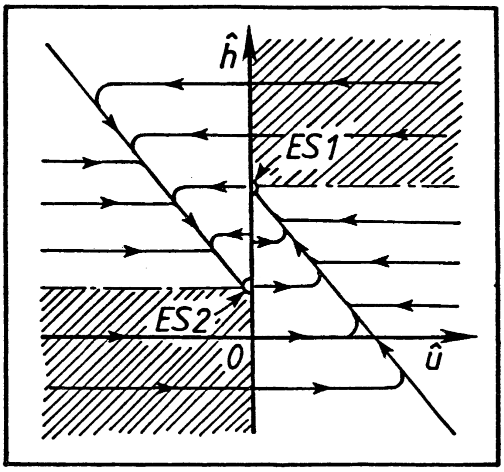

Fig.4 Flows in the Phase Space

model takes the flow direction associated to the head difference between the outside and inside containers. The equilibrium set has the stable nodal type property. Hence all the solutions started in the region $u \geq 0$ (or $u \leq 0$) except in the shaded area converge the equilibrium point in $u \geq 0$ (or $u \leq 0$), and go into the region $u<0$ (or $u>0$) (i.e. the flow direction is changed) after the infinite time passes away. The solutions go and return between two regions repeatedly by spending the eternal time. By the reason the model only produces the oscillation having the infinite period.

From the above analysis, we can estimate the followings.

(1) The total amplitude with regard to the level difference h maximally becomes;

$$\Delta h_{p\text{-}p} = L(\hat{\rho}_s - 1). \qquad (13)$$

(2) The averaged increase of the saltwater surface in the inside container is;

$$\bar{h}_a = L\frac{(\hat{\rho}_s - 1)^2}{\hat{\rho}_s + (\alpha - 1)} \qquad (14)$$

$$\text{where, } \alpha = \frac{\hat{A}_1 + \hat{A}_2}{\hat{A}_2}$$

in one cycle.

(3) The system is stiff, therefore the time constant determined from the eigenvalue which absolute value is smaller governs the period of the oscillation.

Mentioned above, the period of the oscillation employing the proposed model is infinite and not actual. However considering the perturbation observed in the actual system, we can assume that the flow direction change occurs in the neighborhood of the flow velocity being zero. Under the assumption we consider that the amplitude of the oscillation and the averaged increase of the surface in the actual system are approximated by those in the model because of the above (3). In the subsequent section we consider about the switching factor that can change the upward and downward flow alternately and the estimation of the period of the oscillation in the actual system.

5. EXPERIMENTAL RESULTS

5.1 With Regard to the Oscillation

Figure 5 shows the transition of the height of the saltwater in the inside container. The area ratio A_2/A_1 is about 0.04 in our apparatus, therefore the height change of the water in the outside container can be neglected and the wave form in the figure can be considered to indicate the change of the level difference h.

The oscillation can be observed more than one hour, however the amplitude decrease and the period of the oscillation becomes longer as the time passes. The amplitude of the oscillation is from 50% to 80% of the estimated amplitude by eq.(13). With regard to the period, the observed period is from four to five times of the time constant determined by the eigenvalue which absolute value is smaller, except zero eigenvalue, according to the experiment. Figure 6 shows the estimated wave form by the time constant. In the figure we describe the time constant as the non-dimensionalized T_λ and assume that the period has 4.5 T_λ from the experimet. The estimated wave form resembles the experimental wave form, which is also pointed out by Martin.

The other obvious trend of the oscillation is that the averaged saltwater level goes up as the time passes. Although the averaged saltwater level increase estimated by eq.(14) is much greater than the averaged level increase observed in the experiment , the fundamental trend of the increase is qualitatively explained. We consider that it is the reason for the not good concordance of the above level increases that the density of the

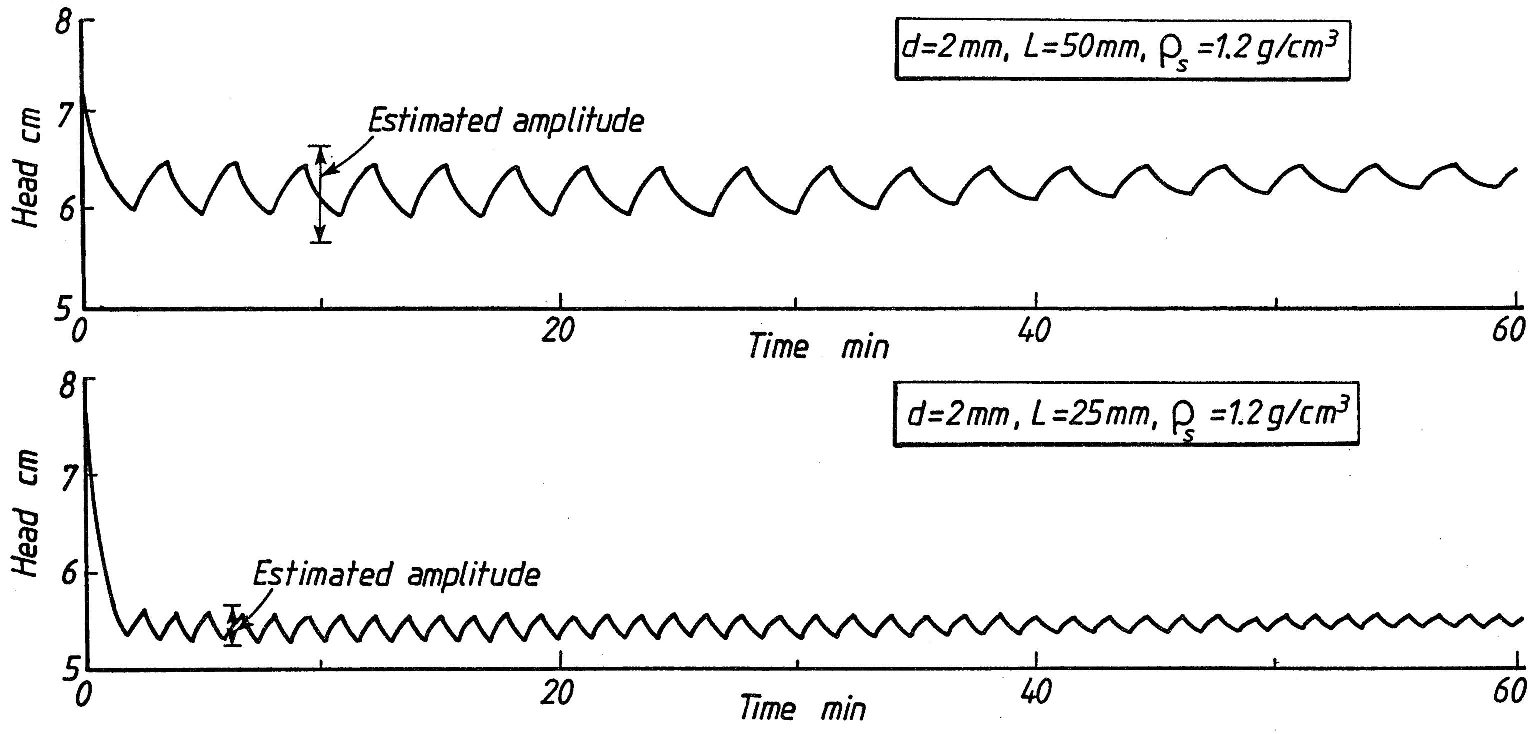

Fig. 5 Observed Level of the Saltwater

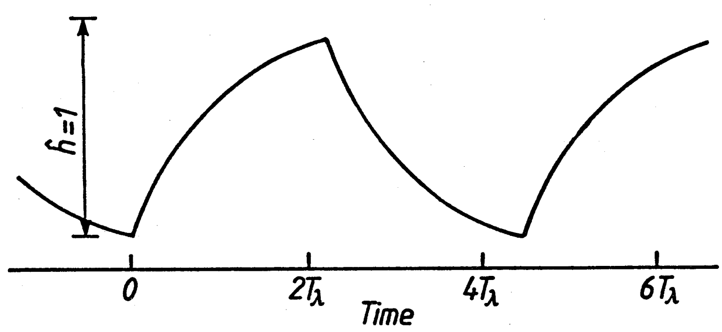

Fig.6 Estimated Wave Form

saltwater in the inside container becomes small because of mixing the water and the saltwater gradually.

5.2 With Regard to the Switching Factor

From the experimental results, observing the upward and downward flow, we notice that the outward flow from the tube exit becomes narrower as the flow velocity becomes slower, and the flow passage is perturbed by itself. The state is sketched in Fig.7. We consider that such flow behavior triggers the change of the flow direction. The model of the above flow switching factor is indispensable for estimating the period in the actual system by the proposed model. We think that the model of the factor can be constructed from the two-dimensional flow at the tube exit or in the flow except in the narrow tube.

In order to construct the model of the switching factor and to estimate the period and the level increase in the actual system, more experiment and observation must be carried out.

6. CONCLUSIONS

The dynamic behavior of the saltwater oscillator is examined using the piece-wise linear model and the experiment. From the result, we have the following.

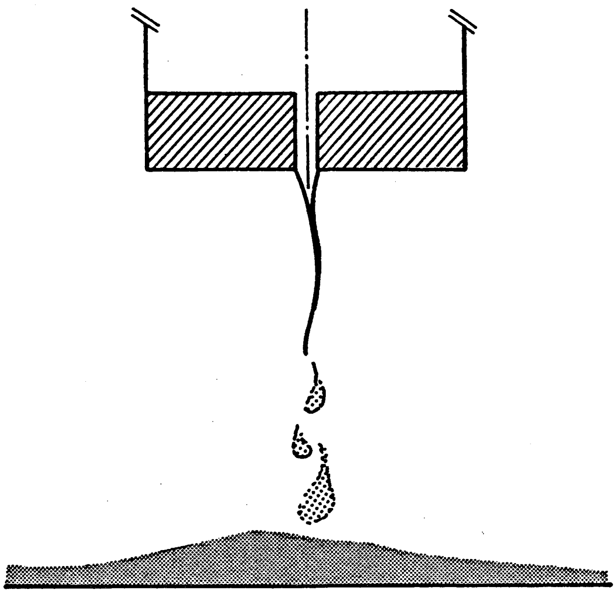

Fig.7 Perturbed Flow Passage

(1) The amplitude of the oscillation of saltwater surface is proportional to the tube length and the density difference between the water and saltwater.
(2) The increase of the averaged saltwater level in the inside container in one cycle is proportional to the tube length and the square of the density difference.
(3) The system is stiff, therefore the downward(upward) flow in the tube continues in the long time.
(4) The proposed model can not determine the period of the oscillation. However, it is obtained from the experiment that the period is from four to five times of the time constant determined by the eigenvalue.
(5) We think that the model of the switching factor can be constructed from the observation of the two-dimensional flow pattern at the tube end.

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APPENDIX

The Schlieren method is one of the flow visualization method using the density difference among the fluids. The principle in detail is written in the technical handbook, for instance Ref.[4]. Our optical setup is illustrated in Fig.A. The picture on the screen is taken using the camera and video camera.

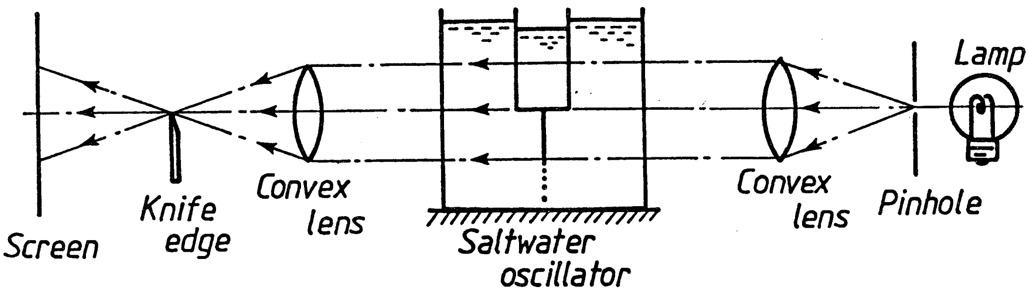

Fig.A Optical Setup