数理解析研究所講究録
第 765 巻 1991 年 126-135

On an affine space partition of the variety of N-stable flags and a generalization of the length–MAJ symmetry

ITARU TERADA

Department of Mathematics, University of Tokyo

1. Introduction. J. Matsuzawa introduced in his talk at Nagoya Conference for Commutative Algebra and Combinatorics, August 1990 (or even earlier at the AMS Summer Institute at Arcata 1986) the following two-variable polynomial $G_\mu(q,t)$ which could be regarded as a simultaneous "q-analogue" of the Poincaré polynomials of two varieties. Let μ be a partition of n (namely $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ with $\mu_i \in \mathbb{Z}_{>0}$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l$ and $\sum_{i=1}^n \mu_i = n$). We fix such μ once and for all in this note. Then his polynomial is:

$$G_\mu(t,q) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda\mu}(q) K_{\lambda'(1^n)}(t).$$

In this expression $\lambda \vdash n$ means that λ is a partition of n, and λ' is the conjugate partition of λ defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$, $l' = \lambda_1$, $\lambda'_j = \#\{\, i \mid \lambda_i \geq j \,\}$ $(1 \leq j \leq l')$ (see [**Mac**, p. 2]).

Then an interesting property is the following:

$$G_\mu(t^2, 1) = P_{\mathcal{P}_\mu}(t), \quad \text{and} \quad G_\mu(1, q^2) = P_{\mathcal{B}_N}(q).$$

The right hand sides denote the Poincaré polynomials of the varieties $\mathcal{P}_\mu$ and $\mathcal{B}_N$ respectively. $\mathcal{P}_\mu$ is a generalized flag variety of $GL(n, \mathbb{C})$ associated to its parabolic subgroup of type μ; namely the variety consisting of all chains $V_1 \subset V_2 \subset \cdots \subset V_l$ of linear subspaces of $\mathbb{C}^n$ with $\dim V_i = \mu_1 + \mu_2 + \cdots + \mu_i$ $(1 \leq i \leq l)$.

The other variety $\mathcal{B}_N$ is the key subject of this note. Let N be a nilpotent $n \times n$ matrix with Jordan cells of size $\mu_1, \mu_2, \ldots, \mu_l$. (Such N will be called of Jordan type μ.) Then $\mathcal{B}_N$

is defined to be the variety of all N-*stable complete flags*, i.e. chains $V_1 \subset V_2 \subset \cdots \subset V_n$, $\dim V_i = i$ $(1 \leq i \leq n)$, of subspaces of $\mathbf{C}^n$ each of which is stable under the transformation N. Since all such N are conjugate (for a fixed μ) under conjugation by $GL(n, \mathbf{C})$, $\mathcal{B}_N$ is isomorphic for all such N.

In this note we give a combinatorial interpretation of this polynomial. We use a result on connection between a partition of the variety $\mathcal{B}_N$ into affine spaces and the Schubert cell decomposition of the variety $\mathcal{B}$ of all complete flags, and we borrow a recent theorem on the Springer representation due to G. Lehrer–T. Shoji and N. Spaltenstein.

The main result is described as follows.

THEOREM. *Let μ be a partition of n, and $G_\mu(q,t)$ be defined as above. Then we have:*

$$G_\mu(t,q) = \sum_{T \in \mathrm{RDT}(\mu)} q^{l(T)} t^{\mathrm{MAJ}(w_T)}, \tag{1}$$

where the notation is explained below.

Notation. $\mathrm{RDT}(\mu)$ is the set of row-decreasing tableaux of shape μ. By a row-decreasing tableau here we mean a tableau in which each letter in the range 1 through n appears once and the entries in each row decrease from left to right. (The row-decreasing tableau is a temporary term used in this note.)

l in the right-hand side is a function $\mathrm{RDT}(\mu) \to \mathbf{Z}_{\geq 0}$ defined in §3. It reduces to the usual length function on $\mathfrak{S}_n$ ($l(w) = \#\{\,(i,j) \mid 1 \leq i < j \leq n,\ w(i) > w(j)\,\}$) in the case $\mu = (1^n)$. $T \mapsto w_T$ is an injective map $\mathrm{RDT}(\mu) \to \mathfrak{S}_n$ also defined in §3. $\mathrm{MAJ}(w)$ denotes the major index (also called the greater index) of $w \in \mathfrak{S}_n$, namely $\mathrm{MAJ}(w) = \sum_{\substack{1 \leq i \leq n-1 \\ w(i) > w(i+1)}} i$ (see [**St**, p. 23]).

The formula (1) reduces to the following expression which represents the length–MAJ symmetry proved by D. Foata and M.-P. Schützenberger in [**FS**]:

$$\sum_{w \in \mathfrak{S}_n} q^{l(w)} t^{\mathrm{MAJ}(w)} = \sum_{\lambda \vdash n} K_{\lambda (1^n)}(q) K_{\lambda (1^n)}(t). \tag{2}$$

Their method was to construct a bijection $\phi\colon \mathfrak{S}_n \to \mathfrak{S}_n$ preserving the "inverse" descent set $D(w^{-1}) = \{\, i \mid 1 \leq i \leq n-1,\ w^{-1}(i) > w^{-1}(i+1)\,\}$ (see [**St**, p. 21] for $D(\cdot)$) and satisfying $l(\phi(w)) = \mathrm{MAJ}(w)$.

H. Naruse gave another proof of (2) using the representation of $\mathfrak{S}_n$ on $H^*(\mathcal{B}, \mathbb{C})$. He also gave some suggestions as to a partition of $\mathcal{B}_N$ into affine spaces and a definition of the function l above. This brief note is a realization of his idea. The detailed version will be published elsewhere.

2. Kostka-Foulkes polynomials and nice bases for $\mathbb{C}[\mathfrak{S}_n]$-modules. First let us recall some properties of the Kostka-Foulkes polynomials. The $\widetilde{K}_{\lambda\mu}(q)$ are defined from the $K_{\lambda\mu}(q)$ by the relation

$$\widetilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1}) \quad \text{where} \quad n(\mu) = \sum_{i=1}^{l} (i-1)\lambda_i.$$

For the definition of $K_{\lambda\mu}(q)$, we refer the reader to [**Mac**, §III.6].

Here are some properties of the Kostka-Foulkes polynomials. Let λ and μ be partitions of a positive integer n.

PROPERTY 1. $K_{\lambda\mu}(1)$ is equal to $K_{\lambda\mu}$ (the Kostka number), which can be counted as the number of semistandard tableaux (called just tablaux in [**Mac**]) with shape λ and weight μ (see [**Mac**, §III.6]).

PROPERTY 2. $\widetilde{K}_{\lambda\mu}(t) = \sum_i \langle H^{2i}(\mathcal{B}_N, \mathbb{C}), V_\lambda \rangle_{\mathfrak{S}_n} q^i$ (see [**Mac**, Ex. III.7.9]. Caution: In [**Mac**] $\mathcal{B}_N$ is denoted as X_μ). Here $H^{2i}(\mathcal{B}_N, \mathbb{C})$ is regarded as a $\mathbb{C}[\mathfrak{S}_n]$-module via the so-called Springer representation. There seems to be two kinds of the Springer representations differing from each other by the signature character. Here we use the one in which the trivial representation appears in H^0. The symbol V_λ denotes the irreducible $\mathbb{C}[\mathfrak{S}_n]$-module indexed by the partition λ. The angular bracket $\langle \ , \ \rangle_{\mathfrak{S}_n}$ denotes the intertwining number of $\mathbb{C}[\mathfrak{S}_n]$-modules.

PROPERTY 3. $K_{\lambda(1^n)}(t) = \sum_{T \in \mathrm{STab}(\lambda)} t^{c'(T)}$ (see [**Mac**, Ex. III.6.2]). Here STab(λ) is the set of the *standard* tableaux of shape λ, namely tableaux containing each letter from 1 to n once and in which the entries increase (a) from left to right along each row and (b) from top to bottom along each column. If $T \in \mathrm{STab}(\lambda)$, then $c'(T)$ is the sum of i ($1 \le i \le n-1$) such that $i+1$ lies to the right in T (in the shaded part of Fig. 1). There is a similar (but more complicated) interpretation of $K_{\lambda\mu}(t)$ for a general μ, shown by A. Lascoux and M.-P. Schützenberger, as a sum of some powers of t determined by the positions of entries in the semistandard tableaux of shape λ and weight μ (see [**Mac**, §III.6]), but we don't need that. Using this, we also know that $K_{\lambda(1^n)}(t) = \Xi(V_\lambda)$ defined just below.

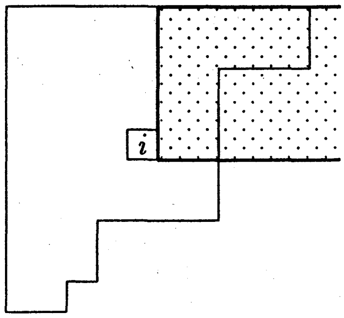

Fig. 1.

Definition (nice bases). Let (ρ, V) be a representation of $\mathfrak{S}_n$ over $\mathbf{C}$, and let s_i $(1 \leq i \leq n-1)$ denote the transposition $(i, i+1) \in \mathfrak{S}_n$. A basis $\{e_k\}_{k\in K}$ of V (where K is some index set) is called *nice* if there exists a subset K_i of K for each i $(1 \leq i \leq n-1)$ for which the $\rho(s_i)$-fixed part of V is precisely spun by the basis vectors indexed by the elements of K_i: $V^{\rho(s_i)} = \bigoplus_{k\in K_i} \mathbf{C}e_k$.

Remark (existence). It is known that any $\mathbf{C}[\mathfrak{S}_n]$-module admits a nice basis. In fact, since any $\mathbf{C}[\mathfrak{S}_n]$-module is semisimple, it suffices to show that any irreducible $\mathbf{C}[\mathfrak{S}_n]$-module has one. Let $(\rho_{\lambda'}, V_{\lambda'})$ be the irreducible representation of $\mathfrak{S}_n$ indexed by the conjugate partition of λ. The representation of $\mathfrak{S}_n$ on $V_{\lambda'}$ obtained by sending s_i to $-\rho_{\lambda'}(s_i)$ is also irreducible and is equivlent to the one indexed by λ. Then any W-graph basis of $V_{\lambda'}$ serves as a nice basis for ρ_λ (not $\rho_{\lambda'}$).

Definition ($\Xi(V)$). Let (ρ, V) be as above, and let $\{e_k\}_{k\in K}$ be a nice basis. We define $\Xi(V)$ to be a polynomial in t obtained by summing up, for $k \in K$, the monomial obtained by raising t to the power $\displaystyle\sum_{\substack{1\leq i\leq n-1\\ \rho(s_i)e_k=e_k}} i$.

Remark. $\Xi(V)$ is independent of the choice of the nice basis. $\Xi(V)$ is clearly additive with respect to V.

3. Reduced lengths or folded lengths of row-decreasing tableaux and the representatives w_T. The reduced (or folded) length is a temporary term used in this note.

Definition ($l(T)$). Let T be a row-decreasing tableau of shape μ. We define its *reduced* or *folded length* $l(T)$ to be the sum for i in the range $1 \leq i \leq n-1$ of the number $l_i(T)$ of entries greater than i sitting in the shaded area in Fig. 2.

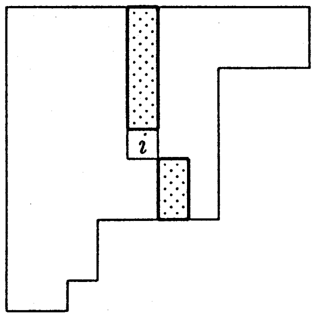

Fig. 2.

Example ($l(T)$). Let $T = \begin{array}{llll} 10 & 8 & 6 & 4 \\ 11 & 3 & 1 & \\ 9 & 7 & 2 & \\ 5 & & & \end{array}$. This is a row-decreasing tableau of shape $(4,3,3,1)$. We have $l_1(T) = 1$, $l_2(T) = 1$, $l_3(T) = 1$, $l_4(T) = 0$, $l_5(T) = 3$, $l_6(T) = 0$, $l_7(T) = 1$, $l_8(T) = 0$, $l_9(T) = 2$ and $l_{10}(T) = l_{11}(T) = 0$, so that $l(T) = 9$.

Definition (w_T). Let μ be a partition of n. Then we denote by T^0_μ the row-decreasing tableau of shape μ obtained by putting the letters 1 through n starting from the rightmost column and proceeding to the left, filling each column from top to bottom. For any row-decreasing tableau T of shape μ, we denote by w_T the element of $\mathfrak{S}_n$ obtained by reading the entries of T^0_μ in the order designated by T. In other words, if the position (p,q) in T is filled by i, then the same position (p,q) in T^0_μ is filled by $w_T(i)$.

Example (w_T). If $\mu = (4,3,3,1)$, then $T^0_\mu = \begin{array}{llll} 8 & 5 & 2 & 1 \\ 9 & 6 & 3 & \\ 10 & 7 & 4 & \\ 11 & & & \end{array}$. For the row-decreasing tableau T shown in the above example, we have $w_T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 6 & 1 & 11 & 2 & 7 & 5 & 10 & 8 & 9 \end{pmatrix}$.

Remark. If $\mu = (1^n)$, then any tableau of shape (1^n) containing each of letters 1 though n exactly once is clearly a row-decreasing tableau. If we denote the entry in the i-th row by $\sigma(i)$, then $w_T = \sigma^{-1}$ and $l(T) = l(\sigma)$. This shows that, in this case, our result reduces to the identity (2) in §1 describing the length–MAJ symmetry.

4. Preparation of the proof of the identity. As can easily be seen from Property 3, we have $K_{\lambda'(1^n)}(t) = t^{\frac{n(n-1)}{2}} K_{\lambda(1^n)}(t^{-1})$ $(= \widetilde{K}_{\lambda(1^n)}(t))$ and the *lesser index* LES(w) if w defined by $\mathrm{LES}(w) = \sum_{\substack{1\le i\le n-1\\ w(i)<w(i+1)}} i$ satisfies $\mathrm{LES}(w) = \dfrac{n(n-1)}{2} - \mathrm{MAJ}(w)$, our assertion is equivalent to the following identity:

CLAIM. $\displaystyle\sum_{\lambda\vdash\mathfrak{S}_n} \widetilde{K}_{\lambda\mu}(q)K_{\lambda(1^n)}(t) = \sum_{T\in\mathrm{RDT}(\mu)} q^{l(T)}t^{\mathrm{LES}(w_T)}.$

We prove this identity by computing

$$G'(q,t) = \sum_j \Xi\left(H^{2j}(\mathcal{B}_N, \mathbf{C})\right) q^j$$

(where these cohomology groups are regarded as $\mathbf{C}[\mathfrak{S}_n]$-modules via the Springer representation) in two different ways.

First, we compute $G'(q,t)$ according to the irreducible decomposition of $H^*(\mathcal{B}_N, \mathbf{C})$ and show that it gives the left-hand side of the claim. We have

$$\begin{aligned}
G'(q,t) &= \sum_j\sum_\lambda \langle H^{2j}(\mathcal{B}_N,\mathbf{C}), V_\lambda\rangle \Xi(V_\lambda) q^j \\
&= \sum_\lambda \widetilde{K}_{\lambda\mu}(q)\Xi(V_\lambda) && \text{(by Property 2)} \\
&= \sum_\lambda \widetilde{K}_{\lambda\mu}(q)K_{\lambda(1^n)}(t), && \text{(by Property 1)}
\end{aligned}$$

which equals the left-hand side of the claim.

5. An affine space partition of $\mathcal{B}_N$ and the Schubert cells. Now we use a partition of $\mathcal{B}_N$ into affine spaces to show that $G'(q,t)$ is equal to the right-hand side of the claim. Such a partition has been given by N. Spaltenstein **[Sp]** for $\mathcal{B}_N$ and by N. Shimomura **[Sh]** for a similar variety consisting of N-stable generalized flags.

Our point here is to clarify the relationship between such a partition and the Schubert cell decomposition of $\mathcal{B}$, the variety consisting of all complete flags in $\mathbf{C}^n$. Let

$$\mathcal{B} = \coprod_{w \in \mathfrak{S}_n} X_w, \quad X_w \approx \mathbf{C}^{l(w)}$$

be a Schubert cell decomposition of $\mathcal{B}$. (See [H, p. 121–122] for example, although there is considerable difference in notation.) It is quite natural to ask the following question.

PROBLEM. *Put $X_{w,N} = X_w \cap \mathcal{B}_N$. Does $\mathcal{B}_N = \coprod_{w \in \mathfrak{S}_n} X_{w,N}$ give a partition into affine spaces?*

In general, this is not true. More precisely, it depends on the position of the "reference flag" (the unique element of X_e, where e denotes the identity element of $\mathfrak{S}_n$) with respect to the chosen transformation N. If one takes the usual Jordan canonical form for N and the "canonical" flag $(V_1^0, V_2^0, \ldots, V_n^0)$ defined by $V_j^0 = \bigoplus_{i=1}^{j} \mathbf{C}e_i$ ($j = 1, \ldots, n$) where $e_i = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$, then we have a negative answer for $\mu = (3,3)$. (Recall that μ is the Jordan type of N.)

However, if we take the following particular transformation N_μ for N (and keep the canonical reference flag) then the answer is positive.

We specify N_μ using the tableau T_μ^0 defined in §3. We present this rule through an example. Let $\mu = (4,3,3,1)$, then N_μ is defined by reading the rows of T_μ^0 as follows

$$T_\mu^0 = \begin{matrix} 8 & 5 & 2 & 1 \\ 9 & 6 & 3 & \\ 10 & 7 & 4 & \\ 11 & & & \end{matrix} \qquad N_\mu : \begin{cases} e_8 \mapsto e_5 \mapsto e_2 \mapsto e_1 \mapsto 0 \\ e_9 \mapsto e_6 \mapsto e_3 \mapsto 0 \\ e_{10} \mapsto e_7 \mapsto e_4 \mapsto 0 \\ e_{11} \mapsto 0 \end{cases}$$

Now we have the following result:

THEOREM. *Let $\mu \vdash n$ and N_μ be defined as above. Let X_w be the Schubert cell with respect to the canonical reference flag and put $X_{w,N_\mu} = X_w \cap \mathcal{B}_{N_\mu}$ ($w \in \mathfrak{S}_n$). Then*

(1) *$X_{w,N_\mu} \neq \emptyset$ if and only if $w = w_T$ for some $T \in \mathrm{RDT}(\mu)$,*

(2) $X_{w_T,N_\mu} \approx \mathbb{C}^{l(T)}$ *for* $T \in \mathrm{RDT}(\mu)$, *where the function* $l(T)$ *is defined in the earlier section.*

Remark. (1) $(V_1, V_2, \ldots, V_n) \in X_{w_T,N_\mu}$ if and only if all V_i are stable under N (i.e. this flag belongs to $\mathcal{B}_{N_\mu}$) and the sizes of the Jordan cells of N_μ acting on V/V_i are the lengths of the rows of the tableau obtained from T by removing the squares that are marked as 1 through i.

(2) For $T \in \mathrm{RDT}(\mu)$, the subset $\coprod_{\substack{T' \in \mathrm{RDT}(\mu) \\ w_{T'} \prec w_T}} X_{w_{T'},N_\mu}$ is closed in $\mathcal{B}_{N_\mu}$ ($\prec$ denotes the Bruhat order).

Due to (2) above, the fundamental classes of X_{w_T,N_μ} form a basis of the homology groups $H_*(\mathcal{B}_{N_\mu}, \mathbb{C})$. Therefore $H^*(\mathcal{B}_{N_\mu}, \mathbb{C})$ has a dual basis:

$$H^*(\mathcal{B}_{N_\mu}, \mathbb{C}) = \bigoplus_{T \in \mathrm{RDT}(\mu)} \mathbb{C}[X_{w_T,N_\mu}]^*.$$

Note that $[X_{w_T,N_\mu}]^* \in H^{2l(T)}(\mathcal{B}_{N_\mu}, \mathbb{C})$.

6. A result of Lehrer-Shoji and Spaltenstein. Next we consider varieties $\mathcal{P}^j$ for $1 \le j \le n-1$ defined as follows:

$$\mathcal{P}^j = \left\{ (V_1, \ldots, V_{n-1}) \;\middle|\; \begin{array}{l} V_1 \subset \cdots \subset V_{n-1} \text{ (linear subspaces of } \mathbb{C}^n) \\ \dim V_k = \begin{cases} k & (k < j) \\ k+1 & (k \ge j) \end{cases} \end{array} \right\}.$$

Then $\mathcal{P}^j$ has a similar classical decomposition:

$$\mathcal{P}^j = \coprod_{\substack{w \in \mathfrak{S}_n \\ w(j) < w(j+1)}} Y^j_w \quad \text{and for such } w \text{ we have} \quad Y^j_w \approx X_w \approx \mathbb{C}^{l(w)}.$$

Now let $\mathcal{P}^j_N$ be the subvariety of $\mathcal{P}^j$ consisting of N-stable elements, and put $Y^j_{w,N} = Y^j_w \cap \mathcal{P}^j_N$. Then we can show that, if $N = N_\mu$, then $\mathcal{P}^j_{N_\mu}$ has a similar decomposition as follows:

$$\mathcal{P}^j_{N_\mu} = \coprod_{\substack{T \in \mathrm{RDT}(\mu) \\ w_T(j) < w_T(j+1)}} Y^j_{w_T,N_\mu} \quad \text{and for such } w \text{ we have} \quad Y^j_{w_T,N_\mu} \approx X_{w_T,N_\mu} \approx \mathbb{C}^{l(T)}.$$

We have a natural projection $\pi\colon \mathcal{B}_N \to \mathcal{P}_N^j$ which induces a map on the cohomology groups $\pi^*\colon H^*(\mathcal{P}_N^j, \mathbf{C}) \to H^*(\mathcal{B}_N, \mathbf{C})$. If $N = N_\mu$, then the $[Y^j_{w_T, N_\mu}]^*$ ($T \in \mathrm{RDT}(\mu)$, $w_T(j) < w_T(j+1)$) form a basis of $H^*(\mathcal{P}^j_{N_\mu}, \mathbf{C})$. The map π^* sends $[Y^j_{w_T,N_\mu}]^*$ onto $[X_{w_T,N_\mu}]^*$ if w_T appears in the decomposition of $\mathcal{P}^j_{N_\mu}$.

The following fact has been shown by T. Shoji, G. I. Lehrer [**ShoL**] and N. Spaltenstein [**Sp2**].

THEOREM (Shoji-Lehrer, Spaltenstein). *Let N, $\mathcal{B}_N$, j, $\mathcal{P}_N^j$, π, s_j be all as above. Then we have:*

$$\pi^*\colon H^*(\mathcal{P}_N^j, \mathbf{C}) \xrightarrow{\simeq} H^*(\mathcal{B}_N, \mathbf{C})^{s_j}.$$

7. Conclusion of the proof. From the above theorem, it follows that the set of $\{[X_{w_T,N_\mu}]^*\}$, $T \in \mathrm{RDT}(\mu)$, is a nice basis of $H^*(\mathcal{B}_{N_\mu}, \mathbf{C})$. $[X_{w_T,N_\mu}]^*$ is fixed by s_j if and only if $w_T(j) < w_T(j+1)$. Therefore we have

$$\begin{aligned} G'(q,t) &= \sum_j \sum_{\substack{T \in \mathrm{RDT}(\mu) \\ l(T)=j}} t^{\mathrm{LES}(w_T)} q^j \\ &= \sum_{T \in \mathrm{RDT}(\mu)} t^{\mathrm{LES}(w_T)} q^{l(T)}, \end{aligned}$$

which concludes our proof.

8. Discussion. (1) Can one characterize (up to conjugacy) the pairs (N, F) (N a nilpotent $n \times n$ matrix of Jordan type μ, F the reference flag for the Schubert cell decomposition of $\mathcal{B}_N$) for which $\{X_{w,N}\}$ gives a partition of $\mathcal{B}_N$ into affine spaces?

(2) Can one find a Foata–Schützenberger type proof of the identity (1)?

(3) (suggested by R. Stanley) Can one find an interpretation of a more general polynomial $\sum_{\lambda \vdash n} \widetilde{K}_{\lambda\mu}(q) \widetilde{K}_{\lambda\nu}(t)$ for partitions μ, ν of n in general? (This polynomial has also been investigated by J. Matsuzawa.) A first step would be to find some interpretation of $\widetilde{K}_{\lambda\mu}(t)$ in the space V_λ which would generalize $\Xi(V_\lambda)$.

REFERENCES

[**H**] H. Hiller, "Geometry of Coxeter groups," Pitman, Boston/London/Melbourne, 1982.

[**Mac**] I. G. Macdonald, "Symmetric functions and Hall polynomials," Oxford University Press, Oxford, 1979.

[**FS**] D. Foata and M.-P. Schützenberger, *Major index and inversion number of permutations*, Math. Nachr. **83** (1978), 143–150.

[**LSho**] G. I. Lehrer and T. Shoji, *On flag varieties, hyperplane complements and Springer representations of Weyl groups*, J. Amer. Math. Soc. (to appear).

[**Sh**] N. Shimomura, *A theorem on the fixed point set of a unipotent transformation on the flag manifold*, J. Math. Soc. Japan **32** (1980), 55–64.

[**Sp**] N. Spaltenstein, *The fixed point set of a unipotent transformation on the flag manifold*, Proc. Kon. Ak. v. Wet. **79** (1976), 452–456.

[**Sp2**] ______________, *On the reflection representation in Springer's theory*, preprint.

[**St**] R. P. Stanley, "Enumerative combinatorics, vol. 1," Wadsworth & Brooks/Cole, Monterey, 1986.