数理解析研究所講究録
第766巻 1991年 24-48

Non-convex curves of constant angle of double type

RIMS, Kyoto University
Junpei SHEN (沈君佩)

§1. Introduction

In this article we study and develop a theory of curves of constant angle of double type. First we summarize the theory of those curves of single type and twin type. Next, we shall give double type theory as a generalization and study the extremal values of various geometric functionals. A duality in those curves will also be treated in the double type theory. Professor Shigetake Matsuura first developed the theory of curves of constant angle of single and convex type for several years (see [1] $\sim$ [17]). It had a very simple intuitive meaning. He, however, introduced the concept of the admissible curves and developed the theory of single and twin type. The duality is now complete in curves of these types.

But twin type theory suggests that there is room to develop more general complete theory of curves of constant angle. Thus we realize the generalization in the form of double type theory. It includes the theory of single and twin type.

Our main purposes are to investigate the interelations of these geometric quantities and to obtain their extremal values. Our main tools are the theory of distributions of L.Schwartz, Fourier series in $\mathbf{L}^2$ apace and geometrical inequalities. Further, the method of linearization is extensively exploited. To obtain the concrete extremal values, the construction of extremelizing sequence of generators is important.

According to the lack of space, we should omit to give proofs of almost all results.

§2. Preliminaries

§2.1. Admissible curve

We consider the director circle C in $\mathbf{R}^2$, whose center is the origin and the radius is the unity 1. Let Λ be a curve inside the circle C. Since we are obliged to treat non-convex curves, something which plays the role of supporting lines to convex curves, we introduce generalized tangent lines to admissible curves defined below.

Definiton 2.1.1. Let $\mathbf{T}$ be one dimensional torus $\mathbf{R}/2\pi\mathbf{Z}$. An admissible curve is a curve Λ which satisfies the following three conditions:

(i) Λ is a closed continuous curve parametrized by $\theta \in \mathbf{T}$,

$$\begin{cases} x = x(\theta) \\ y = y(\theta). \end{cases}$$

(ii) Λ is a rectifiable curve. i.e. for any partion Δ of $[0, 2\pi]$,

$$\Delta : 0 = \theta_o < \theta_1 < ... < \theta_N = 2\pi,$$

the length of a polygonal line with respect to Δ,

$$|L|(\Lambda_\Delta) = \sum_{j=1}^{N} \sqrt{(x(\theta_j) - x(\theta_{j-1}))^2 + (y(\theta_j) - y(\theta_{j-1}))^2} \qquad (2.1.1)$$

is bounded when Δ runs over all partitions, the length of Λ is defined by

$$|L|[\Lambda] = \sup_{\Delta} |L|[\Lambda_\Delta]. \qquad (2.1.2)$$

(iii) For every fixed θ, consider the straight line l_θ, the equation of which takes the form

$$l_\theta : x\cos\theta + y\sin\theta = p(\theta). \qquad (2.1.3)$$

Lemma 2.1.2. Λ is a rectifiable curve. $\Longleftrightarrow$ $x(\theta), y(\theta)$ are of bounded variation.

§2.2. Non-convex curves and its generators

In general, we shall need to consider non-convex curves. Denote a generator of the non-convex curve by $p(\theta)$, and its period be 2π. Differentiate

$$p(\theta) = x(\theta)\cos\theta + y(\theta)\sin\theta. \qquad (2.2.1)$$

Since $x(\theta), y(\theta)$ are of bounded variation, $\dot{x}(\theta), \dot{y}(\theta)$ are Radon measures. The above equality shows that $p(\theta)$ is continuous in θ. Thus we get

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta + [\dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta].$$

Since the derivatives $\dot{x}(\theta), \dot{y}(\theta)$ are taken in the sense of $D'(\mathbf{T})$, we have $\dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta = 0$ in an open neighbourhood of the fixed value θ_0. Thus, we get

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta. \tag{2.2.2}$$

It is obvious that $\dot{p}(\theta)$ is continuous, which means that $p \in C^1(\mathbf{T})$. By differentiating (2.2.2), we have

$$\ddot{p}(\theta) = -x(\theta)\cos\theta - y(\theta)\sin\theta + [-\dot{x}\sin\theta + \dot{y}\cos\theta].$$

Hence

$$p + \ddot{p} = -\dot{x}\sin\theta + \dot{y}\cos\theta. \tag{2.2.3}$$

It implies that $p + \ddot{p}$ is a Radon measure. Put

$$T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

For each point $(x(\theta), y(\theta))$, by formula (2.2.1) and (2.2.2), it easily follows that,

$$\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix} = T(-\theta) \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = T(\theta) \begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix}.$$

Hence, the parametric representation of Λ takes the form:

$$\begin{cases} x = x(\theta) = p(\theta)\cos\theta - \dot{p}(\theta)\sin\theta \\ y = y(\theta) = p(\theta)\sin\theta + \dot{p}(\theta)\cos\theta \end{cases}, \theta \in (0, 2\pi).$$

Since $T(\theta)$ is an orthogonal transformation, it is clear that,

$$p^2 + \dot{p}^2 = x^2 + y^2. \tag{2.2.4}$$

Before to consider the curves of constant angle α of non-convex of double type, we need to use the following notations :

$$\begin{cases} c_1 = c_1(\alpha) = \sin\alpha \\ c_2 = c_2(\alpha) = \cos\alpha, \end{cases} \qquad \begin{cases} \tilde{c}_1 = \tilde{c}_1(\alpha) = \sin\frac{\alpha}{2} \\ \tilde{c}_2 = \tilde{c}_2(\alpha) = \cos\frac{\alpha}{2}. \end{cases}$$

Here α is a given angle (which is constant) and $0 < \alpha < \pi$. Let I_α and J_α be open intervals defined by

$$I_\alpha = (-\Omega_\alpha, \Omega_\alpha),$$

where $\Omega_\alpha = min(\tilde{c}_1, \tilde{c}_2)$, and

$$J_\alpha = \begin{cases} (0, c_1), & \text{if } 0 < \alpha \leq \frac{\pi}{2}; \\ (-c_2, 1), & \text{if } \frac{\pi}{2} \leq \alpha < \pi. \end{cases}$$

the characteristic function χ_α is as follows:

$$\chi_\alpha(t) = c_1\sqrt{1-t^2} - c_2 t, \qquad t \in J_\alpha. \tag{2.2.5}$$

Then χ_α maps J_α onto J_α and is strictly monotone decreasing.

Let $\xi, \eta \in J_\alpha$, such that $\eta = \chi_\alpha(\xi)$, then we get the characteristic ellipse as

$$\xi^2 + \eta^2 + 2c_2\xi\eta = c_1^2$$

whose graph is the thick line parts of Figure [2.1].

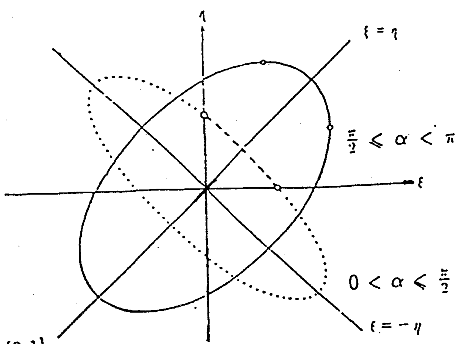

Figure [2.1]

Theorem 2.2.1. For a pair of continuous functions $p(\theta) = (p_1, p_2)$ of θ to be the gererator of a curve of constant angle of double type, it is necessary and sufficient that

(a) $p_j(\theta)$, $(j = 1, 2)$ is a C^1-function with period 2π;

(b) $\begin{cases} p_j(\theta) \in J_\alpha, \forall\theta; \\ p_j(\theta)^2 + \dot{p}_j(\theta)^2 < 1; \end{cases}$ [differential inequality]

(c) $p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1(\theta))$; [functional equation]

(d) $p_j + \ddot{p}_j \in M(\mathbf{T})$.

Let $p = (p_1, p_2)$ be a pair of generators of admissible curves, α be a given angle, $0 < \alpha < \pi$, we observe that the following conditions are equivalent:

(i)

$$\begin{cases} p_1(\theta) \in J_\alpha \\ p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1(\theta)) \end{cases} \tag{2.2.6}$$

(ii)

$$\begin{cases} p_2(\theta) \in J_\alpha \\ p_1(\theta - (\pi - \alpha)) = \chi_\alpha(p_2(\theta)) \end{cases} \tag{2.2.7}$$

(iii)

$$\begin{cases} p_1(\theta - (\pi - \alpha)) = \chi_\alpha(p_2(\theta)) \\ p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1(\theta)) \end{cases} \tag{2.2.8}$$

Definition 2.2.2. We denote by $\wp_\alpha^{double}$, the totality of pairs $p = (p_1, p_2)$, where p_1, p_2 satisfy the four conditions in Theorem 2.2.1. Then p is called the generator of curves of constant angle α of general double type.

Definition 2.2.3. We denote by $\wp_\alpha^{\overset{double}{symmetric}}$, the totality of pairs $p = (p_1, p_2)$ such that both p and $\check{p} = (p_2, p_1)$ are in $\wp_\alpha^{double}$. Then p is called the generator of curves of constant angle α of symmetric double type.

Definition 2.2.4. We denote by $\wp_\alpha^{twin}$, the totality of pairs $p = (p_1, p_2)$ satisfying $p_2(\theta) \equiv p_1(\theta + \pi)$. Then p is called the generator of curves of constant angle α of twin type.

Definition 2.2.5. We denote by $\wp_\alpha^{single}$, the totality of pairs $p = (p_1, p_2)$ satisfying $p_1(\theta) \equiv p_2(\theta)$. Then p is called the generator of curves of constant angle α of single type.

In view of these definitions, we obtain the inclusion relations among them as follows:

$$\wp_\alpha^{single} \subseteq \wp_\alpha^{twin} \subseteq \wp_\alpha^{\substack{double\\symmetric}} \subseteq \wp_\alpha^{double}.$$

Lemma 2.2.6. Let $p \in \wp_\alpha^{\substack{double\\symmetric}}$, then both 2π and 2α are periods of p_1 and p_2.

Theorem 2.2.7. α be given, let $\alpha/\pi \in \mathbf{Q}$, and $p \in \wp_\alpha^{\substack{double\\symmetric}}$, then the curves $\Lambda_j \quad (j = 1, 2)$ are concentric circles with C.

Remark 2.2.8. Let (p_1, p_2) be a generator of general double type, one of components can be chosen rather arbitrarily. More precisely, for instance, if p_1 satisfies the conditions: $p_1(\theta) \in J_\alpha, p_1(\theta)^2 + \dot{p}_1(\theta)^2 < 1$, and $\ddot{p}_1 \in M(\mathbf{T})$, then we can define $p_2(\theta)$ by the formula $p_2(\theta) = \chi_\alpha(p_1(\theta - \pi + \alpha))$. Then we get $p = (p_1, p_2) \in \wp_\alpha^{double}$.

§2.3. The modified characteristic function $\tilde{\chi}_\alpha$

The modified characteristic function $\tilde{\chi}_\alpha$ is defined by the formula

$$\tilde{\chi}_\alpha(s) = \tilde{c}_1\sqrt{1 - s^2} - \tilde{c}_2 s, \qquad s \in I_\alpha \quad \text{or} \quad s \in J_\alpha \tag{2.3.1}$$

$\tilde{\chi}_\alpha$ maps J_α onto I_α, and is strictly monotone decreasing.

Let $p_j \in J_\alpha, w_j \in I_\alpha, (j = 1, 2)$, such that $w_j = \tilde{\chi}_\alpha(p_j)$, then we get the ellipse equation as

$$p_j^2 + w_j^2 + 2\tilde{c}_2 p_j w_j = \tilde{c}_1^2$$

whose graph is the thick line part of Figure [2.2],

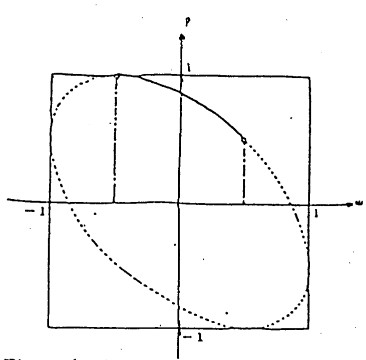

Figure [2.2]

To exploit the linearization effect of $\tilde{\chi}_\alpha$, it is convenient to introduce the space W_α of functions of the double type defined by

$$W_\alpha^{double} = \tilde{\chi}_\alpha(\wp_\alpha^{double}).$$

The precise meaning of this definition is

$$W_\alpha^{double} = \{w = (w_1, w_2); w_j = \tilde{\chi}_\alpha(p_j), \quad j = 1, 2, \quad p = (p_1, p_2) \in \wp_\alpha^{double}\}.$$

Since

$$p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1), \qquad \text{[non linear relation in } p_1 \text{ and } p_2\text{]}$$

We get,

$$w_2(\theta + \pi - \alpha) = -w_1(\theta). \qquad \text{[linear relation in } w_1 \text{ and } w_2\text{]}$$

This simplifies much calculations especially when we try to obtain extremal values of the oriented length $L[p]$ and the oriented area $A[p]$.

§2.4. Duality relation and its linearization

Let (Λ_1, Λ_2) be admissible curves inside C, and θ be given. For every point P on the director circle C, there exists a unique pair of values (θ_1, θ_2), such that P is the target point of l_{θ_1}, which is called the generalized tangent line incoming to P, and P is the source point of l_{θ_2}, which is called generalized tangent line outgoing from P.

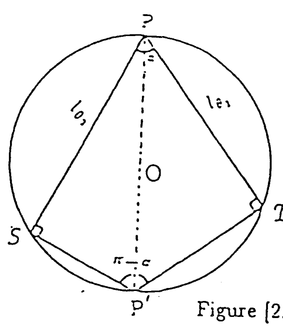

Figure [2.3]

The two edge lines $l_{\theta_1}, l_{\theta_2}$ of the sector intersect with C at points S and T, respectively. Consider the point P' antipodal to P, i.e. the other end of the diameter of C passing P. Put $\hat{\alpha} = \pi - \alpha$. Then the angle $SP'T$ is equal to $\hat{\alpha}$. Now let P move around on the circle C, then P' also moves around on C. We obtain the new sector of the angle $\hat{\alpha}$ with vertex P' and with the edge lines passing through points S and T respectively.(See the Figure [2.3]).

Definition 2.4.1. Let (Λ_1, Λ_2) be a pair of admissible curves inside C. For every point $P \in C$, let $l_1(P)$ be the incoming generalized tangent line to Λ_1, $l_2(P)$ be the outgoing generalized tangent line to Λ_2. α be an angle $0 < \alpha < \pi$, suppose that $l_1(P)$ and $l_2(P)$ span the angle α for all $P \in C$. Then we call the pair (Λ_1, Λ_2) a curve of constant angle α of double type. Denote by X_α^{double} be the totality of those curves. With the notation $\hat{\alpha} = \pi - \alpha$, and for $(p_1, p_2) \in \wp_\alpha^{double}$, we denote the duality mapping by $\wedge$,

$$\wedge : \wp_\alpha^{double} \ni (p_1, p_2) \longleftrightarrow (\hat{p}_1, \hat{p}_2) \in \wp_{\hat{\alpha}}^{double};$$

with the following properties:

(i) $\hat{p} = (\hat{p}_1, \hat{p}_2)$ is given by the formulas:

$$\begin{cases} \hat{p}_1(\theta) = \sqrt{1 - p_2(\theta - \frac{\pi}{2})^2} \\ \hat{p}_2(\theta) = \sqrt{1 - p_1(\theta + \frac{\pi}{2})^2} \end{cases}. \tag{2.4.1}$$

(ii) $\wedge$ is a bijective mapping such that $\wedge$ is the identity on $\wp_\alpha^{double}$, i.e. $\hat{\hat{p}} = p$.

The pair of curves $\hat{\Lambda} = (\hat{\Lambda}_1, \hat{\Lambda}_2)$ ($\hat{\Lambda}_j$ being the curve defined by generator $\hat{p}_j$) would be the dual of Λ. Schematically,

$$\wedge : X_\alpha^{double} \ni (\Lambda_1, \Lambda_2) \longleftrightarrow (\hat{\Lambda}_1, \hat{\Lambda}_2) \in X_{\hat{\alpha}}^{double}.$$

Remark 2.4.2. The bijectivity of duality mapping can be stated schematically

$$\wedge : W_\alpha^{double} \ni (w_1, w_2) \longleftrightarrow (\hat{w}_1, \hat{w}_2) \in W_{\hat{\alpha}}^{double}.$$

Now we state the duality relation in the linearized form.

Theorem 2.4.3. $\hat{w}_1, \hat{w}_2$ are calculated as follows,

$$\begin{cases} \hat{w}_1(\theta) = -w_2(\theta - \frac{\pi}{2}) \\ \hat{w}_2(\theta) = -w_1(\theta + \frac{\pi}{2}). \end{cases} \tag{2.4.2}$$

And $\hat{w}_j = w_j \quad (j = 1, 2)$.

Summarizing the arguments above on the linearization effect of $\tilde{\chi}_\alpha$ on the relations of generators involving χ_α, we give a scheme with $p_j(\theta) \in J_\alpha$, $\hat{p}_j(\theta) \in J_{\hat{\alpha}}$, $w_j(\theta) \in I_\alpha$, and $\hat{w}_j(\theta) \in I_{\hat{\alpha}}$ as follows:

$$\begin{array}{ccc} \alpha & & \hat{\alpha} \\ (p_1, p_2) & \overset{\wedge}{\longleftrightarrow} & (\hat{p}_1, \hat{p}_2) \\ \updownarrow \tilde{\chi}_\alpha & & \updownarrow \tilde{\chi}_{\hat{\alpha}} \\ (w_1, w_2) & \overset{\wedge}{\longleftrightarrow} & (\hat{w}_1, \hat{w}_2) \end{array}$$

Noting that for $J_\alpha = (0, \sin\alpha)$, we have $J_{\hat{\alpha}} = (-\cos\hat{\alpha}, 1) = (\cos\alpha, 1)$. It implies that $J_{\hat{\alpha}}$ is different from J_α in general. However, we have always $I_{\hat{\alpha}} = I_\alpha$, because $\Omega_\alpha = Min(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}) = Min(\sin\frac{\hat{\alpha}}{2}, \cos\frac{\hat{\alpha}}{2})$, therefore $(-\Omega_\alpha, \Omega_\alpha) = (-\Omega_{\hat{\alpha}}, \Omega_{\hat{\alpha}})$.

§3. The oriented length and the oriented area

Our main purpose of this chapter is to introduce structural properties of the oriented length $L[p]$ of the generators p of the admissible curves of double type. We shall obtain the concrete extremal values of them. In section 3.3 easy calculators show the oriented length $L[\hat{p}]$ of the dual curves. A similar argument applies to the oriented area $A[p]$ of the generators of the admissible curves of double type which will be treated in separate sections.

§3.1. The orientated length

In section 2.2, we have discussed the admissible curves Λ with generator $p \in C(\mathbf{T})$,then by the formula (2.2.3)

$$p + \ddot{p} = -\dot{x}\sin\theta + \dot{y}\cos\theta$$

we see $< (\dot{x}(\theta), \dot{y}(\theta)), \tilde{e} >= \pm|(\dot{x}(\theta), \dot{y}(\theta))|$, if $x(\theta), y(\theta)$ are differentiable.

On the other hand, we have already defined the non-oriented length of admissible curves in section 2.1. For a given rectifiable curve Λ, the non-oriented length of Λ has been defined by

$$|L|[\Lambda] = \sup_{\Delta} |L|[\Lambda_\Delta].$$

However, for generator $p \in C^1(\mathbf{T})$, these observations give us that for $x, y \in C(\mathbf{T})$,then heuristically assuming the differentiability of $x(\theta), y(\theta)$,.we get

$$\pm\sqrt{\dot{x}(\theta)^2 + \dot{y}(\theta)^2} = p(\theta) + \ddot{p}(\theta),$$

$$ds = \sqrt{\dot{x}(\theta)^2 + \dot{y}(\theta)^2}d\theta = \pm(p(\theta) + \ddot{p}(\theta))d\theta. \tag{3.1.1}$$

Thus, we define the oriented length of Λ by the following formula

$$\begin{aligned} L[\Lambda] &\stackrel{\text{def}}{=} \text{total mass of } \mathbf{T} \text{ measured by } \quad p(\theta) + \ddot{p}(\theta) \\ &= (p + \ddot{p})[1] \\ &= \int_0^{2\pi} p(\theta) d\theta, \end{aligned}$$

since $\ddot{p}[1] = \dot{p}[\dot{1}] = -\dot{p}[0] = 0$.

Definition 3.1.1. For an admissible curve Λ with generator $p = (p_1, p_2) \in \wp_\alpha^{double}$, the oriented length $L[p]$ is defined by:

$$L[p] \stackrel{\text{def}}{=} \frac{1}{2}(L[p_1] + L[p_2]).$$

§3.2. Extremal values of $L[p]$

In this section, we intend to discuss the concrete extremal values of $L[p]$. The idea is to apply the linearization to the relation of generator involving $\tilde{\chi}_\alpha$ which is given in section 2.3.

For $p = (p_1, p_2) \in \wp_\alpha^{double}$, we have

$$p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1(\theta)).$$

By the definition of W_α^{double}, for $(w_1, w_2) \in W_\alpha^{double}$, we have

$$w_j(\theta) = \tilde{\chi}_\alpha(p_j(\theta)), \qquad j = 1, 2.$$

Then

$$w_2(\theta + \pi - \alpha) = -w_1(\theta).$$

Hence,

$$p_1(\theta) = \tilde{\chi}_\alpha(w_1(\theta)) = \tilde{c}_1\sqrt{1 - w_1(\theta)^2} - \tilde{c}_2 w_1(\theta);$$
$$p_2(\theta + \pi - \alpha) = \tilde{\chi}_\alpha(w_2(\theta + \pi - \alpha)) = \tilde{\chi}_\alpha(-w_1(\theta))$$
$$= \tilde{c}_1\sqrt{1 - w_1(\theta)^2} + \tilde{c}_2 w_1(\theta).$$

Thus,

$$L[p_1] = \int_0^{2\pi} p_1(\theta)d\theta = \int_0^{2\pi} [\tilde{c}_1\sqrt{1 - w_1(\theta)^2} - \tilde{c}_2 w_1(\theta)]d\theta.$$

Now we review that if $f(\theta)$ is a function with period 2π, then

$$\int_0^{2\pi} f(\theta + \beta)d\theta = \int_0^{2\pi} f(\theta)d\theta.$$

Since $p_2(\theta)$ has 2π as a period and Lebesgue measure $d\theta$ is translation in variant, we can also obtain

$$L[p_2] = \int_0^{2\pi} p_2(\theta)d\theta$$
$$= \int_0^{2\pi} p_2(\theta + \pi - \alpha)d\theta$$
$$= \int_0^{2\pi} [\tilde{c}_1\sqrt{1 - w_1(\theta)^2} + \tilde{c}_2 w_1(\theta)]d\theta.$$

Then we have following:

Theorem 3.2.1. For an admissible curve Λ with generator $p = (p_1, p_2) \in \wp_\alpha^{double}$, the oriented length $L[p]$ is given by the formula:

$$L[p] = \frac{1}{2}(L[p_1] + L[p_2])$$
$$= \tilde{c}_1 \int_0^{2\pi} \sqrt{1 - w_1(\theta)^2}d\theta. \qquad (3.2.1)$$

Obviously, the maximum of the right-hand side of the quantity (3.2.1) is attained if we take $w_1 = 0$, that is,

$$\max_{p \in \wp_\alpha^{double}} = 2\pi\tilde{c}_1.$$

However, to minimize this quantity (4.2.1) is delicated, since $w_1 \in I_\alpha = (-\Omega_\alpha, \Omega_\alpha)$, where $\Omega_\alpha = Min(\tilde{c}_1(\alpha), \tilde{c}_2(\alpha))$, therefore we need to separate two cases.

Case 1. If $0 < \alpha \leq \frac{\pi}{2}$; then $\tilde{c}_2 \geq \tilde{c}_1$. So $\Omega_\alpha = \tilde{c}_1(\alpha)$. Letting $w_1 \to \tilde{c}_1(\alpha)$, then we have

$$\begin{aligned}\inf_{p \in \wp_\alpha^{double}} L[p] &= \tilde{c}_1 \int_0^{2\pi} \sqrt{1 - {\Omega_\alpha}^2}\, d\theta \\ &= \tilde{c}_1 \int_0^{2\pi} \sqrt{1 - \tilde{c}_1^2}\, d\theta \\ &= 2\pi\tilde{c}_1\tilde{c}_2.\end{aligned}$$

Case 2. If $\frac{\pi}{2} \leq \alpha < \pi$; then $\tilde{c}_1 \geq \tilde{c}_2$. So $\Omega_\alpha = \tilde{c}_2(\alpha)$. Letting $w_1 \to \tilde{c}_2(\alpha)$, then we have

$$\begin{aligned}\inf_{p \in \wp_\alpha^{double}} L[p] &= \tilde{c}_1 \int_0^{2\pi} \sqrt{1 - \Omega_\alpha^2}\, d\theta \\ &= \tilde{c}_1 \int_0^{2\pi} \sqrt{1 - \tilde{c}_2^2}\, d\theta = 2\pi\tilde{c}_1^2.\end{aligned}$$

§3.3. Relations between the oriented length $L[p]$ and $L[\hat{p}]$

It is interesting to compare $L[p]$ and that of the dual curve $L[\hat{p}]$. Now we give their relations in the following.

Theorem 3.3.1. Let $\hat{p}$ be duality generator of p. Then

$$\frac{L[p]}{L[\hat{p}]} = \text{constant}.$$

§3.4. The oriented area

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}, p$ be a generator of admissible curve, $p \in C^1$ and $\dot{p}$ be bounded variation. By Leibniz formula we have

$$(p_1\dot{p}_1)^{\cdot} = \dot{p}_1^2 + p\ddot{p}_1. \tag{3.4.1}$$

Consider the formula $p_1(p_1 + \ddot{p}_1)$, by (3.4.1) we get

$$p_1(p_1 + \ddot{p}_1) = p_1^2 + p_1\ddot{p}_1 = p_1^2 - \dot{p}_1^2 + (p_1\dot{p}_1)^{\cdot}.$$

Heuristically, the oriented area of Λ_1 would be $\frac{1}{2}\int\limits_0^{2\pi} p_1(\theta)ds$.

Thus we define

$$\begin{aligned} A[p_1] &= \frac{1}{2}p_1(p_1 + \ddot{p}_1)[1] \\ &= \frac{1}{2}(p_1^2 - \dot{p}_1^2 + (p_1\dot{p}_1)^{\cdot})[1] \\ &= \frac{1}{2}\int\limits_0^{2\pi} p_1^2 - \dot{p}_1^2 d\theta. \end{aligned}$$

Definition 3.4.1. For an admissible curve Λ with generator $p = (p_1, p_2) \in \wp_\alpha^{double}$, the oriented area $A[p]$ is defined by

$$A[p] \overset{\text{def}}{=} \frac{1}{2}(A[p_1] + A[p_2]).$$

Thus, the oriented area $A[p]$ is given by the formula

$$A[p] = \frac{1}{4}\int\limits_0^{2\pi} (p_1^2 + p_2^2 - \dot{p}_1^2 - \dot{p}_2^2)d\theta. \tag{3.4.2}$$

§3.5. Extramal values of the oriented area

To maximize the functional $A[p]$ described as formula (3.4.2), we use Fourier series expansion in the Hilbert space $\mathbf{H} = \mathbf{L}^2[0, 2\pi]$ or the same thing $\mathbf{L}^2(\mathbf{T})$. Since

$$e_0 = \frac{1}{\sqrt{2\pi}}, \cdots, e_{2k+1} = \frac{1}{\sqrt{\pi}}\cos k\theta, e_{2k+2} = \frac{1}{\sqrt{\pi}}\sin k\theta, \quad k = 1, 2, \cdots$$

constitute a complete orthonomal system in $\mathbf{L}^2(\mathbf{T})$, every element f of $\mathbf{H}$ can be expanded as a strongly converging series

$$f = a_0 + \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta$$

and

$$f = \sum_{k=1}^{\infty} c_k e_k,$$

($<, >$ indicates the inner product).

And the Parseval equality gives

$$\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2.$$

Applying this to the generator p_1, we have

$$p_1 = a_0^{(1)} + \sum_{k=1}^{\infty} a_k^{(1)} \cos k\theta + b_k^{(1)} \sin k\theta$$

and

$$\|p_1\|^2 = 2\pi a_0^{(1)2} + \pi \sum_{k=1}^{\infty} a_k^{(1)2} + b_k^{(1)2}. \tag{3.5.1}$$

In the space D' we can differentiate both sides term by term, thus we get

$$\dot{p_1}(\theta) = \sum_{k=1}^{\infty} -k a_k^{(1)} \sin k\theta + k b_k^{(1)} \cos k\theta. \tag{3.5.2}$$

Since $\dot{p}_1 \in \mathbf{H}$, this expansion coincides the usual Fourier series expansion in $\mathbf{H}$ because the topology of D' is weaker than that of $\mathbf{H}$. Thus, according to the Parseval equality, we have

$$\|\dot{p}_1\|^2 = \pi \sum_{k=1}^{\infty} k^2 (a_k^{(1)2} + b_k^{(1)2}).$$

Hence, we have

$$\begin{aligned} A[p_1] &= \frac{1}{2}\int_0^{2\pi} p_1^2 - \dot{p}_1^2 d\theta \\ &= \frac{1}{2}(\|p_1\|^2 - \|\dot{p}_1\|^2) \\ &= \frac{1}{2}\{2\pi a_0^{(1)2} + \pi \sum_{k=1}^{\infty}(1-k^2)a_k^{(1)2} + b_k^{(1)2}\}. \end{aligned}$$

Since the length of the curve with generator $p_1(\theta)$ is

$$L[p_1] = \int_0^{2\pi} p_1(\theta)d\theta = 2\pi a_0^{(1)},$$

therefore,

$$\begin{aligned} A[p_1] &= \frac{1}{2}\{\frac{1}{2\pi}L[p_1]^2 + \pi \sum_{k=1}^{\infty}(1-k^2)a_k^{(1)2} + b_k^{(1)2}\} \\ &\leq \frac{1}{4\pi}L[p_1]^2. \end{aligned}$$

So we obtain the following formula

$$A[p] = \frac{1}{2}(A[p_1] + A[p_2]) \leq \frac{1}{8\pi}\{L[p_1]^2 + L[p_2]^2\}. \tag{3.5.3}$$

Here the equality holds, if and only if Λ_1, Λ_2 are concentric circles with C, according to the following

Lemma 3.5.1. Let Λ_1, Λ_2 be circles in $C, x \in \Lambda_1, y \in \Lambda_2, (p_1, p_2)$ be generators of curves of constant angle. Then Λ_1, Λ_2, C are concentric circles.

Let $p_1(\theta) = d_1, p_2(\theta) = d_2$, and d_1, d_2 be constants such that $d_2 = \chi_\alpha(d_1)$.Then by the formula (3.5.3), we have

$$\begin{aligned} A[p] &\leq \frac{1}{8\pi}\{(2\pi d_1)^2 + (2\pi d_2)^2\} \\ &= \frac{1}{2}[\pi d_1{}^2 + \pi d_2{}^2] \\ &= \frac{\pi}{2}(d_1{}^2 + d_2{}^2). \end{aligned}$$

Thus the problem to maximize $A[p]$ is reduced to the choice of constants d_1, d_2. Since $d_2 = \chi_\alpha(d_1) = c_1\sqrt{1 - d_1{}^2} - c_2 d_1$, we have

$$d_1{}^2 + d_2{}^2 = c_1{}^2 - 2c_2 d_1 d_2.$$

Now we treat the problem in three different cases.

Case 1. $\alpha = \frac{\pi}{2}$. Then $c_2 = 0$, $d_1{}^2 + d_2{}^2 = c_1{}^2$. Thus $A[p] \leq \frac{\pi}{2}c_1{}^2$. This $\frac{\pi}{2}c_1{}^2$ is the maximum value of $A[p]$ in this case.

Case 2. $\frac{\pi}{2} < \alpha < \pi$. Then $c_2 \leq 0$. In this case the diameter of the ellipse $d_1{}^2 + d_2{}^2 = c_1{}^2 + (-2c_2)d_1 d_2$ is on the diagonal line in (d_1, d_2)-plane. To maximize $d_1{}^2 + d_2{}^2$, it is the same to maximize $d_1 d_2$. Now consider the family of hyperbolas $d_1 d_2 = c$ (c :parameter). On the ellipse $d_1{}^2 + d_2{}^2 = c_1{}^2 + (-2c_2)d_1 d_2$, the value c attains its maximum on the edge point of the diameter which lies on the diagonal $d_1 = d_2$. Thus $A[p]$ attains the maximum at $d_1 = d_2 = \tilde{c}_1$. And

$$\max_{p \in \wp_\alpha^{double}} A[p] = \pi \tilde{c}_1^2.$$

Case 3. $0 < \alpha < \frac{\pi}{2}$. Then $c_2 > 0$. Thus to maximize $d_1{}^2 + d_2{}^2 = c_1{}^2 - 2c_2 d_1 d_2$, we minimize $d_1 d_2$. Now consider the family of hyperbolas $d_1 d_2 = c$ (c :parameter). To minimize $d_1 d_2$ on the ellipse, hyperbolas $d_1 d_2 = c$ should, in the limit, passes through the two points on the axes in the first quadrant. Therefore $A[p]$ does not attain the maximum. But its supremum value is attained as the limit when $(d_1, d_2) \to (0, c_1)$ or $\to (c_1, 0)$. And

$$\sup_{p \in \wp_\alpha^{double}} A[p] = \frac{\pi}{2} c_1^2.$$

Case 2.

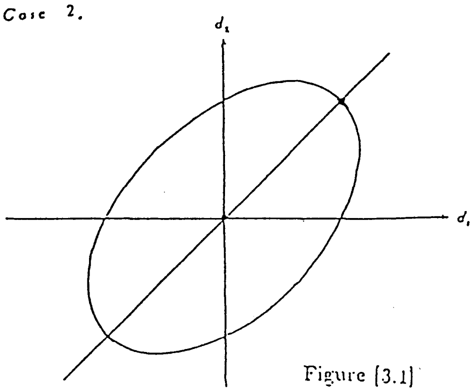

Figure [3.1]

Case 3.

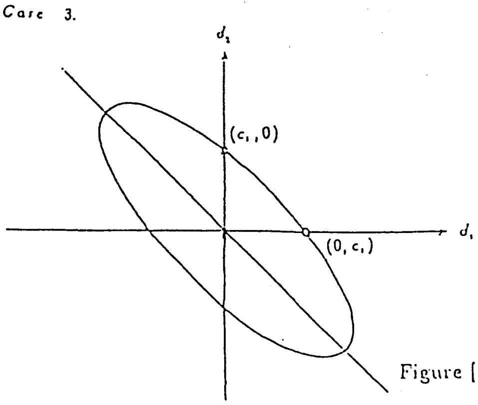

Figure [3.2]

Now we treat the problem of minimizing $A[p]$. For that purpose we exploit the linearization effect of the modified characteristic function $\tilde{\chi}_\alpha$. We put

$$w_j(\theta) = \tilde{\chi}_\alpha(p_j(\theta)) \qquad (j = 1, 2).$$

Then we have

$$p_j{}^2 = \{\tilde{c}_1 \sqrt{1 - w_j{}^2} - \tilde{c}_2 w_j\}^2, \tag{3.5.4}$$

and

$$\begin{aligned} \dot{p}_j^2 &= \{-\tilde{c}_1 \frac{w_j}{\sqrt{1 - w_j{}^2}} - \tilde{c}_2\}^2 \dot{w}_j^2 \\ &= \{\tilde{c}_1 w_j + \tilde{c}_2 \sqrt{1 - w_j{}^2}\}^2 \frac{\dot{w}_j^2}{1 - w_j{}^2}. \end{aligned} \tag{3.5.5}$$

Since

$$p_2(\theta + \pi - \alpha) = \chi_\alpha(p_1),$$

the linearization effect of $\tilde{\chi}_\alpha$ gives

$$w_2(\theta + \pi - \alpha) = -w_1(\theta).$$

Putting $\pi - \alpha = \hat{\alpha}$, we get

$$A[p] = \frac{1}{4} \int_0^{2\pi} (p_1^2(\theta) - \dot{p}_1^2(\theta) + p_2^2(\theta + \hat{\alpha}) - \dot{p}_2^2(\theta + \hat{\alpha})) d\theta \tag{3.5.6}$$

since the Lebesgue measure $d\theta$ is translation invariant and p_2 has 2π as a period.

To simplify the notations, we put

$$w(\theta) = w_1(\theta),$$

then we have

$$-w(\theta) = w_2(\theta + \hat{\alpha}),$$

so that the equation (3.5.4),(3.5.5) take the forms

$$\begin{cases} p_1(\theta)^2 = \{\tilde{c}_1\sqrt{1 - w(\theta)^2} - \tilde{c}_2 w(\theta)\}^2, \\ \dot{p}_1(\theta)^2 = \{\tilde{c}_1 w(\theta) + \tilde{c}_2\sqrt{1 - w(\theta)^2}\}^2 \dfrac{\dot{w}(\theta)^2}{1 - w(\theta)^2}. \end{cases}$$

$$\begin{cases} p_2(\theta + \hat{\alpha})^2 = \{\tilde{c}_1\sqrt{1 - w(\theta)^2} + \tilde{c}_2 w(\theta)\}^2, \\ \dot{p}_2(\theta + \hat{\alpha})^2 = \{-\tilde{c}_1 w(\theta) + \tilde{c}_2\sqrt{1 - w(\theta)^2}\}^2 \dfrac{\dot{w}(\theta)^2}{1 - w(\theta)^2}. \end{cases}$$

Therefore, we have

$$A[p] = \frac{1}{2}\int_0^{2\pi} [\tilde{c}_1^2 - (\tilde{c}_1^2 - \tilde{c}_2^2)w^2] - [\tilde{c}_2^2 + (\tilde{c}_1^2 - \tilde{c}_2^2)w^2]\frac{\dot{w}^2}{1 - w^2} d\theta. \qquad (3.5.7)$$

To minimize this quantity, we need the following

Lemma 3.5.2. (Explicit construction of extremizing sequence) Consider a functional of the form

$$\Phi[w] = \frac{1}{2}\int_0^{2\pi} F_1(w) - F_2(w)\frac{\dot{w}^2}{1 - w^2} d\theta$$

on the function space $W_\alpha = \{\tilde{\chi}_\alpha(p_1(\theta)); (p_1, p_2) \in \wp_\alpha^{double}\}$, where F_1, F_2 are non-negative functions of w. Note that the inequality

$$0 \leq \frac{\dot{w}^2}{1 - w^2} < 1.$$

Then, we have the results.

(1) If $p = (p_1, p_2) \in \wp_\alpha^{\substack{double \\ symmetric}}$, and $\alpha/\pi \notin \mathbf{Q}$, then the elements of $\wp_\alpha^{\substack{double \\ symmetric}}$ are all constant functions. Thus the same holds for all $w \in W_\alpha$. Therefore $\dfrac{\dot{w}^2}{1-w^2} = 0$ and the function $\Phi[w]$ takes the form

$$\Phi[w] = \Phi[c] = \frac{1}{2}\int_0^{2\pi} F_1(c)d\theta = 2\pi F_1(c),$$

where c is a suitable constant.

(2) Except the case (1), namely in the case that $p = (p_1, p_2) \notin \wp_\alpha^{\substack{double \\ symmetric}}$ or $\alpha/\pi \in \mathbf{Q}$, we can explicitly construct a sequence of functions $w_k \in W_\alpha$, $(k = 1, 2, 3, \cdots)$ such that

(i) For every constant $c \in \bar{I}_\alpha = [-\Omega_\alpha, \Omega_\alpha]$, $w_k \to c$, uniformly, (as $k \to \infty$).

(ii) $\dfrac{\dot{w}(\theta)^2}{1-w(\theta)^2} \to 1$, almost everywhere in $[0, 2\pi]$, (as $k \to \infty$).

To minimize the quantity given by the right hand side of (3.5.7), we firstly assume that the condition (2) of the above lemma be fulfiled, then we need to separate two cases.

Case 1. If $\frac{\pi}{2} \leq \alpha < \pi$. Then $\tilde{c}_1 \geq \tilde{c}_2$. Taking the inequality $0 \leq \dfrac{\dot{w}^2}{1-w^2} < 1$ into account, we get that

$$A[p] > \frac{1}{2}\int_0^{2\pi} (\tilde{c}_1^2 - \tilde{c}_2^2)(1 - 2w^2)d\theta.$$

Since $w^2 < \Omega_\alpha^2 = \tilde{c}_2^2$, we get further

$$A[p] > \frac{1}{2}\int_0^{2\pi} (\tilde{c}_1^2 - \tilde{c}_2^2)(1 - 2\tilde{c}_2^2)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (\tilde{c}_1^2 - \tilde{c}_2^2)^2 d\theta = \pi c_2{}^2.$$

However, if we take an extremalizing sequence w_k such that $w_k(\theta) \to \Omega_\alpha$ uniformly, we get the result

$$\inf_{p \in \wp_\alpha^{double}} A[p] = \pi {c_2}^2.$$

Case 2. If $0 < \alpha \leq \frac{\pi}{2}$. Then $\tilde{c}_2 \geq \tilde{c}_1$. We rewrite (3.5.7) for $A[p]$ in the form

$$A[p] = \frac{1}{2}\int_0^{2\pi} [\tilde{c}_1^2 + (\tilde{c}_2^2 - \tilde{c}_1^2)w^2] + (\tilde{c}_2^2 - \tilde{c}_1^2)w^2\frac{\dot{w}^2}{1-w^2} - \tilde{c}_2^2\frac{\dot{w}^2}{1-w^2}d\theta.$$

Now it is clear that this quantity tends to its infimum, when we employ an extremizing sequence w_k such that $w_k(\theta) \to 0$ uniformly as $k \to \infty$. Thus we get the result

$$\inf_{p \in \wp_\alpha^{double}} A[p] = \frac{1}{2}\int_0^{2\pi} (\tilde{c}_1^2 - \tilde{c}_2^2)d\theta = -\pi c_2.$$

Secondly, we assume that the condition (1) of the above lemma is fulfiled. In this case, putting $w(\theta) = d \in I_\alpha$, ($d$: constant), we get

$$\begin{aligned} A[p] &= \frac{1}{2}\int_0^{2\pi} [\tilde{c}_1^2 - (\tilde{c}_1^2 - \tilde{c}_2^2)d^2]\mathrm{d}\theta \\ &= \pi[\tilde{c}_1^2 - (\tilde{c}_1^2 - \tilde{c}_2^2)d^2]. \end{aligned} \tag{3.5.8}$$

Case 1′. If $\frac{\pi}{2} \leq \alpha < \pi$. Then letting $d \to \tilde{c}_2$, we get

$$\begin{aligned} \inf_{p \in \wp_\alpha^{double}} A[p] &= \pi[\tilde{c}_1^2 - (\tilde{c}_1^2 - \tilde{c}_2^2)\tilde{c}_2^2] \\ &= \pi[\tilde{c}_1^2(1 - \tilde{c}_2^2) + \tilde{c}_2^4] \\ &= \pi(1 - \frac{1}{2}{c_1}^2) \\ &= \frac{\pi}{2}(1 + {c_2}^2). \end{aligned}$$

Case 2′. If $0 < \alpha \leq \frac{\pi}{2}$. Then putting $d = 0$, we get

$$\min_{p \in \wp_{\alpha}^{double}} A[p] = \pi \tilde{c}_1^2.$$

Thus, the minimizing arguments of $A[p]$ are completed.

§3.6. Relations between the area $A[p]$ and the area $A[\hat{p}]$

We may suppose without loss of generality $\frac{\pi}{2} \leq \alpha < \pi$. This condition is convenient for discussions, since $A[p] > 0$, if $p \in \wp_{\alpha}^{double}$. Then $\hat{p} \in \wp_{\hat{\alpha}}^{double}, \hat{\alpha} = \pi - \alpha, 0 < \hat{\alpha} \leq \frac{\pi}{2}$. We have

$$A[\hat{p}] = \frac{1}{2} \int_0^{2\pi} [\tilde{c}_2^2 - (\tilde{c}_2^2 - \tilde{c}_1^2)w^2] - [\tilde{c}_1^2 + (\tilde{c}_2^2 - \tilde{c}_1^2)w^2] \frac{\dot{w}^2}{1 - w^2} d\theta. \tag{3.6.1}$$

Comparing $A[p]$ and $A[\hat{p}]$, we get the result,

Theorem 3.6.1. If $\alpha \in \mathbf{Q}$, then for $\forall m \in \mathbf{R}, m > 0$, we have

$$\frac{|A[\hat{p}]|^m}{|A[p]|^m} = \text{not constant}.$$

Now we consider the quantity $A[p] + A[\hat{p}]$. By formulas (3.5.7) and (3.6.1), we get

$$\begin{aligned} 0 &< A[p] + A[\hat{p}] \\ &= \frac{1}{2} \int_0^{2\pi} (1 - \frac{\dot{w}^2}{1 - w^2}) d\theta \\ &= \pi - \frac{1}{2} \int_0^{2\pi} \frac{\dot{w}^2}{1 - w^2} d\theta \leq \pi. \end{aligned}$$

Therefore, we obtain

$$\max_{p \in \wp_{\alpha}^{double}} (A[p] + A[\hat{p}]) = \pi;$$

$$\inf_{p\in\wp_\alpha^{double}}(A[p]+A[\hat{p}])=0.$$

If we consider the quantity $A[p]-A[\hat{p}]$, then we see that

$$A[p]-A[\hat{p}]=\frac{(\tilde{c}_1^2-\tilde{c}_2^2)}{2}\int_0^{2\pi}[(1-2w^2)+(1-2w^2)\frac{\dot{w}^2}{1-w^2}]d\theta$$
$$=(\pi\tilde{c}_1^2-\pi\tilde{c}_2^2)+\frac{(\tilde{c}_1^2-\tilde{c}_2^2)}{2}\int_0^{2\pi}[(1-2w^2)\frac{\dot{w}^2}{1-w^2}-2w^2]d\theta,$$

then

$$A[p]-A[\hat{p}]-(\pi\tilde{c}_1^2-\pi\tilde{c}_2^2)$$
$$=\frac{(\tilde{c}_1^2-\tilde{c}_2^2)}{2}\int_0^{2\pi}[(1-2w^2)\frac{\dot{w}^2}{1-w^2}-2w^2]d\theta. \qquad (3.6.2)$$

If $\alpha\in\mathbf{Q}$, by using the Lemma 3.5.2, we can obtain the concrete extremal values of $A[p]-A[\hat{p}]$.

If we take an extremalizing sequence w_k such that $w_k(\theta)\to 0$, uniformly, as $k\to\infty$, and taking the inequality $0\le\frac{\dot{w}^2}{1-w^2}<1$ into account, by (3.6.2), we get

$$A[p]-A[\hat{p}]-(\pi\tilde{c}_1^2-\pi\tilde{c}_2^2)<\pi\tilde{c}_1^2-\pi\tilde{c}_2^2.$$

Thus, we obtain

$$\sup_{p\in\wp_\alpha^{double}}(A[p]-A[\hat{p}])=2(\pi\tilde{c}_1^2-\pi\tilde{c}_2^2)=-2\pi c_2.$$

However, if we take an extremalizing sequence w_k such that $w_k(\theta)\to\Omega_\alpha$, uniformly, as $k\to\infty$. Since $\frac{\pi}{2}\le\alpha<\pi$, then $\tilde{c}_1\ge\tilde{c}_2$, $\Omega_\alpha^2=Min(\tilde{c}_1^2,\tilde{c}_2^2)=\tilde{c}_2^2$. We get

$$A[p]-A[\hat{p}]-(\pi\tilde{c}_1^2-\pi\tilde{c}_2^2)>-2\pi(\tilde{c}_1^2-\tilde{c}_2^2)\tilde{c}_2^2,$$

therefore,

$$A[p] - A[\hat{p}] > \pi(\tilde{c}_1^2 - \tilde{c}_2^2)(1 - 2\tilde{c}_2^2) = \pi c_2^2(\alpha).$$

Thus, we obtain

$$\inf_{p \in \wp_\alpha^{double}} (A[p] - A[\hat{p}]) = \pi {c_2}^2(\alpha).$$

Remark 3.6.2. Assume that $p = (p_1, p_2) \in \wp_\alpha^{\substack{double \\ symmetric}}$, and $\alpha/\pi \notin \mathbf{Q}$. In this case, the condition (2) of Lemma 3.5.2 is fulfiled. Then putting $w(\theta) = d \in I_\alpha$, by (3.5.7) and (3.6.1) we have

$$A[\hat{p}] = \frac{1}{2}\int_0^{2\pi}[\tilde{c}_2^2 - (\tilde{c}_2^2 - \tilde{c}_1^2)d^2]\mathrm{d}\theta = \pi[\tilde{c}_2^2 - (\tilde{c}_2^2 - \tilde{c}_1^2)d^2],$$

and

$$A[p] = \pi[\tilde{c}_1^2 - (\tilde{c}_1^2 - \tilde{c}_2^2)d^2].$$

Hence

$$A[\hat{p}] + A[p] = \pi = \text{ constant}$$

holds.

Acknowledgement

The author would like to thank Professor Shigetake Matsuura for his many useful suggestions and continuous encouragement during the present work.

References

[1] Matsuura,S., $I \sim VII$, Mathematical Seminar (1981, August ~ 1982,Feb.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[2] ———, $VIII \sim XI$, Mathematical Seminar (1982,July ~ 1982, Oct.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[3] ———, XII, Mathematical Seminar (1982,Dec.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[4] ———, $XIII$, Mathematical Seminar (1983,Feb.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[5] ———, XVI, Mathematical Seminar (1983,Apr.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[6] ———, $XV \sim XVI$, Mathematical Seminar (1983,June ~ 1983, July) (Nihon Hyoronsha Co.Ltd) in Japanese.

[7] ———, $XVII \sim XVIII$, Mathematical Seminar (1983,Sept. ~ 1983,Oct.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[8] ———, $XIX \sim XXIII$, Mathematical Seminar (1984,Jan. ~ 1984,May) (Nihon Hyoronsha Co.Ltd) in Japanese.

[9] ———, $XXIV \sim XXVII$, Mathematical Seminar (1984,Aug. ~ 1984,Nov.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[10] ———, $XXVIII$, Mathematical Seminar (1985,Jan.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[11] ———, $XXIX \sim XXXIV$, Mathematical Seminar (1985,Mar. ~ 1985,Aug.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[12] ———, $XXXV \sim XXXVI$, Mathematical Seminar (1985,Oct. ~ 1985,Nov.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[13] ———, $XXXVII \sim XXXX$, Mathematical Seminar (1986,Jan. ~ 1986,Apr.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[14] ———, $XXXXI \sim XXXXVII$, Mathematical Seminar (1986,

June ~ 1986,Dec.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[15] ———, $XXXXVIII \sim XXXXIX$, Mathematical Seminar (19 87,Feb. ~ 1987,Mar.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[16] ———, L, Mathematical Seminar (1987,Apr.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[17] Matsuura,S. and Kasahara, K., Mathematical Seminar (1988,Mar.) (Nihon Hyoronsha Co.Ltd) in Japanese.

[18] Matsuura,S., Theory of non-convex curves of constant angle (22 expositions) notes taken by M.Kametani (1990).

[19] Matsuura,S., Remarks on non-convex curves of constant angle, Kyoto Univ. RIMS, Kōkyūroku (1991).

[20] Blaschke,W., Über eine Ellipsen eigenschaft und über gewisse Eilinien, Ark. der Maths und Phy. III, Reihe 26 (1917).

[21] Schwartz,L., Théorie des distributions, Hermann, Paris (1966).

[22] Hőrmander,L., The analysis of linear partial differential operators I (2nd ed.), Springer-Verlag (1990).

[23] Bourbaki,N., Espaces vectoriels topologiques I,II, Hermann, Paris (1966).