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Finite size approximation for representations of $U_q(\widehat{\mathfrak{sl}}(n))$

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1. The present note is an elucidation of an observation made in [1] concerning the crystal base of integrable representations of $U_q(\widehat{\mathfrak{sl}}(n))$.

Let $U_q = U_q(\widehat{\mathfrak{sl}}(2))$ denote the quantized affine algebra of type $A_1^{(1)}$. Just as in the classical case $q = 1$, it admits the following two classes of representations of particular interest:

(1) Highest weight representations. These are irreducible modules $L(\Lambda)$ with dominant integral highest weight Λ. For simplicity we consider here the level 1 representations $L(\Lambda_i)$ $(i = 0, 1)$ where the Λ_i denote the fundamental weights.

(2) Finite dimensional representations. These are level 0, non-highest weight representations (cf.[C]). For example, the natural representation $V = \mathbf{C}^2$ of $U_q(\mathfrak{sl}(2))$ can be made a $U_q(\widehat{\mathfrak{sl}}(2))$–module by letting the Chevalley generators act on V as follows:

$$e_0 = f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_1 = f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; t_0 = t_1^{-1} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

where $t_i = q^{h_i}$. (Here we follow the notations of [2]).

Given two modules L, L' over U_q one can form their tensor product $L \otimes L'$ via the comultiplication

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.$$

Let us consider the N–fold tensor product $V^{\otimes N}$ of $V = \mathbf{C}^2$. Our objective here is to show the following fact

$$\lim_{N\to\infty} V^{\otimes N} \sim L(\Lambda_0) \sqcup L(\Lambda_1), \qquad (*)$$

whose meaning will be made clear below.

2. The algebra U_q loses meaning at $q = 0$. However, Kashiwara's theory of crystal base [2] tells that on each integrable module L one can define the action of 'the Chevalley generators at $q = 0$' $\tilde{e}_i$, $\tilde{f}_i$. Moreover there exists a unique canonical base $B = B(L)$ of L 'at $q = 0$', such that

$$\text{If } u, v \in B, \text{ then } \quad \tilde{f}_i u = v \Longleftrightarrow u = \tilde{e}_i v$$

holds. For precise statements see [2]. The above situation is represented as

$$u \xrightarrow{i} v.$$

This equips B with a structure of colored (by the index $i = 0, 1$), oriented graph, called the crystal graph of L. It is known also that the crystal base B has a unique canonical extention to nonzero q [3].

There are some subtle points for finite-dimensional representations, since they are not integrable in the sense of [2]; but one can still consider crystal graphs for them. For instance $V = \mathbf{C}^2$ has the crystal graph

$$u_0 \underset{0}{\overset{1}{\rightleftarrows}} u_1$$

with u_i denoting the natural base of V.

According to [2] the crystal graph behaves remarkably nicely under tensor products. The vertices of $B(L_1 \otimes L_2)$ are simply $B(L_1) \times B(L_2)$ as a set. The edges of the graph are described by a simple rule [2], color–by–color. It is an amusing exercise to work out the crystal graphs for $B(V^{\otimes N})$ using this rule. Their vertices consist of sequences $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$ with $\xi_i \in \{0, 1\}$, representing the vectors $u_{\xi_i} \otimes \cdots \otimes u_{\xi_N}$. We show how they look like at the end of this note.

3. Let B_i^N ($i = 0$, 1) be the full subgraph of the crystal graph $B(V^{\otimes N})$, whose vertices consist of sequences $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$ with $\xi_N = i$. From the figure for $N = 2, 3, 4$ the following is already apparent:

Theorem. *There is an imbedding of graphs $B_i^N \hookrightarrow B_{i+1}^{N+1}$ given by $v \mapsto v \otimes u_{i+1}$, where the suffix i is to be read modulo 2. As N even $\to \infty$, B_i^N converges to the crystal graph $B(L(\Lambda_i))$ of the highest weight representation $L(\Lambda_i)$ (with the arrows reversed, because of conventions).*

Thus the equality $(*)$ makes sense in the language of crystal base. The proof of the theorem can be done by straightforward induction using Kashiwara's rule. As a consequence, $L(\Lambda_i)$ has a basis labeled by infinite sequences (called paths)

$\xi = (\xi_1, \xi_2, \cdots)$, whose 'tail' is $\cdots 010101 \cdots$ (i.e. $\xi_j \equiv j + i - 1 \bmod 2$ for $j \gg 0$). Though we have omitted here, there is also a formula for the weight of these base vectors given in terms of the paths [1]. This type of result has an important application in solvable lattice models of statistical mechanics [4]; in fact the whole story was motivated by the latter.

4. In [1] a similar result is established for $U_q(\widehat{\mathfrak{sl}}(n))$. Integrable representations of aribitrary level l can be 'approximated' by taking $V = S^l(\mathbf{C}^n)$, the l-th symmetric power of the standard representation $\mathbf{C}^n$.

Remark. At the stage of writing this note, Kashiwara found a simple explanation to this phenomenom.

References

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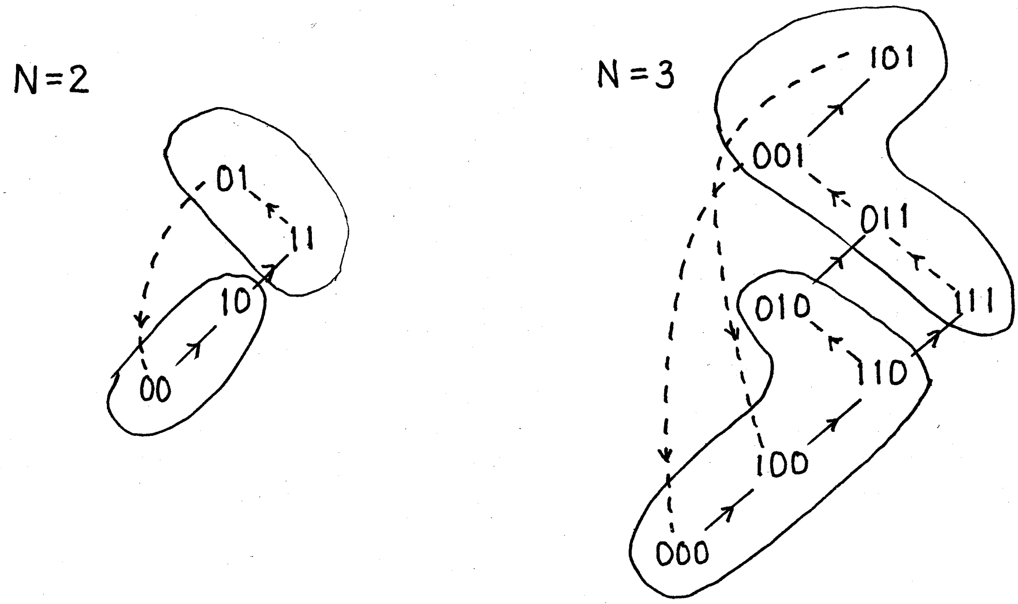
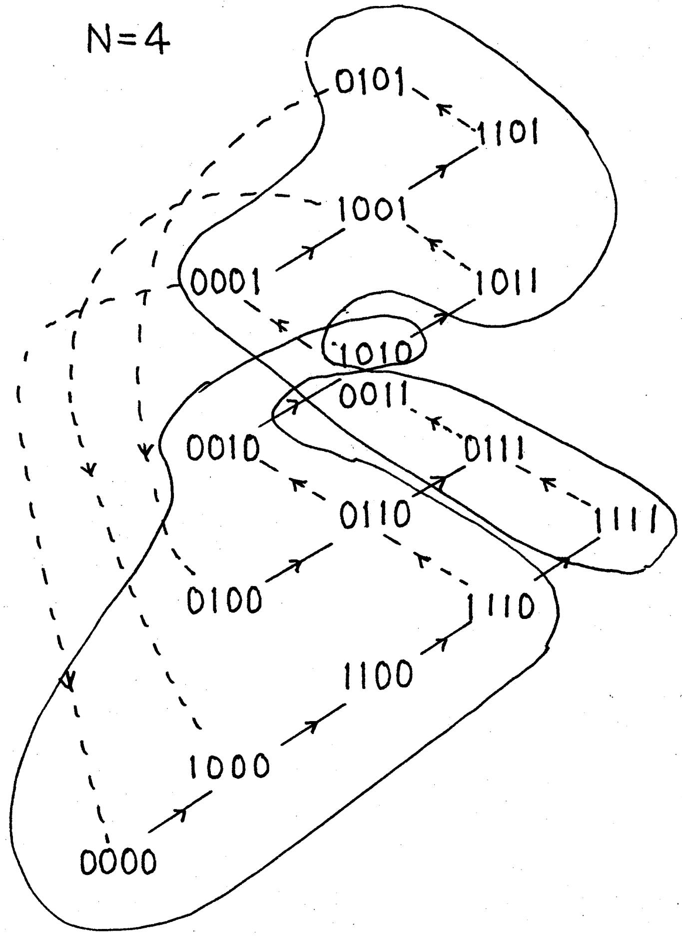
[4] See e.g. the review 'Solvable Lattice Models', *Proceedings of Symposia in Pure Mathematics*, **49** (1989) 295–331.





⟨ Convergence of crystal graphs for $V^{\otimes N}$ ⟩

$\longrightarrow \; = \; \xrightarrow{1}$

$\dashrightarrow \; = \; \xrightarrow{0}$

$0110 = u_0 \otimes u_1 \otimes u_1 \otimes u_0$ etc.