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On special generic maps of simply connected 2n-manifolds into $\mathbb{R}^3$

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Abstract. The purpose of this paper is to study special generic maps into $\mathbb{R}^3$. We prove the congruence formula and equality which show relations between the source manifold and singular point set. As corollaries, we determine the homeomorphism type of the source manifold in four-dimensional case and give an unknotting result for a special generic map S^4 into $\mathbb{R}^3$.

1. Introduction

Let f be a smooth map from n-dimensional manifold M^n into p-dimensional manifold N^p ($n \geq p$). Homological properties of the singular point set of f are one of the most interesting problems in singularity theory. However, most of known results are in mod 2 (e.g. the real Thom polynomial [14], Whitney-Thom-Levine's result on the number of cusp points [7], [14], [15]). We want to know their homological properties in finer forms (i.e. modulo 4, 8, etc) and to evaluate the number of connected components of the singular point set. We restrict ourselves to special generic maps. Since we see that the singular point set consists of only 2-spheres (lemma 6.1), we have the following

Theorem A

Let M^4 be a closed, simply connected 4-dimensional manifold and $f: M^4 \to \mathbb{R}^3$ be a special generic map. Then we have

$$\sigma(M^4) \equiv S(f)\cdot S(f) \pmod{16},$$

where $S(f)$ is the singular point set of f and $\sigma(M^4)$ denotes the signature of M. And $S(f)\cdot S(f)$ stands for the self-intersection number of $S(f)$ in M^4.

Theorem B

Let M be a closed, simply connected 2n-dimensional manifold ($n \geqq 2$). For a spcial generic map $f:M \to \mathbb{R}^3$, we have

$$\chi(M) = 2\#S(f),$$

where $\#S(f)$ denotes the number of connected components of $S(f)$ and $\chi(M)$ is the Eulercharacteristic of M.

As corollaries, we determine the homeomorphism type of the source manifold in 4-dimensional case and show in section 6 that the set of singular points of special generic maps over S^4 into $\mathbb{R}^3$ is unknotted.

In a more generalized setting, we have the following congruence formula for a stable map

Theorem C

Let M^4 be a closed, oriented 4-dimensional manifold with $H_1(M^4;\mathbb{Z})=0$ and $f:M^4 \to \mathbb{R}^3$ be a stable map. Then we have

$$\sigma(M^4) \equiv -S(f)\cdot S(f) \pmod 4,$$

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2. Euler characteristics of the source manifold and singular point sets

In this section we recall Fukuda's results on the relations between the sour cemanifold and set of singular points when the map has only fold singularity. At the end of this section we will study the unorientability of the singular point set $S(f)$ of a map which has only folds. Let $f:M^n \to \mathbb{R}^p$ $(n \geqq p)$ be such a smooth map. If $p \in S(f)$, then we can choose local coordinates $(x_1, x_2, \ldots, x_n)$ centered at p and $(y_1, y_2, \ldots, y_p)$ centered at $f(p)$ so that f has the following nomal forms

$$y_i = x_i \qquad (1 \leqq i \leqq p-1)$$

$$y_p = \pm x_p^2 \ldots \pm x_n^2.$$

Then the Jacobian matrix at p is

$$J_f(p) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ 0 & \cdots & 0 & \pm 2x_p & \cdots & \pm 2x_n \end{pmatrix}.$$

Hence $S(f) = \{x_p = \ldots = x_n = 0\}$, and rank $J_f(p) = p-1$ and $S(f)$ ia a $p-1$ dimensional manifold. Furthermore, the restricted map $f|S(f)$ is a smooth immersion.

If a smooth map $f:M \to \mathbb{R}^p$ $(n \geqq p)$ admits only definite fold points, such a map is called special generic (This terminology is originally due to [2]).

Now we recall Fukuda's results in [4]. Let $A_k(f)$ be the set of A_k-type singularity $(1 \leqq k \leqq p)$ for a smooth map $f: M^n \to \mathbb{R}^p$(See [9], in which A_k-type singularities are referred as $\Sigma^{n-p+1,1,\ldots 1,0}$ in the language of the Thom-Boardman symbols).

Lemma 2.1 ([4])

Let $f: M^n \to \mathbb{R}^p (n \geq p)$ be a smooth map which has only A_k-type singularities $(1 \leq k \leq p-1)$. Then we have

$$\chi(M^n) \equiv \Sigma \chi(\overline{A_k(f)}) \pmod 2,$$

where $\overline{A_k(f)}$ is the topological closure of $A_k(f)$.

In particullar, if f has only folds (A_1-type), then the Euler characteristic of M^n has the same parity a that of the singular point set $S(f)$.

Definition 2.2

Suppose that n-p+1 is even. For a smooth map $f: M^n \to \mathbb{R}^p (n \geq p)$ which admits only fold singularity, such a point p is called a fold point with index λ (mod 2) if f has the following normal form using local coordinates at p and f(p)

$$y_i = x_i \qquad (1 \leq i \leq p-1)$$

$$y_p = -x_p^2 \ldots -x_{p+\lambda-1} + x_{p+\lambda}^2 + \ldots + x_n^2.$$

We set

$$S^+(f) = \{p \in S(f);\ \text{index } \lambda \text{ is even}\}$$

$$S^-(f) = \{p \in S(f);\ \text{index } \lambda \text{ is odd}\}.$$

These two sets are clearly well-defined for being n-p+1 being even.

Lemma 2.3 ([4])

Let $f: M^n \to \mathbb{R}^p (n \geq p, n-p+1 : \text{even})$ be a smooth map which has only folds. Then we have

$$\chi(M^n) = \chi(S^+(f)) - \chi(S^-(f)).$$

Remark 2.4

When $f: M^n \to \mathbb{R}^3$ has only folds, lemma 2.1 says if the Euler characteristic of M^n is odd, then the singularpoint set $S(f)$ contains unorientable surfaces with odd genus.

Lemma 2.3 plays a fundamental role in proof of Theorem B stated in section6.

We end this section by generalizing this remark.

Proposition 2.5

Let $f:M^n \to \mathbb{R}^p$ ($n \geq p \geq 3$) be a smooth map which admits only folds. If $\chi(M^n)$ is odd, then $S(f)$ is unorientable.

proof. As usual we define the normal bundle of the immersion $f|S(f)$ by the exactness of

$$0 \to \nu(f) \to \tau(S(f)) \to f^*\tau(\mathbb{R}^p) \to 0,$$

where $\tau(S(f))$ is the tangent bundle of $S(f)$ and $f^*\tau(\mathbb{R}^p)$ the induced bundle. Since $S(f)$ is a p-1 dimensional manifold, the normal bundle $\nu(f)$ is a line bundle over $S(f)$. Then we set $w(\nu(f)) = 1 + \hat{\alpha}$, where $w(\nu(f))$ is the total Stiefel-Whitney class and $\hat{\alpha} \in H^1(S(f);\mathbb{Z}/2)$. We then have

$$\tau(S(f)) \oplus \nu(f) \cong f^*\tau(\mathbb{R}^p).$$

Note that $f^*\tau(\mathbb{R}^p)$ is trivial. This implies

$$w(S(f))\ w(\nu(f)) = w(f^*\tau(\mathbb{R}^p)) = 1.$$

Thus we have

$$w(S(f)) = 1 + \hat{\alpha} + \hat{\alpha}^2 + \cdots + \hat{\alpha}^{p-1},$$

where the powers are cup products.

Hence we have $w_1(\nu(f)) = \hat{\alpha} = w_1(S(f))$. Using Poincare-Hopf theorem modulo 2 and applying lemma 2.1, we have

$$\begin{aligned} \chi(M^n) &\equiv \chi(S(f)) \pmod 2 \qquad \text{(lemma 2.1)} \\ &\equiv \langle w_{p-1}(S(f)), [S(f)]_2 \rangle \pmod 2 \\ &\equiv \langle \alpha^{p-1}, [S(f)]_2 \rangle \pmod 2 \\ &\equiv \langle (w_1(S(f)))^{p-1}, [S(f)]_2 \rangle \pmod 2. \end{aligned}$$

The assumption that $\chi(M^n)$ be odd implies $w_1(S(f))$ is non-trivial, which means that $S(f)$ is unorientable. This completes the proof.

3. Proof of Theorem C

Let M be a closed n-dimensional manifold and $f:M \to \mathbb{R}^3$ be a stable map. If $p \in S(f)$, then there exist local coordinates $(x, y, z_1, \cdots, z_{n-2})$ and (y_1, y_2, y_3) centered at p and $f(p)$ respectively such that f has the following normal forms:

1) $(x, y, z_1, \cdots, z_{n-2}) \longrightarrow (x, y, \pm z_1^2 \cdots \pm z_{n-2}^2)$, fold
2) $(x, y, z_1, \cdots, z_{n-2}) \longrightarrow (x, y, z_1^3 + xy \pm z_2^2 \cdots \pm z_{n-2}^2)$, cusp
3) $(x, y, z_1, \cdots, z_{n-2}) \longrightarrow (x, y, z_1^4 + xy^2 + xy \pm z_2^2 \cdots \pm z_{n-2}^2)$, swallow tail

In what follows, we will investigate the relation between the self-intersection number of $S(f)$ in M^4 and signature of M^4.

In this section we prove the following Theorem C.

Theorem C

Let M^4 be a closed, oriented 4-dimensional manifold with $H_1(M^4;\mathbb{Z})=0$ and $f:M^4 \to \mathbb{R}^3$ be a stable map. Then we have

$$\sigma(M^4) \equiv -S(f)\cdot S(f) \pmod 4,$$

Lemma 3.1

For a stable map $f: M^4 \to \mathbb{R}^3$ as above, we have

$$\chi(M^4) \equiv \chi(S(f)) \pmod 2.$$

proof. By lemma 2.1 we have

$$\chi(M^4) \equiv \chi(\overline{A_1(f)}) + \chi(\overline{A_2(f)}) + \#A_3(f) \pmod 2, \qquad (*)$$

where $\#A_3(f)$ denotes the number of A_3-type (swallow tail) singular points. Since $\overline{A_2(f)}$ is a union of circles, we have

$$\chi(\overline{A_2(f)}) = 0. \qquad (**)$$

According to Ando [1], the Thom polynomial of $\overline{A_3(f)}$ is $w_1^4 + w_1 w_3$. Hence we have

$$\#A_3(f) \equiv \langle w_1^4 + w_1 w_3, [M^4]_2 \rangle \pmod 2 \qquad (***)$$

Since M^4 is oriented, $w_1 = 0$. Therefore $\#A_3(f) \equiv 0 \pmod 2$. Since $\overline{A_1(f)}$ is $S(f)$, the conclusion follows from (*), (**) and (***).

Definition 3.2

A closed 2-dimensional submanifold F of M is called a characteristic surface of M if the mod 2 cycle $[F]_2 \in H_2(M;\mathbb{Z}/2)$ is Poincare dual to the 2-nd Stiefel-Whitney class $w_2(M) \in H^2(M;\mathbb{Z}/2)$.

The following lemma was first given by Rochlin [13] and fully proved in a generalized form by Guillou and Marin [5].

Lemma 3.3 ([5], [13])

Let M be a closed, oriented 4-dimensional manifold with $H_1(M;\mathbb{Z})=0$ and F be a characteristic surface of M. Then we have

$$\sigma(M) \equiv F\cdot F + 2\chi(F) \pmod 4.$$

Lemma 3.4 ([14])

Let $f:M^4 \to \mathbb{R}^3$ be a stable map. Then S(f) is a mod 2 cycle of M^4 and its Poincare dual class $[S(f)]_2^* \in H^2(M^4;\mathbb{Z}/2)$ coincides with the 2-nd Stiefel-Whitney class $w_2(M^4)$.

(proof of Theorem C)

Let $f: M^4 \to \mathbb{R}^3$ be a stable map. From lemma 3.4 S(f) is a characteristic surface of M^4. Then from lemma 3.3 we have

$$\sigma(M^4) \equiv S(f)\cdot S(f) + 2\chi(S(f)) \pmod 4. \qquad (1)$$

As we will see later, we have

$$\sigma(M^4) \equiv \chi(S(f)) \pmod 2. \qquad (2)$$

Hence

$$2\sigma(M^4) \equiv 2\chi(S(f)) \pmod 4. \qquad (3)$$

Combining (1) and (3), we obtain the required result

$$\sigma(M^4) \equiv -S(f)\cdot S(f) \pmod 4.$$

We have the above congruence (2) as follows.
We decompose $H^2(M^4;\mathbb{Q})$ into the positive eigen space H^+ and the negative eigen space H^- of the symmetric bilinear form defining the signature of M^4:

$$H^2(M^4;\mathbb{Q}) = H^+ \oplus H^-.$$

Then we have

$$\begin{aligned}\sigma(M^4) = \dim H^+ - \dim H^- &\equiv \dim H^+ + \dim H^- \pmod 2\\ &= \text{2-nd betti number of } M^4\\ &\equiv \chi(M^4) \pmod 2\\ &\equiv \chi(S(f)) \pmod 2,\end{aligned}$$

where the last congruence follows from lemma 3.1. This completes the proof of Theorem C.

The above congruence (2) implies

Corollary 3.5

Let M^4 be an oriented 4-dimensional manifold and $f:M^4 \to \mathbb{R}^3$ be a stable map. If the signature of M^4 is odd, then $S(f)$ contains unorientable surface with odd genus.

4. Proof of Theorem A

In this section we prove the following

Theorem 4.1

Let M^4 be a closed, oriented 4-manifold and N^3 be an oriented 3-manifold. If $f:M^4 \to N^3$ is a stable map whose singular point set is a union of 2-spheres, then we have

$$\sigma(M^4) \equiv S(f)\cdot S(f) \pmod{16}.$$

As we will see later in section 6, for a special generic map over simply connected 4-manifold M^4 into $\mathbb{R}^3$ the singular point set is a disjoint union of 2-spheres. Therefore Theorem 4.1 implies Theorem B.

Lemma 4.2 ([14])

Let M^4 be a closed, oriented 4-manifold and N^3 be an oriented 3-manifold. For a stable map $f:M^4 \to N^3$, the dual class $[S(f)]_2^*$ coincides wih the 2-nd Stiefel-Whitney class $w_2(M^4)$.

proof) Since any oriented 3manifold is parallelizable, $w_j(N^3) = 0$ $(1 \leqq j \leqq 3)$. Hence $f^*w_j(N^3)$ do not appear in the Thom polynomial $P(\overline{\Sigma^{2,0}}) = P(w_i(M^4, f^*w_j(N^3))$ Therefore the same conclusion as in lemma 3.4 follows.

(proof of Theorem B)

The method of the proof is similar to [6]. First fix an orientation of M^4. We assume that $S(f)$ has k connected components and set $S(f) = S_1 \cup \cdots \cup S_k$. Moreover we set

$$n_i = S_i \cdot S_i \geqq 0 \qquad (1 \leqq i \leqq p)$$

$$m_j = S_j \cdot S_j < 0 \qquad (p+1 \leqq j \leqq k),$$

where $S_x \cdot S_x$ is the self-intersection number of S_x for $1 \leqq x \leqq k$ in M^4.
We construct a spin manifold M_k by surgering the singular point set out and by induction on i ad j.

As the first step we construct a manifold $\widetilde{M}_1$ such that $w_2(\widetilde{M}_1) = [S_2 \cup \cdots S_k]_2^* \in H^2(\widetilde{M}_1;\mathbb{Z}/2)$ and that $\sigma(\widetilde{M}_1) = \sigma(M^4) - S_1 \cdot S_1$. Let $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ be the complex projective plane and the one with the opposite orientation, respectively. Then $\mathbb{C}P_i^1 \subseteq \overline{\mathbb{C}P_i^2}$ $(1 \leqq i \leqq n_1+1)$ and $[\mathbb{C}P_i^1] = w_2(\overline{\mathbb{C}P_i^2})$. Set $M_1 = M^4 \# \overline{\mathbb{C}P_1^2} \# \cdots \# \overline{\mathbb{C}P_{n_1+1}^2}$. We construct $\widetilde{M}_1$ from M_1 as follows.

Consider the connected sum $S_1 \# \mathbb{C}P_1{}^1 \# \cdots \# \mathbb{C}P_{n+1}$ in M_1. Set $\widetilde{S}_1 = S_1 \# \mathbb{C}P_1{}^1 \# \cdots \# \mathbb{C}P^1_{n_1+1}$. Then S_1 is a smoothly embedded 2-sphere in M_1. Let $\xi \in H_2(M^4;\mathbb{Z})$ be the homology class represented by S_1 and $\eta_i \in H_2(\overline{\mathbb{C}P_i{}^2};\mathbb{Z})$ $(1 \leqq i \leqq n_i+1)$ the homology class represented by $\mathbb{C}P_i{}^1$, respectively. Then the homology class $\zeta = \xi + \Sigma\eta_i \in H_2(M_1;\mathbb{Z})$ can be represented by $\widetilde{S}_1$, using the natural isomorphism

$$H_2(M^4;\mathbb{Z}) \oplus H_2(\overline{\mathbb{C}P_1{}^2};\mathbb{Z}) \oplus \cdots \oplus H_2(\overline{\mathbb{C}P^2_{n_1+1}};\mathbb{Z}) \cong H_2(M_1;\mathbb{Z}).$$

The self-intersection number of S_1 in M_1 is

$$\widetilde{S}_1 \cdot \widetilde{S}_1 = \xi\cdot\xi + \Sigma\eta_i\cdot\eta_i = n_i - (n_i+1) = -1.$$

Hence the tubular neighborhood of $\widetilde{S}_1$ in M_1 is the D^2-bundle over $\widetilde{S}_1$ with Euler number $-1 \in \pi_1(SO(2))$, which is denoted by $N(\widetilde{S}_1)$. Then $\partial N(\widetilde{S}_1)$ is the (-1)-Hopf bundle and diffeomorphic to S^3. We now set $\widetilde{M}_1 = (M_1 - \mathrm{Int}N(\widetilde{S}_1)) \cup_\partial D^4$. Note that

$$\begin{aligned} \widetilde{M}_1 \# \overline{\mathbb{C}P^2} &= (\widetilde{M}_1 - \mathrm{Int}D^4) \cup_\partial (\overline{\mathbb{C}P^2} - \mathrm{Int}D^4) \\ &= (M_1 - \mathrm{Int}N(\widetilde{S}_1)) \cup_{id} N(\widetilde{S}_1) = M_1 \qquad (*) \\ &= M^4 \# \overline{\mathbb{C}P_1{}^2} \# \cdots \# \overline{\mathbb{C}P^2_{n_1+1}}. \end{aligned}$$

From the above construction we see

<u>Lemma 4.2</u>

$S_1 \cup \mathbb{C}P_1{}^1 \cup \cdots \cup \mathbb{C}P_{n+1}$ $(\subset M_1)$ lies in $N(\widetilde{S}_1) = \overline{\mathbb{C}P^2} - \mathrm{Int}D^4$ of the decomposition (*) of $M_1 = \widetilde{M}_1 \# \overline{\mathbb{C}P^2}$.

This lemma will be used at the end of this section.

The additivity of the signature implies

$$\sigma(\widetilde{M}_1) - 1 = \sigma(M^4) - (n_1+1).$$

Hence we have

$$\sigma(\widetilde{M}_1) = \sigma(M^4) - S_1\cdot S_1. \qquad (X_1)$$

Moreover, as we will see later, we have

$$w_2(\widetilde{M}_1) = [S_2 \cup \cdots \cup S_k]_2{}^* \in H^2(\widetilde{M}_1;\mathbb{Z}/2). \qquad (W_1)$$

This completes the first step of our induction.

Next for i= 2,⋯,p we can construct $\widetilde{M}_i$ and M_i from $\widetilde{M}_{i-1}$ inductively in the same way such that

$$\sigma(\widetilde{M}_i) = \sigma(\widetilde{M}_{i-1}) - n_i = \sigma(M^4) - \Sigma S_t \cdot S_t. \qquad (X_i)$$

$$w_2(\widetilde{M}_i) = [S_{i+1} \cup \cdots \cup S_k]_2^* \in H^2(\widetilde{M}_i;\mathbb{Z}/2). \qquad (W_i)$$

Hence we have

$$\sigma(\widetilde{M}_p) = \sigma(M^4) - (n_1+\cdots+n_p) = \sigma(M^4) - \Sigma S_i \cdot S_i. \qquad (X_p)$$

$$w_2(\widetilde{M}_p) = [S_{p+1} \cup \cdots \cup S_k]_2^* \in H^2(\widetilde{M}_p;\mathbb{Z}/2). \qquad (W_p)$$

Next for j = p+1,⋯,k we will make similar process s above.
Let $M_{p+1} = M_p \# \mathbb{C}P_1{}^2 \# \cdots \# \mathbb{C}P_m{}^2$, where $m_1 = |m_{p+1}|+1$ and consider the connected sum $\widetilde{S}_{p+1} = S_{p+1} \# \mathbb{C}P_1{}^1 \# \cdots \# \mathbb{C}P_{m_1}{}^1$. Then $\widetilde{S}_{p+1}$ is also a smoothly embedded 2-sphere with self intersection number +1 in M_{p+1}. Then we set

$$\widetilde{M}_{p+1} = (M_{p+1} - \mathrm{Int}N(\widetilde{S}_{p+1})) \cup_\partial D^4.$$

We see

$$\widetilde{M}_{p+1} \# \mathbb{C}P^2 = M_{p+1} = \widetilde{M}_p \# \mathbb{C}P_1{}^2 \# \cdots \# \mathbb{C}P_{m_1}{}^2.$$

Moreover, we see in the same way as (X_i)

$$\sigma(\widetilde{M}_{p+1}) = \sigma(\widetilde{M}_p) + |m_{p+1}| = \sigma(\widetilde{M}_p) - \Sigma S_{p+1} \cdot S_{p+1}. \qquad (X_{p+1})$$

$$w_2(\widetilde{M}_{p+1}) = [S_{p+2} \cup \cdots \cup S_k]_2^* \in H^2(\widetilde{M}_{p+1};\mathbb{Z}/2). \qquad (W_{p+1})$$

Repeating the same constructions until surgering out all the 2-spheres as an obstruction of a spin structure, we have

$$\begin{aligned} \sigma(\widetilde{M}_k) = \sigma(\widetilde{M}_{k-1}) - m_k = \cdots &= \sigma(M^4) - \Sigma n_i - \Sigma m_j \\ &= \sigma(M^4) - S(f) \cdot S(f), \end{aligned} \qquad (X_k)$$

$$w_2(\widetilde{M}_k) = 0. \qquad (W_k)$$

Hence M_k is spin. From the clasical Rochlin's theorem [11], $\sigma(\widetilde{M}_k) \equiv 0 \pmod{16}$. Thus from (X_k) we have the required result.

(proof of (W_i))

First we prove (W_1). According to Wu's formula([10],p.136), on a closed, oriented smooth 4-manifold, w_2 is characterized by $w_2 \cup v = v \cup v$ for any $v \in H^2(M;\mathbb{Z}/2)$ So it is sufficient to show that $[S_2 \cup \cdots \cup S_k]_2{}^* \cup v = v \cup v$ for all $v \in H^2(M;\mathbb{Z}/2)$. Equivalently, by the Poincare duality, it suffices to show $[S_2 \cup \cdots \cup S_k]_2 \cup y = y \cup y$ (mod 2) for all $y \in H_2(M;\mathbb{Z}/2)$. From lemma 4.2 we have

$$[S(f)]_2 \cdot x = x \cdot x \pmod 2 \quad \text{for all } x \in H_2(\widetilde{M}_1;\mathbb{Z}/2).$$

We set $[F] = [S_2 \cup \cdots \cup S_k]_2$. We have the follwing isomorphism.

$$H_2(\widetilde{M}_1) \oplus H_2(\overline{\mathbb{CP}^2}) \cong H_2(M^4) \oplus H_2(\mathbb{CP}_1{}^2) \oplus \cdots \oplus H_2(\mathbb{CP}_{n+1}).$$

Then every element $y \in H_2(\widetilde{M}_1)$ has the form

$$y \equiv x + a_1 v_1 + \cdots + a_m v_m \pmod 2,$$

where $x \in H_2(M^4)$, $v_i \in H_2(\overline{\mathbb{CP}_i{}^2})$ and $m = n_1 + 1$.

Since $(S_2 \cup \cdots \cup S_k) \cap (CP_1{}^1 \cup \cdots \cup CP_m{}^1) = \phi$, we see that $[F] \cdot v_i = 0$ for $i = 1, \cdots, m$. Hence we have

$$[F] \cdot y = [F] \cdot x = [S(f)]_2 \cdot x - x \cdot x . \tag{1}$$

On the other hand,

$$\begin{aligned} [S_1] \cdot x + a_1 + \cdots + a_m &= [S_1] \cdot x + a_1 v_1 \cdot v_1 + \cdots + a_m v_m \cdot v_m \\ &= [S_1] \cdot x + v_1 \cdot a_1 v_1 + \cdots + v_m \cdot a_m v_m \\ &= ([S_1] + v_1 + \cdots + v_m) \cdot y \\ &= 0, \end{aligned} \tag{2}$$

where note that $v_i \cdot x = 0$ and $v_i \cdot v_j = 0$ $(i \neq j)$, since $S_1 \cap \mathbb{CP}_i{}^1 = \phi$ and $\mathbb{CP}_i{}^1 \cap \mathbb{CP}_j{}^1 = \phi$ for $i \neq j$.

The last equality in (2) can be seen as follows: From lemma 4.3 we see $([S_1] + v_1 + \cdots + v_m) \in H_2(\overline{\mathbb{CP}^2};\mathbb{Z}/2) \subset H_2(\widetilde{M}_1;\mathbb{Z}/2) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z}/2) \cong H_2(M_1;\mathbb{Z}/2)$.

On the other hand, $y \in H_2(\widetilde{M}_1;\mathbb{Z}/2) \subset H_2(\widetilde{M}_1;\mathbb{Z}/2) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z}/2) \cong H_2(M_1;\mathbb{Z}/2)$.

Thus $([S_1] + v_1 + \cdots + v_m) \cdot y = 0$.

Moreover, we get

$$a_1^2+ \cdots + a_m^2 = a_1+ \cdots +a_m +a_1(a_1-1)+ \cdots +a_m(a_m-1)$$

$$\equiv a_1+ \cdots +a_m \pmod 2. \qquad (3)$$

Therefore, from (1), (2) and (3) we have

$$[F]\cdot y \equiv [S(f)]_2\cdot x + a_1^2+ \cdots + a_m^2$$

$$\equiv x\cdot x + a_1v_1\cdot a_1v_1 + \cdots + a_mv_m\cdot a_mv_m$$

$$\equiv y\cdot y \pmod 2.$$

Thus from the haracterization of w_2, we have $[F]^*= [S_2\cup\cdots\cup S_k]_2^*=w_2(\widetilde{M}_1)$. This completes the proof of (W_1).

In the same way we can prove (W_i) for $i=2, \cdots ,k$.

This completes the proof of Theorem B.

5. Special generic maps and their Stein factorization

Let $f:M \to \mathbb{R}^p$ be a stable map. It induces on M an equivalence relation, that is: $x\sim x'$ if and only if $f(x) = f(x') = y$ and x,x' belong to the same connected component of $f^{-1}(y)$. We denote the natural projection by $q:M \to M/\sim = W_f$ and let $q':W_f \to \mathbb{R}^p$ be the map defined by $q'\circ q$. This factorization of f is known in algebraic geometry as the Stein factorization.

In what follows, we restrict ourselves to the special generic map into $\mathbb{R}^3$.

<u>Lemma 5.1</u>

For a special generic map $f:M \to \mathbb{R}^3$, W_f is a compact 3-manifold with boundary such that ∂W_f is homeomorphic to $S(f)$ and q' ia an immersion, hence $S(f)$ is orientable.

proof) Recall that if $p \in S(f)$, then there exist local coordinates $(x_1, \cdots, x_n)$ and (y_1, y_2, y_3) centered at p and f(p) respectively, such that f is given by the following normal form

$$y_i = x_i \qquad (i=1,2)$$
$$y_3 = x_3^2 + \cdots + x_n^2.$$

Then we choose open ε-ball neighborhood U centered at p of S(f) in M, $U = \{x_3^2 + \cdots + x_n^2 < \varepsilon^2\}$. Then f sends to $V = \{y_1^2 + y_2^2 + y_3 < \varepsilon^2, y_3 \geqq 0\}$.
In particullar, the open 2-disk $\{x_3 = \cdots = x_n = 0\}$, which is coordinate nighborhood of p in S(f)corresponds homeoorphically to $\{y_1^2 + y_2^2 < \varepsilon^2\}$. From the definition of the Stein factorization, q(U) is homeomorphic to V. We denote this homeomorphism by ψ_U. Then $\{q(U), \psi_U\}$ is a chart of W_f and hence W_f ia a 3-manifold with boundary. Evidently, q(S(f)) is homeomorphic to ∂W_f and q' is an immersion.

Remark 5.2

It is easy to see from the normal form that the quotient map $q: M \to W_f$ induces the surjective homomorphism $q_*: \pi_1(M) \to \pi_1(W_f)$.

6. Proof of Teorem B

In this section we prove the following equality

Theorem B

Let M b a closed, simply connected 2n-dimensional manifold ($n \geqq 2$). For a special generic map $f: M \to \mathbb{R}^3$, we have

$$\chi(M) = 2\#S(f),$$

where #S(f) denotes the number of connected components of S(f).

This theorem is an immediate concluion combining the following lemma 6.1 and lemma 2.3.

Lemma 6.1

Let M b a closed, simply connected 2n-dimensional manifold ($n \geqq 2$). For a special generic map $f:M \to \mathbb{R}^3$, $S(f)$ consists of only 2-spheres.

(proof of Theorem C)

From lemma 2.3 we have the following equality for a special generic map $f:M \to \mathbb{R}^3$, since $S^+(f) = S(f)$ and $S^-(f) = \phi$.

$$\chi(M) = \chi(S(f)).$$

Then by the above lemma, we have

$$\chi(S(f)) = 2\#S(f).$$

This completes the proof.

(proof of lemma 6.1)

Since $q^*:\pi_1(M) \to \pi_1(W_f)$ is surjective and M is simply connected , W_f is also simply connected. Hence $H_1(W_f;\mathbb{Z}) = 0$ and $H^1(W_f;\mathbb{Z}) = 0$. Consider the homology exact sequence of the pair $(W_f, \partial W_f)$

$$\cdots \longrightarrow H_2(W_f,\partial W_f;\mathbb{Z}) \longrightarrow H_1(\partial W_f;\mathbb{Z}) \longrightarrow H_1(W_f;\mathbb{Z}) \longrightarrow \cdots$$

From the Poncare-Lefschetz duality,

$$H_2(W_f,\partial W_f;\mathbb{Z}) \sim H_1(W_f;\mathbb{Z}) = 0.$$

Therefore we have

$$H_1(\partial W_f;\mathbb{Z}) = 0.$$

By the classification of 2-manifolds, ∂W_f consists of only 2-spheres. Hence by lemma 6.1 S(f) is a union of 2-spheres. This completes the proof.

As stated in the introduction, we obtain the following corollary.

Corollary 6.3

Let M^4 be a closed, oriented, simply connected 4-manifold. If M^4 admits a special generic map $f:M^4 \to \mathbb{R}^3$ such that $S(f)$ is connected, then M^4 is homeomorphic to 4-sphere.

proof) By Theorem B we have $\chi(M^4)=2$. Since M^4 is simply connected, M^4 is a homotopy 4-sphere. The conclusion follows from [3].

Corollary 6.4

Let M be a closed, simply connected 2n-manifold ($n \geqq 2$). If the Euler characteristic of M is odd, then there exist no special generic maps over M into $\mathbb{R}^3$. For example, $\mathbb{C}P^2$ admits no special generic maps into $\mathbb{R}^3$.

Corollary 6.5

For a special generic map $f:S^4 \to \mathbb{R}^3$, $S(f)$ is unknot.

proof) Under the assumption $S(f)$ is a 2-sphere smoothly embedded in S^4. If W_f is diffeomorphic to D^3, we define the composite map

$$(S^4, S(f)) \xrightarrow{q} (W_f, \partial W_f) \xrightarrow{\psi} (D^3, S^2) \xrightarrow{h} \mathbb{R},$$

where ψ is a diffemorphism and h is a height function. We set $\rho = h \circ \psi \circ q$. Then $\rho | S(f)$ has only two critical points. On the other hand, the Poincare conjecture is still open. However, by Poenaru [16] (p.484)

$$S^2 = \partial\Delta^3 \longrightarrow \partial(\Delta^3 X D^2) = S^4$$

is smoothly unknotted. This completes the proof.

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