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Torelli theorem for certain rational surfaces and root system of type A

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For an integer $n \geqslant 2$, let Σ_n be the $n-$th Hirzebruch surface defined by

$$\{(\zeta_0 : \zeta_1 : \zeta_2)(s : t) \in \mathbf{P}^2 \times \mathbf{P}^1 | s^n\zeta_0 = t^n\zeta_1\}, \tag{0.1}$$

where $\mathbf{P}^k$ is n-dimensional complex projective space. Let X_n be a surface obtained by blowing up $n+1$ points of Σ_n and D be an anti-canonical divisor on X_n such that D consists of four nonsingular rational curves and its intersection diagram is a circle (thus D forms a square).

We study the isomorphism classes of the pairs (X_n, D). The isomorphism classes can be characterized in terms of the root system of type A. E.Looijenga investigated the isomorphism classes of rational surfaces with anti-canonical divisors [L]. We deal with another class of rational surfaces. The method and formulation are very similar to those of Looijenga's.

1. HIRZEBRUCH SURFACES

We assume $n \geqslant 3$. Σ_n is a subvariety of $\mathbf{P}^2 \times \mathbf{P}^1$ (cf (0.1)). Let $\pi : \Sigma_n \longrightarrow \mathbf{P}^1$ be the second projection. Σ_n is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. Let F be a fiber of the projection $\pi : \Sigma_n \longrightarrow \mathbf{P}^1$ and S be the section defined by $\zeta_0 = \zeta_1 = 0$.

DEFINITION. *we say that $n+1$ points $P_1, \ldots, P_{n+1}$ of Σ_n are in 'general position'if they satisfy the following conditions: (1)$P_i \neq P_j$ for $i \neq j$ and (2) there exists a nonsingular curve in the complete linear system $|nF + S|$ passing through $P_1, \ldots, P_{n+1}$.*

REMARK. *If $P_1, \ldots, P_{n+1}$ are in general position, then $P_i \notin S$ and no two of P_i are on a fiber.*

Let $p : X_n \longrightarrow \Sigma_n$ be the morphism obtained by blowing up $n+1$ points $P_1, \ldots, P_{n+1}$ in general position.

LEMMA 1.1. *If D is an anti-canonical divisor on X_n and satisfies the following conditions:*

(1) *D is the strict transform of an anti-canonical divisor D' on Σ_n,*

(2) D' *consists of four irreducible components and its intersection diagram is a circle,*

(3) $P_1, \ldots, P_{n+1}$ *are on only one component of* D' *and not on other components,*

then

$$D = F_1 + F_2 + S + C,$$

where F_i *is a strict transform of a fiber of the projection* $\pi : \Sigma_n \longrightarrow \mathbf{P}^1$, S *is the strict transform of the* $(-n)-$ *section of* Σ_n *and* C *is the strict transform of the unique nonsingular curve of* $|nF + S|$ *passing through* $P_1, \ldots, P_{n+1}$.

NOTATION. *We say that an anti-canonical divisor* D *on* X_n *is of '#-type' if it satisfies the condition of lemma 1.1. We denote by* F_0 *and* F_∞ *the components of* D *which are the strict transforms of the fibers of* π.

2. Homology and root system

Let X_n and $D = F_0 + F_\infty + S + C$ be as in §1. Consider the homology exact sequence:

$$\begin{array}{ccccc} \cdots & \rightarrow & H_3(X_n;\mathbb{Z}) & \rightarrow & H_3(X_n, X_n - D;\mathbb{Z}) \\ & & \| & & \\ & & 0 & & \\ \xrightarrow{\partial_*} H_2(X_n - D;\mathbb{Z}) & \xrightarrow{i_*} & H_2(X_n;\mathbb{Z}) & \xrightarrow{j_*} & H_2(X_n, X_n - D;\mathbb{Z}) \\ \rightarrow & & \cdots & & \end{array}$$

We extend the intersection form in $H_2(X_n;\mathbb{Z})$ to $H_2(X_n;\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. Let

$$Q = \ker j_* \subset H_2(X_n;\mathbb{Z})$$

and

$$R = \{\alpha \in Q | \alpha \cdot \alpha = -2\}.$$

LEMMA 2.1. *R is a root system of type A_n in $Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ and Q is generated by R. The set $\{e_i - e_{i-1} | 1 \leqslant i \leqslant n\}$ is the basis of R, where e_i is the class of the exceptional curve $E_i = p^{-1}(P_i)$.*

We now have the short exact sequence:

$$0 \rightarrow H_3(X_n, X_n - D;\mathbb{Z}) \xrightarrow{\partial_*} H_2(X_n - D;\mathbb{Z}) \xrightarrow{i_*} Q \rightarrow 0. \tag{2.1}$$

LEMMA 2.2 (K.IRIE).

$$H_3(X_n, X_n - D;\mathbb{Z}) \simeq \mathbb{Z}$$

Let ε be the generator of $H_3(X_n, X_n - D; \mathbb{Z})$. We next consider a meromorphic 2-form on X_n which has poles only along D.

Lemma 2.3. *There exists a unique meromorphic 2-form ω on X_n such that*

(1) *ω has poles only along D,*

(2) $\omega(\partial_*(\varepsilon)) = 1$.

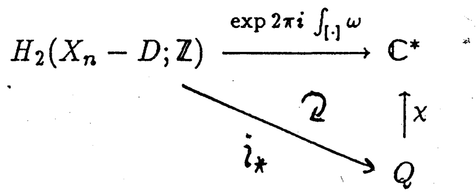
Furthermore, we can chose an affine coordinate z on $C(\subset D)$ such that $F_0 \cap C = 0$, $F_\infty \cap C = \infty$ and

$$\operatorname{Res}_C \omega = \frac{1}{(2\pi i)^2} \frac{dz}{z}.$$

It follows from this lemma, we can define a character $\chi : Q \longrightarrow \mathbb{C}^*$ by

$$\chi(i_*[\Gamma]) = \exp 2\pi i \int_\Gamma \omega,$$

where $\Gamma \in H_2(X_n - D; \mathbb{Z})$.



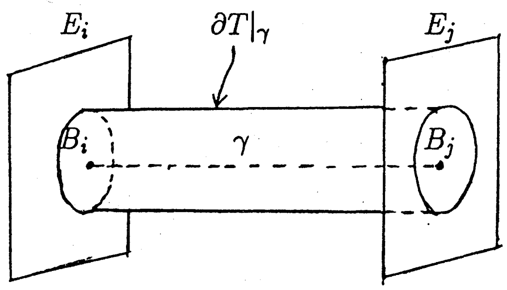
3. Torelli theorem for the pair (X_n, D)

We first consider the value of χ at the class $e_i - e_j \in Q$, where e_i and e_j are the homology classes of the exceptional curves $E_i = p^{-1}(P_i)$ and $E_j = p^{-1}(P_j)$ respectively. Let $B_i = E_i \cap C$ and let T be a closed tubular neighborhood of C in X_n such that $T \cap E_i$ and $T \cap E_j$ are fibers. Let γ be an injective path in C from B_i to B_j and let

$$\Gamma_{i,j} = (E_i \setminus (E_i \cap T)) \cup \partial T|_\gamma \cup (E_j \setminus (E_j \cap T)).$$

We can take the orientation such that $\Gamma_{i,j}$ is homologous to $E_i - E_j$ in X_n. Hence we have

$$i_*([\Gamma_{i,j}]) = e_i - e_j.$$



Since E_i and E_j are the inverse image of the points P_i and P_j respectively, we have

$$\int_{E_i\setminus(E_i\cap T)} \omega = \int_{E_j\setminus(E_j\cap T)} \omega = 0.$$

Therefore

$$\int_{\Gamma_{i,j}} \omega = \int_{\partial T|_\gamma} \omega.$$

By the residue formula, we have

$$\begin{aligned}\int_{\partial T|_\gamma} \omega &= 2\pi i \int_\gamma \operatorname{Res}_C \omega \\ &= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_{t_i}^{t_j} \frac{dz}{z} \\ &= \frac{1}{2\pi i}\log\frac{t_j}{t_i} \qquad (\mathrm{mod}\ \mathbb{Z}),\end{aligned}$$

where t_i and t_j are the affine coordinates of the points B_i and B_j respectively. Then we now have

$$\begin{aligned}\chi(e_i - e_j) &= \exp 2\pi i \int_{\Gamma_{i,j}} \omega \\ &= \frac{t_j}{t_i}\end{aligned}$$

The important point is that this is the cross ratio of $C \cap F_0, C \cap F_\infty, B_j$ and B_i. Thus we have the theorem of Torelli type.

Theorem. *Let X_n and X'_n be the surfaces defined in §1 and let D and D' be anti-canonical divisors of #-type on X_n and X'_n respectively (cf. notation in §1). Let denote root lattices by Q and Q', root systems by R and R', and characters by χ and χ' defined as in §2 for X_n and X'_n respectively. If $\varphi : H_2(X_n;\mathbb{Z}) \to H_2(X'_n;\mathbb{Z})$ is an isometry such that*

(1) $\varphi([F_i]) = [F'_i]$,
(2) $\varphi([C]) = [C']$,
(3) $\varphi(R) = R'$,
(4) $\varphi^*(\chi') = \chi$,

then there exists a unique isomorphism $\Phi : X_n \to X'_n$ which maps F_i to F'_i and C to C' and induces φ.

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[M] J.Matsuzawa, *Monoidal transformations of Hirzebruch surfaces and Weyl groups of type C*, J.Fac.Sci.Univ. Tokyo **35** (1988), 425–429; *Correction*, J.Fac.Sci.Univ. Tokyo **36** (1989), p. 827.