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Non-collision solutions for a second order singular Hamiltonian system with weak force

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0. Introduction

We study the existence of T-periodic solutions of the following Hamiltonian system:

$$\begin{aligned} &\ddot{q} + V_q(q,t) = 0, \\ &q(t+T) = q(t), \qquad t \in \mathbf{R}, \\ &q(t) \neq 0, \end{aligned} \tag{HS}$$

where $q = (q_1, q_2, \cdots, q_N) \in \mathbf{R}^N$ $(N \geq 3)$ and $V(q,t) : (\mathbf{R}^N \setminus \{0\}) \times \mathbf{R} \to \mathbf{R}$ is a T-periodic (in t) function such that $V(q,t)$, $V_q(q,t) \to 0$ as $|q| \to \infty$ and $V(q,t) \to -\infty$ as $q \to 0$.

Classical solutions of (HS) can be characterized as critical points of functional:

$$I(q) = \int_0^T [\frac{1}{2} |\dot{q}|^2 - V(q,t)]dt \ : \ \Lambda \to \mathbf{R}$$

where

$$\Lambda = \{q(t) \in H^1_{loc}(\mathbf{R}, \mathbf{R}^N); q(t+T) = q(t),\ q(t) \neq 0 \text{ for all } t\}.$$

In case $V(q,t)$ satisfies the *strong force condition* (SF) of Gordon [**Go**]:

(SF) there is a neighborhood Ω of 0 in $\mathbf{R}^N$ and a function $W(q) \in C^1(\Omega \setminus \{0\}, \mathbf{R})$ such that

$$\begin{aligned} &W(q) \to \infty \quad \text{as } q \to 0, \\ &-V(q,t) \geq |W_q(q)|^2 \quad \text{for all } q \in \Omega \setminus \{0\} \text{ and } t, \end{aligned}$$

the functional $I(q)$ satisfies the Palais-Smale compactness condition and we can apply minimax arguments to $I(q)$. Especially under the assumptions of (SF) and

(V1) $V(q,t) \in C^1((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T-periodic in t;

(V2) $V(q,t) < 0$ and $V(q,t)$, $V_q(q,t) \to 0$ as $|q| \to \infty$;

(V3) $-V(q,t) \to \infty$ as $q \to 0$,

Bahri and Rabinowitz [**BR**] introduced a minimax method and obtained the existence of classical solutions (non-collision solutions) of (HS). See also [**AC1**, **Gr1**]. But in case (SF) does not hold, we cannot verify the Palais-Smale compactness condition for $I(q)$ and we cannot apply minimax argument directly to $I(q)$. However, using a suitable approximation argument, Bahri and Rabinowitz [**BR**] proved the existence of *generalized T-periodic solutions*, that may enter the *singularity* 0 (i.e., *collision*) under the conditions (V1)–(V3) (without (SF)).

For the study of the existence of non-collision solutions in case of *weak forces* (i.e., the case where (SF) does not hold), we refer to [**AC3,DGM,DG,C,ST**]. In [**AC3,DGM,DG**], they found critical points of $I(q)$, whose critical values are less than

$$\inf_{q(t)\in\partial\Lambda} I(q) = \inf\{I(q); q \in H^1_{loc}(\mathbf{R},\mathbf{R}^N),\ q(t+T) = q(t) \text{ for all } t \text{ and } q(t) = 0 \text{ for some } t\}.$$

In [**C,ST**], they studied (HS) through minimization problems. They studied the behavior of solutions near collisions (especially [**ST**] studied the Morse index) and they obtained the existence of non-collision solutions.

This work is largely motivated by the works [**BR,C,ST**] and we study the existence of non-collision solutions under the weak force condition through *minimax problem*. We study the following class of weak force potentials; for $0 < \alpha < 2$ we assume the potential $V(q,t)$ is of a form:

(W1) $V(q,t) = -\dfrac{1}{|q|^\alpha} + U(q,t);$

where

(W2) $U(q,t) \in C^2((\mathbf{R}^N\backslash\{0\}) \times \mathbf{R},\mathbf{R})$ is T-periodic in t;

(W3) $|q|^\alpha U(q,t)$, $|q|^{\alpha+1} U_q(q,t)$, $|q|^{\alpha+2} U_{qq}(q,t)$, $|q|^\alpha U_t(q,t) \to 0$ as $|q| \to 0$ uniformly in t.

We remark (V1) and (V3) follow from (W1)–(W3). We also remark (SF) holds if $\alpha \geq 2$.

Our main result is as follows:

Theorem 0.1. *Assume $N \geq 3$, (W1)–(W3), (V2) and $1 < \alpha < 2$. Then (HS) has at least one T-periodic (non-collision) solution.*

In case $0 < \alpha \leq 1$, we cannot show the existence of non-collision solution. However we can estimate the number of collisions of the generalized T-periodic solutions due to Bahri and Rabinowitz [**BR**]. More precisely, we get

Theorem 0.2. *Assume $N \geq 3$, (W1)–(W3), (V2) and $0 < \alpha \leq 1$. Then (HS) has a generalized T-periodic solution, which has at most one collision, i.e., which enters the singularity 0 at most one time in its period T.*

The existence of a non-collision solution of (HS) will be obtained as follows; first we consider modified functional:

$$I_\epsilon(q) = \int_0^T [\frac{1}{2} \mid \dot{q} \mid^2 - V(q,t) + \frac{\epsilon}{\mid q \mid^4}] \, dt \quad \text{for } \epsilon \in (0,1]$$

and obtain critical points $q_\epsilon \in \Lambda$ of $I_\epsilon(q)$. Second, we try to pass to the limit $\epsilon \to 0$. Here we remark $I_\epsilon(q)$ satisfies the strong force condition (SF) for each $\epsilon \in (0,1]$.

The proof of Theorem 0.1 will be given in the following sections; in Section 1, we study the modified functional $I_\epsilon(q)$. We apply the minimax method of Bahri and Rabinowitz [**BR**] and get a critical point $q_\epsilon(t)$ of $I_\epsilon(q)$ for $\epsilon \in (0,1]$. Moreover we obtain the following uniform bounds

$$m \le I_\epsilon(q_\epsilon) \le M, \tag{0.1}$$

$$I_\epsilon'(q_\epsilon) = 0, \tag{0.2}$$

$$\operatorname{index} I_\epsilon''(q_\epsilon) \le N-2, \tag{0.3}$$

for $\epsilon \in (0,1]$, where m, $M > 0$ are independent of ϵ. Here we denote by $\operatorname{index} I_\epsilon''(q_\epsilon)$, the Morse index of $I_\epsilon''(q_\epsilon)$.

From (0.1) and (0.2), we can deduce the uniform H^1-bound for $(q_\epsilon(t))_{\epsilon\in(0,1]}$. Thus we may assume

$$q_{\epsilon_n} \to q_\infty \quad \text{weakly in } H^1 \text{ and strongly in } L^\infty \tag{0.4}$$

for some sequence $\epsilon_n \to 0$. However $q_\infty(t)$ *may enter the singularity* 0.

In Sections 2–4, we study the behavior of critical points $(q_{\epsilon_n})_{n=1}^\infty$ of $I_{\epsilon_n}(q)$ with properties (0.1), (0.2) and (0.4). We will establish the following estimate of the Morse index

Proposition 0.3. *Let $(q_n(t))_{n=1}^\infty \subset \Lambda$ be a sequence of critical points of $I_{\epsilon_n}(q)$ satisfying*
(i) $\epsilon_n \to 0$;
(ii) there are constants $0 < m < M$ independent of n such that

$$I_{\epsilon_n}(q_n) \in [m, M] \quad \textit{for all } n;$$

(iii) $I_{\epsilon_n}'(q_n) = 0$;
(iv) $q_n \to q_\infty(t)$ weakly in H^1 and strongly in L^∞;

and let ν be the number of times $q_\infty(t)$ enters the singularity 0; that is,

$$\nu = \#\{t \in (0,T];\ q_\infty(t) = 0\} \in \mathbf{N} \cup \{\infty\}. \tag{0.5}$$

Then

$$\liminf_{n\to\infty} \operatorname{index} I''_{\epsilon_n}(q_n) \geq (N-2)i(\alpha)\nu, \tag{0.6}$$

where $i(\alpha) \in \mathbf{N}$ *is an integer defined by*

$$i(\alpha) = \max\{m \in \mathbf{N};\ m < \frac{2}{2-\alpha}\}. \tag{0.7}$$

We remark that $i(\alpha) = 1$ for $\alpha \in (0,1]$ and $i(\alpha) \geq 2$ for $\alpha \in (1,2)$. To prove the above proposition, we use re-scaling argument, which is based on the scale-invariance of the equation:

$$\ddot{q} + \frac{\alpha q}{|\, q\,|^{\alpha+2}} = 0 \quad \text{in } \mathbf{R}, \tag{0.8}$$

that is, (0.8) is invariant by the scale changes:

$$q(\cdot) \to \delta^{-1} q(\delta^{(\alpha+2)/2}\cdot).$$

In Section 5, we combine results obtained in Sections 1-4 and give proofs of our theorems 0.1 and 0.2.

1. Modified functional and minimax procedure

In this section, we study the following functional

$$I_\epsilon(q) = \int_0^T [\frac{1}{2} |\, \dot{q}\,|^2 - V(q,t) + \frac{\epsilon}{|\, q\,|^4}]dt \quad \text{for } \epsilon \in (0,1]. \tag{1.1}$$

Here we assume only (V2), (V3) and

(V1') $V(q,t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T-periodic in t.

We need the following notations; let $E = H^1_T(\mathbf{R}, \mathbf{R}^N)$ denote the space of T-periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^N$ under the norm:

$$\|q\|_E = (\int_0^T |\, \dot{q}\,|^2\, dt + [q]^2)^{1/2},$$

where $[q] = \frac{1}{T}\int_0^T q(t)dt$. We remark that

$$\Lambda = \{q \in E;\ q(t) \neq 0 \text{ for all } t\}$$

is open in E and $I_\epsilon(q) \in C^2(\Lambda, \mathbf{R})$. We also use the notation:

$$\|q\|_{L^2} = (\int_0^T |\, q(t)\,|^2\, dt)^{1/2}.$$

There is a one-to-one correspondence between critical points of $I_\epsilon(q)$ and classical T-periodic solutions of the following equation:

$$\begin{aligned}
&\ddot{q} + V_q(q,t) + \frac{4\epsilon q}{|q|^6} = 0, \\
&q(t+T) = q(t), \qquad \text{in } \mathbf{R}, \\
&q(t) \neq 0.
\end{aligned} \tag{1.2}$$

We remark the potential $V(q,t) - \dfrac{\epsilon}{|q|^4}$ satisfies the strong force condition (SF) with $W(q) = \dfrac{\sqrt{\epsilon}}{|q|}$.

First we state some properties of $I_\epsilon(q)$.

Lemma 1.1. *Assume (V1'), (V2) and (V3).*

(i) For any $M > 0$, there exist constants $C_i(M) > 0$ $(i = 1, 2)$ independent of $\epsilon \in (0,1]$ such that

$$\|\dot{q}\|_{L^2}, \ \int_0^T -V(q,t)dt, \ \int_0^T \frac{\epsilon}{|q|^4}dt \leq C_1(M), \tag{1.3}$$

$$\min_{t\in[0,T]} |q(t)| \geq C_2(M)\epsilon^{1/2} \tag{1.4}$$

for all $q \in \Lambda$ and $\epsilon \in (0,1]$ with $I_\epsilon(q) \leq M$.

(ii) For any $M > m > 0$, there exists a constant $C_3(m,M) > 0$ independent of $\epsilon \in (0,1]$ such that

$$\|q\|_E \leq C_3(m,M) \tag{1.5}$$

for all $q \in \Lambda$ and $\epsilon \in (0,1]$ with $I_\epsilon(q) \in [m,M]$ and $\|I'_\epsilon(q)\|_{E^*} \leq m/\sqrt{2M}$.

(iii) For any $\epsilon \in (0,1]$, $I_\epsilon(q)$ satisfies the condition (PS^+) on Λ:

(PS^+): for any $s > 0$, if $(q_n) \subset \Lambda$, $I_\epsilon(q_n) \to s$ and $I'_\epsilon(q_n) \to 0$, then q_n possesses a subsequence converging to some $q \in \Lambda$ in E.

Proof. (i) By (V2) and (V3), it follows from $I_\epsilon(q) \leq M$ that

$$\|\dot{q}\|_{L^2} \leq \sqrt{2M}, \tag{1.6}$$

$$\int_0^T -V(q,t)dt \leq M, \tag{1.7}$$

$$\int_0^T \frac{\epsilon}{|q|^4}dt \leq M. \tag{1.8}$$

Thus we get (1.3). Next we deal with (1.4). We get for all s, $t \in [0,T]$ that

$$\begin{aligned}\frac{1}{|q(t)|}-\frac{1}{|q(s)|} &\leq \int_0^T \left|\frac{d}{d\tau}\frac{1}{|q(\tau)|}\right| d\tau \\ &\leq \left(\int_0^T |\dot{q}|^2\, d\tau\right)^{1/2}\left(\int_0^T \frac{1}{|q(\tau)|^4} d\tau\right)^{1/2} \leq \frac{\sqrt{2}M}{\sqrt{\epsilon}}.\end{aligned} \tag{1.9}$$

By (V3), we can find a constant $c(M)>0$ with the following property; for any $q \in \Lambda$ with (1.7) there is a $t_0 = t_0(q) \in [0,T]$ such that

$$|q(t_0)| \geq c(M).$$

We set $s = t_0$ in (1.9), then we get for all $t \in [0,T]$

$$\frac{1}{|q(t)|} \leq \frac{\sqrt{2}M}{\sqrt{\epsilon}} + \frac{1}{c(M)} \leq \frac{1}{\sqrt{\epsilon}}\left(\sqrt{2}M + \frac{1}{c(M)}\right).$$

Thus

$$|q(t)| \geq \left(\sqrt{2}M + \frac{1}{c(M)}\right)^{-1}\epsilon^{1/2} \equiv C_2(M)\epsilon^{1/2}.$$

Hence we get (1.4).

(ii) By (1.6), it suffices to prove $\|q\|_{L^\infty} \leq C_3(m,M)$. We have for $q \in \Lambda$ with $\|I'_\epsilon(q)\|_{E^*} \leq m/\sqrt{2M}$ and $I_\epsilon(q) \leq M$ that

$$\begin{aligned} I_\epsilon(q) =& \frac{1}{2}I'_\epsilon(q)(q-[q]) + \frac{1}{2}\int_0^T V_q(q,t)(q-[q])dt \\ &+ \int_0^T -V(q,t)dt + 2\int_0^T \frac{\epsilon}{|q|^6}(q, q-[q])dt \\ &+ \int_0^T \frac{\epsilon}{|q|^4}dt \\ \leq& \frac{1}{2}\frac{m}{\sqrt{2M}}\|\dot{q}\|_{L^2} + \frac{1}{2}\int_0^T |V_q(q,t)|\,|q-[q]|\,dt \\ &+ \int_0^T -V(q,t)dt + \int_0^T \frac{2|q-[q]|}{|q|^5}dt + \int_0^T \frac{1}{|q|^4}dt. \end{aligned}$$

Note that we have from (1.6)

$$\|q(t)-[q]\|_{L^\infty} \leq \sqrt{T}\|\dot{q}\|_{L^2} \leq \sqrt{2TM}.$$

Thus we get

$$
\begin{aligned}
I_\epsilon(q) \leq & \frac{1}{2}\frac{m}{\sqrt{2M}}\sqrt{2M} + \frac{\sqrt{2TM}}{2}\int_0^T \mid V_q(q,t) \mid dt \\
& + \int_0^T -V(q,t)dt + \int_0^T \frac{2}{\mid q \mid^5}\sqrt{2TM}dt + \int_0^T \frac{1}{\mid q \mid^4}dt \\
\leq & \frac{1}{2}m + T\Phi([q] - \sqrt{2TM}),
\end{aligned}
\tag{1.10}
$$

where

$$
\Phi(R) = \max_{|y|\geq R,\ t\in[0,T]}[\frac{\sqrt{2TM}}{2} \mid V_q(y,t) \mid - V(y,t)] + \frac{2\sqrt{2TM}}{R^5} + \frac{1}{R^4}.
$$

We remark that

$$
\Phi(R) \to 0 \quad \text{as } R \to \infty. \tag{1.11}
$$

Now we assume $I_\epsilon(q) \in [m, M]$, then we have from (1.10) that

$$
\frac{1}{2}m \leq T\Phi([q] - \sqrt{2TM}).
$$

By (1.11), we can see there is a constant $C_3(m, M) > 0$ independent of $\epsilon \in (0, 1]$ such that

$$
\mid [q] \mid \leq C_3(m, M),
$$

i.e.,

$$
\|q\|_{L^\infty} \leq C_3(m, M).
$$

Thus we get (1.5).

(iii) Assume $(q_n) \subset \Lambda$ satisfies $I_\epsilon(q_n) \to s > 0$ and $I'_\epsilon(q_n) \to 0$ in E^*. From (1.4)–(1.6), we can extract a subsequence — we denote it still by q_n — such that

$$
q_n \to q \in \Lambda \quad \text{weakly in } E \text{ and strongly in } L^\infty.
$$

Thus the form of $I'_\epsilon(q)$ shows $q_n \to q$ strongly in E. ∎

Next we apply minimax method, which is essentially due to Bahri and Rabinowitz [**BR**], to $I_\epsilon(q)$ for each $\epsilon \in (0, 1]$. Consider the family of mappings $C(S^{N-2}, \Lambda)$. Identifying $[0, T]/\{0, T\} \simeq S^1$, we can associate each $\gamma \in C(S^{N-2}, \Lambda)$ with a mapping $\tilde{\gamma} : S^{N-2} \times S^1 \to S^{N-1}$ by

$$
\tilde{\gamma}(x, t) = \frac{\gamma(x)(t)}{\mid \gamma(x)(t) \mid} \quad \text{for } x \in S^{N-2} \text{ and } t \in S^1 \simeq [0, T]/\{0, T\}.
$$

We denote the Brouwer degree of $\widetilde{\gamma}$ by $\deg\widetilde{\gamma}$. We define

$$\Gamma^* = \{\gamma \in C(S^{N-2}, \Lambda);\ \deg\widetilde{\gamma} \neq 0\}. \tag{1.12}$$

We can see $\Gamma^* \neq \emptyset$ as in [**BR**, Lemma 1.2].
We define minimax values of $I_\epsilon(q)$ as follows:

$$b_\epsilon = \inf_{\gamma\in\Gamma^*} \max_{x\in S^{N-2}} I_\epsilon(\gamma(x)) \quad \text{for } \epsilon \in (0,1], \tag{1.13}$$

$$b_0 = \inf_{\gamma\in\Gamma^*} \max_{x\in S^{N-2}} I(\gamma(x)). \tag{1.14}$$

Since $I(q) \leq I_\epsilon(q) \leq I_1(q)$ for all $q \in \Lambda$ and $\epsilon \in (0,1]$, we have

$$b_0 \leq b_\epsilon \leq b_1 \qquad \text{for } \epsilon \in (0,1]. \tag{1.15}$$

We argue as in [**BR**, Proposition 1.4], we get

Proposition 1.2. $b_0 > 0$. ∎

Thus we have

Proposition 1.3. *For $\epsilon \in (0,1]$, there is a critical point $q_\epsilon(t) \in \Lambda$ of $I_\epsilon(q)$ such that*
(i) $I_\epsilon(q_\epsilon) = b_\epsilon$, (1.16)
(ii) $I'_\epsilon(q_\epsilon) = 0$, (1.17)
(iii) $\operatorname{index} I''_\epsilon(q_\epsilon) \leq N-2$, (1.18)

where $\operatorname{index} I''_\epsilon(q_\epsilon)$ *is the Morse index of* $I''_\epsilon(q_\epsilon)$.
Moreover there are constants $M > m > 0$ such that

$$m \leq b_\epsilon = I_\epsilon(q_\epsilon) \leq M \quad \textit{for } \epsilon \in (0,1]. \tag{1.19}$$

Proof. (1.19) follows from (1.15) and Proposition 1.2. Since $I_\epsilon(q)$ satisfies the strong force condition (SF) for $\epsilon \in (0,1]$, we have the following "Deformation Theorem":

Proposition 1.4 ([**BR**, Proposition 1.17]). *Suppose $\epsilon \in (0,1]$ and assume $s > 0$ is not a critical value of $I_\epsilon(q)$. Then for each $\overline{a} > 0$ there is an $a \in (0, \overline{a})$ and $\eta \in C([0,1] \times \Lambda, \Lambda)$ such that*
1° $\eta(1,q) = q$ *if* $I_\epsilon(q) \notin (s-\overline{a}, s+\overline{a})$,
2° $I_\epsilon(\eta(\tau,q)) \leq I_\epsilon(q)$ *for* $\tau \in [0,1]$,
3° $\eta(1, [I_\epsilon \leq s+a]) \subset [I_\epsilon \leq s-a]$, where $[I_\epsilon \leq \sigma] = \{q \in \Lambda;\ I_\epsilon(q) \leq \sigma\}$. ∎

By Proposition 1.2 and (1.15), we can see

$$b_\epsilon \geq b_0 > 0 \quad \text{for all } \epsilon \in (0,1].$$

Using the property 3° of Proposition 1.4 in a standard way (c.f. [**R**]), we can see $b_\epsilon > 0$ is a critical value of $I_\epsilon(q)$.

As to the property (1.18), we can obtain it in a similar way to the proof of Theorem A of Tanaka [**T**]. In [**T**], we studied properties of Morse indices of critical values related to the symmetric mountain pass theorem and we got (1.16)–(1.18) for the symmetric mountain pass theorem. See also [**BL,Sc,V,LS**]. ∎

The above proposition ensures the existence of approximate solutions $q_\epsilon(t) \in \Lambda$ together with uniform estimates (1.17) and (1.18). We will get a solution of the original problem (HS) as a limit of $q_\epsilon(t)$ as $\epsilon \to 0$.

To do so, we study the behavior of critical points of $I_\epsilon(q)$ whose critical values and Morse indices are uniformly bounded, that is, we study the behavior of critical points $q_n(t) \in \Lambda$ such that

$$\begin{aligned}
&\epsilon_n \to 0,\\
&I_{\epsilon_n}(q_n) \in [m, M],\\
&I'_{\epsilon_n}(q_n) = 0,\\
&\operatorname{index} I''_{\epsilon_n}(q_n) \leq N-2.
\end{aligned}$$

The following proposition, which is due to Bahri and Rabinowitz [**BR**], ensures the existence of convergent subsequence of $(q_n(t))$ and it shows the limit of the subsequence is a *generalized solution* of (HS).

Proposition 1.5 (c.f. [**BR**, Theorem 3.24]). *Let $(\epsilon_n)_{n=1}^\infty \subset (0,1]$ be a sequence such that $\epsilon_n \to 0$. Suppose $(q_n(t))_{n=1}^\infty \subset \Lambda$ is a sequence of critical points of $I_{\epsilon_n}(q)$ such that*

$$I'_{\epsilon_n}(q_n) = 0, \tag{1.20}$$

$$I_{\epsilon_n}(q_n) \in [m, M] \quad \text{for all } n, \tag{1.21}$$

where $0 < m < M$ are constants independent of n.
Then there is a subsequence — still denoted by n — and $q_\infty(t) \in E$ such that
(i) $q_n(t)$ converges to $q_\infty(t)$ weakly in E and strongly in L^∞;
(ii) $\int_0^T -V(q_\infty, t)dt < \infty$;
(iii) $q_\infty(t)$ vanishes on a set D, of measure 0;
(iv) $q_\infty(t) \in C^2(\mathbf{R}\setminus D, \mathbf{R})$;
(v) $q_\infty(t)$ satisfies (HS) on $\mathbf{R}\setminus D$.

Remark 1.6. (i) $(q_\epsilon(t))_{\epsilon\in(0,1]}$ given in Proposition 1.3 satisfies the assumptions of the above proposition.
(ii) $q_\infty(t)$ is a *generalized T-periodic solution* of (HS) in the sense of [**BR**].

Proof of Proposition 1.5. By Lemma 1.1, we get

$$\|q_n\|_E,\ \int_0^T -V(q_n,t)dt \le C_6 \tag{1.22}$$

where $C_6 > 0$ is independent of n.
Thus we get (i). By (1.22) and Fatou's lemma, we get (ii). We have (iii) easily from (ii). Since $q_n(t)$ satisfies (1.2) with $\epsilon = \epsilon_n$ and $q_n(t) \to q_\infty(t)$ in L^∞, we can deduce (iv) and (v). ∎

If $D = \emptyset$ in the above proposition, the limit function $q_\infty(t)$ is a classical solution (non-collision solution) of the original problem (HS). In the following sections, we will show that for the sequence $(q_\epsilon(t))_{\epsilon\in(0,1]}$ given in Proposition 1.3

(i) if $V(q,t)$ satisfies (W1)–(W3) with $\alpha \in (1,2)$ in addition to (V1)–(V3), then $D = \emptyset$;

(ii) if $V(q,t)$ satisfies (W1)–(W3) with $\alpha \in (0,1]$ in addition to (V1)–(V3), then $D \cap (0,T]$ consists of at most one point, that is, $q_\infty(t)$ enters the singularity 0 at most one time in period T.

To get the above properties (i)–(ii), the uniform estimate of Morse indices (1.18) plays an important role. We remark that in Proposition 1.5, we used only the uniform bound of critical values.

Lastly in this section, we assume (W1)–(W3) in addition to (V1)–(V3) and get some a priori estimate, which will be used in the following sections.

Proposition 1.7. *Assume (W1)–(W3) and (V2). For any $0 < m < M$, there are constants $C_7(m,M)$, $C_8(m,M) > 0$ independent of $\epsilon \in (0,1]$ such that for all $q \in \Lambda$ and $\epsilon \in (0,1]$ with $I_\epsilon(q) \in [m,M]$ and $I'_\epsilon(q) = 0$*

(i) $\|q\|_E,\ \int_0^T \frac{1}{|q|^\alpha}dt,\ \int_0^T \frac{\epsilon}{|q|^4}dt \le C_7(m,M),$

(ii) $$\left| \frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{|q|^\alpha} + U(q,t) - \frac{\epsilon}{|q|^4} \right| \le C_8(m,M) \text{ for all } t \in \mathbf{R}. \tag{1.23}$$

Proof. We can get the assertion (i) from (W1)–(W3) and (i), (ii) of Lemma 1.1. To obtain (ii), we set

$$E(t) \equiv \frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{|q|^\alpha} + U(q,t) - \frac{\epsilon}{|q|^4}.$$

By (i), we get

$$\begin{aligned}\int_0^T |E(t)|\,dt &\le \frac{1}{2}\int_0^T |\dot{q}|^2\,dt + \int_0^T -V(q,t)dt + \int_0^T \frac{\epsilon}{|q|^4}dt \\ &\le C'_7(m,M).\end{aligned} \tag{1.24}$$

Since $q(t) \in \Lambda$ is a solution of (1.2),

$$\frac{d}{dt}E(t) = U_t(q,t).$$

Thus by (W3)

$$\int_0^T \mid \frac{d}{dt}E(t) \mid dt \le \int_0^T \mid U_t(q,t) \mid dt \le C \int_0^T \frac{1}{\mid q(t) \mid^\alpha} dt \le C_7''(m,M). \tag{1.25}$$

Combining (1.24) and (1.25), we get

$$\|E(t)\|_{L^\infty} \le C_8(m,M).$$

Therefore we obtain (ii). ∎

2. Asymptotic behavior of $q_n(t)$ near collision

In what follows, we assume (V2) and (W1)–(W3). Suppose $(q_n(t)) \subset \Lambda$ be a sequence of critical points of $I_{\epsilon_n}(q)$ satisfying

$$\epsilon_n \longrightarrow 0, \tag{2.1}$$

$$I_{\epsilon_n}(q_n) \in [m,M], \tag{2.2}$$

$$I'_{\epsilon_n}(q_n) = 0, \tag{2.3}$$

$$q_n(t) \to q_\infty(t) \quad \text{weakly in } E \text{ and strongly in } L^\infty, \tag{2.4}$$

where $0 < m < M$ are constants independent of n. By Proposition 1.5, a suitable subsequence of critical points $(q_\epsilon(t))_{\epsilon\in(0,1]} \subset \Lambda$, which is obtained in Proposition 1.3, satisfies the conditions (2.1)–(2.4).

The main purpose of the following 3 sections is to prove Proposition 0.3, that is, to estimate the Morse index of $I''_{\epsilon_n}(q_n)$ from below by the number of collisions ν:

$$\nu \equiv \#D = \#\{t \in (0,T];\ q_\infty(t) = 0\,\}.$$

We can obtain Theorems 0.1 and 0.2 from Proposition 0.3 and (1.18). First we study the asymptotic behavior of $q_n(t)$ near collisions. Suppose $t_\infty \in (0,T]$ satisfies

$$q_\infty(t_\infty) = 0.$$

We may assume $t_\infty \in (0,T)$ without loss of generality. Extracting a subsequence — still denoted by n —, we can choose $t_n \in (0,T]$ such that

1° $| q_n(t_n) |$ takes its *local minimum* at $t = t_n$, (2.5)

2° $t_n \to t_\infty$ as $n \to \infty$, (2.6)

3° $| q_n(t_n) | \to 0$ as $n \to \infty$. (2.7)

In fact, by (iii) of Proposition 1.5, we can find a sequence $a_n, b_n \in (0, T)$ such that

$$t_\infty - \frac{1}{n} < a_n < t_\infty < b_n < t_\infty + \frac{1}{n}, \qquad q_\infty(a_n) > 0, \qquad q_\infty(b_n) > 0. \tag{2.8}$$

Thus we can find a sequence of integers $m(1) < m(2) < \cdots$ such that

$$| q_{m(n)}(t_\infty) | \leq \frac{1}{2} \min\{| q_{m(n)}(a_n) |, | q_{m(n)}(b_n) |\}. \tag{2.9}$$

Suppose $| q_{m(n)}(t_{m(n)}) | = \min_{[a_n, b_n]} | q_{m(n)}(t) |$ for $t_{m(n)} \in [a_n, b_n]$. By (2.9), $t_{m(n)} \in (a_n, b_n)$. Thus $| q_{m(n)}(t) |$ takes its local minimum at $t = t_{m(n)}$. Moreover we have $t_{m(n)} \to t_\infty$ by (2.8) and $| q_{m(n)}(t_{m(n)}) | \leq | q_{m(n)}(t_\infty) | \to 0$. Therefore we get (2.5)–(2.7) for the subsequence $m(n)$.

By Proposition 1.7, $q_n(t)$ satisfies

$$\ddot{q}_n + \frac{\alpha q_n}{| q_n |^{\alpha+2}} - U_q(q_n, t) + \frac{4\epsilon_n q_n}{| q_n |^6} = 0, \tag{2.10}$$

$$q_n(t + T) = q_n(t), \qquad \text{in } \mathbf{R}, \tag{2.11}$$

$$\left| \frac{1}{2} | \dot{q}_n(t) |^2 - \frac{1}{| q_n |^\alpha} + U(q_n, t) - \frac{\epsilon}{| q_n |^4} \right| \leq C_7(m, M). \tag{2.12}$$

We set

$$\delta_n = | q_n(t_n) | > 0 \tag{2.13}$$

and define $x_n : \mathbf{R} \to \mathbf{R}^N \setminus \{0\}$ by

$$x_n(s) = \delta_n^{-1} q_n(\delta_n^{(\alpha+2)/2} s + t_n) \quad \text{for } s \in \mathbf{R}. \tag{2.14}$$

We consider the asymptotic behavior of $x_n(s)$ as $n \to \infty$. From the definition of $x_n(s)$ and (2.5)–(2.7), (2.10)–(2.13), we can easily see

Lemma 2.1. *$x_n(s)$ and $\delta_n > 0$ satisfies*

(i) $\delta_n \to 0$, *(2.15)*

(ii) $x_n(s)$ takes its local minimum at $s = 0$,

(iii) $| x_n(0) | = 1$, $x_n(0) \perp \dot{x}_n(0)$, *(2.16)*

(iv) $\ddot{x}_n(s) + \dfrac{\alpha x_n}{| x_n |^{\alpha+2}} - \delta_n^{\alpha+1} U_q(\delta_n x_n, \delta_n^{(\alpha+2)/2} s + t_n) + \dfrac{4\epsilon_n}{\delta_n^{4-\alpha}} \dfrac{x_n}{| x_n |^6} = 0$ *in* $\mathbf{R}$ *(2.17)*

(v) $\left| \dfrac{1}{2} | \dot{x}_n(s) |^2 - \dfrac{1}{| x_n |^\alpha} + \delta_n^\alpha U(\delta_n x_n, \delta_n^{(\alpha+2)/2} s + t_n) - \dfrac{\epsilon_n}{\delta_n^{4-\alpha}} \dfrac{1}{| x_n |^4} \right| \leq C_8(m, M) \delta_n^\alpha$
for all $s \in \mathbf{R}$ and $n \in \mathbf{N}$. *(2.18)*

∎

The following lemma gives us an estimate of the coefficient of the equation (2.17).

Lemma 2.2.
$$\limsup_{n\to\infty}\frac{\epsilon_n}{\delta_n^{4-\alpha}}\le\frac{2-\alpha}{2}.$$

Proof. Since $|x_n(s)|^2$ takes its local minimum at $s=0$, we have

$$0\le\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0}|x_n(s)|^2=(\ddot{x}_n(0),x_n(0))+|\dot{x}_n(0)|^2.$$

Using (2.16)–(2.18), we get

$$\begin{aligned}0\le&(2-\alpha)-\frac{2\epsilon_n}{\delta_n^{4-\alpha}}-\delta_n^{\alpha+1}(x_n(0),U_q(\delta_n x_n(0),t_n))\\&-2\delta_n^{\alpha}U(\delta_n x_n(0),t_n)+2C_8(m,M)\delta_n^{\alpha}.\end{aligned}$$

By the assumption (W3), we can see

$$\limsup_{n\to\infty}\frac{\epsilon_n}{\delta_n^{4-\alpha}}\le\frac{2-\alpha}{2}.$$ ∎

By Lemma 2.2, we can extract a subsequence — we still denote it by n — such that

$$\frac{\epsilon_n}{\delta_n^{4-\alpha}}\to d\in[0,\frac{2-\alpha}{2}]\quad\text{as } n\to\infty.$$

Then we can deduce the following from (2.18).

$$|\dot{x}_n(0)|\to\sqrt{2(1+d)}\quad\text{as } n\to\infty. \tag{2.19}$$

We extract a subsequence again — still denoted by n — and by (2.16) we may assume

$$x_n(0)\to e_1, \tag{2.20}$$
$$\dot{x}_n(0)\to\sqrt{2(1+d)}e_2, \tag{2.21}$$

where e_1, e_2, $\cdots$, e_N are an orthonormal basis of $\mathbf{R}^N$.

By the continuous dependence of solutions on initial data and equation, we have

Proposition 2.3. *For any $\ell>0$, $x_n(s)$ converges to a function $y_{\alpha,d}(s)$ in $C^2([-\ell,\ell],\mathbf{R}^N)$, where $y_{\alpha,d}(s)$ is a solution of*

$$\ddot{y}+\frac{\alpha y}{|y|^{\alpha+2}}+4d\frac{y}{|y|^6}=0\quad in\ \mathbf{R}, \tag{2.22}$$
$$y(0)=e_1, \tag{2.23}$$
$$\dot{y}(0)=\sqrt{2(1+d)}e_2. \tag{2.24}$$

Proof. By (W3), we have for any $R > 1$

$$\delta_n^{\alpha+1} U_q(\delta_n x, \delta_n^{(\alpha+2)/2} s + t_n) \to 0$$

in $C^1(\{x \in \mathbf{R}^N;\ 1/R \leq |\ x\ | \leq R\} \times \mathbf{R}, \mathbf{R}^N)$ as $n \to \infty$. On the other hand, (2.22)–(2.24) has a global solution $y_{\alpha,d}(s)$ satisfying

$$|\ y_{\alpha,d}(s)\ | \geq 1 \quad \text{for all } s \in \mathbf{R} \tag{2.25}$$

for $0 < \alpha < 2$ and $d \in [0, \frac{2-\alpha}{2}]$. (The proof of (2.25) will be given in Lemma 4.2.) Therefore we can see

$$x_n(s) \to y_{\alpha,d}(s) \quad \text{in } C^2([-\ell, \ell], \mathbf{R}^N)$$

for any $\ell > 0$. ∎

Using Proposition 2.3, we will estimate the Morse index of $I''_{\epsilon_n}(q_n)$ for large n in the following sections.

3. Morse index of $I''_\epsilon(q)$ and the limit problem

For arbitrary given $\ell > 0$, we define linear operator $T_n : H_0^1(-\ell, \ell; \mathbf{R}) \to H_0^1(0, T; \mathbf{R})$ by

$$(T_n\varphi)(t) = \delta_n \varphi(\delta_n^{-(\alpha+2)/2}(t - t_n)) \tag{3.1}$$

for $n \in \mathbf{N}$ and $\varphi \in H_0^1(-\ell, \ell; \mathbf{R})$. Remark that T_n is well-defined for large n. Extending $(T_n\varphi)(t)$ periodically, we regard it as a T-periodic function on $\mathbf{R}$.

We have for $j = 3, \cdots, N$

$$\begin{aligned}
&\delta_n^{-(2-\alpha)/2} I''_{\epsilon_n}(q_n)((T_n\varphi)e_j, (T_n\varphi)e_j) \\
=&\delta_n^{-(2-\alpha)/2} \int_0^T [|\ \frac{d}{dt}(T_n\varphi)\ |^2 - \frac{\alpha\ |\ T_n\varphi\ |^2}{|\ q_n\ |^{\alpha+2}} + \frac{\alpha(\alpha+2)(q_n, e_j)^2\ |\ T_n\varphi\ |^2}{|\ q_n\ |^{\alpha+4}} \\
&- U_{qq}(q_n, t)((T_n\varphi)e_j, (T_n\varphi)e_j) - \frac{4\epsilon_n\ |\ T_n\varphi\ |^2}{|\ q_n\ |^6} + \frac{24\epsilon_n(q_n, e_j)^2\ |\ T_n\varphi\ |^2}{|\ q_n\ |^8}]dt \\
=&\int_{-\ell}^{\ell} [|\ \dot{\varphi}(s)\ |^2 - \frac{\alpha\ |\ \varphi\ |^2}{|\ x_n\ |^{\alpha+2}} + \frac{\alpha(\alpha+2)(x_n, e_j)^2\ |\ \varphi\ |^2}{|\ x_n\ |^{\alpha+4}} \\
&- \delta_n^{\alpha+2} U_{qq}(\delta_n x_n, \delta_n^{(\alpha+2)/2} s + t_n)(\varphi e_j, \varphi e_j) \\
&- \frac{4\epsilon_n}{\delta_n^{4-\alpha}} \frac{|\ \varphi\ |^2}{|\ x_n\ |^6} + \frac{24\epsilon_n}{\delta_n^{4-\alpha}} \frac{(x_n, e_j)^2\ |\ \varphi\ |^2}{|\ x_n\ |^8}]ds.
\end{aligned}$$

By (W3) and Proposition 2.3, we have

$$\begin{aligned}
&\delta_n^{-(2-\alpha)/2} I''_{\epsilon_n}(q_n)((T_n\varphi)e_j, (T_n\varphi)e_j) \\
&\to \int_{-\ell}^{\ell} [|\, \dot{\varphi}(s)\,|^2 - \frac{\alpha \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^{\alpha+2}} + \frac{\alpha(\alpha+2)(y_{\alpha,d}, e_j)^2 \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^{\alpha+4}} \\
&\quad - \frac{4d \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^6} + \frac{24d(y_{\alpha,d}, e_j)^2 \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^8}] ds \\
&= \int_{-\ell}^{\ell} [|\, \dot{\varphi}(s)\,|^2 - \frac{\alpha \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^{\alpha+2}} - \frac{4d \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^6}] ds
\end{aligned} \tag{3.2}$$

as $n \to \infty$.
Here we used the fact:

$$y_{\alpha,d}(s) \in \mathrm{span}\{e_1, e_2\} \quad \text{for } s \in \mathbf{R}.$$

We set

$$J_{\alpha,d,\ell}(\varphi) = \int_{-\ell}^{\ell} [|\, \dot{\varphi}(s)\,|^2 - \frac{\alpha \,|\, \varphi\,|^2}{|\, y_{\alpha,d}\,|^{\alpha+2}}] ds \;:\; H_0^1(-\ell, \ell; \mathbf{R}) \to \mathbf{R}$$

for $\alpha \in (0,2)$, $d \in [0, (2-\alpha)/2]$ and $\ell > 0$. Then we can see

$$\lim_{n\to\infty} \delta_n^{-(2-\alpha)/2} I''_{\epsilon_n}(q_n)((T_n\varphi)e_j, (T_n\varphi)e_j) \le J_{\alpha,d,\ell}(\varphi) \tag{3.3}$$

for all $\varphi \in H_0^1(-\ell, \ell; \mathbf{R})$.

We define

$$\begin{aligned}
N(\alpha, d, \ell) = \max\{\dim H;\ H \subset H_0^1(-\ell, \ell; \mathbf{R}) \text{ is a subspace such that} \\
J_{\alpha,d,\ell}(\varphi) < 0 \text{ for } \varphi \in H \setminus \{0\}\ \}.
\end{aligned} \tag{3.4}$$

Clearly

$N(\alpha, d, \ell) =$ the number of negative eigenvalues of the following eigenvalue problem:

$$\begin{aligned}
-\ddot{u} - \frac{\alpha}{|\, y_{\alpha,d}(s)\,|^{\alpha+2}} u = \lambda u \quad \text{in } (-\ell, \ell), \\
u(-\ell) = u(\ell) = 0.
\end{aligned} \tag{3.5}$$

We remark that $N(\alpha, d, \ell)$ is a non-decreasing function of ℓ for each α and d. Let $\varphi_i(s) \in H_0^1(-\ell, \ell; \mathbf{R})$ ($i = 1, 2, \cdots, N(\alpha, d, \ell)$) be eigenfunctions of the problem (3.5) with negative eigenvalues, in particular, we have

$$J_{\alpha,d,\ell}(\varphi) < 0 \quad \text{for } \varphi \in \mathrm{span}\{\varphi_i(s);\ i = 1, \cdots, N(\alpha, d, \ell)\} \setminus \{0\}. \tag{3.6}$$

We consider the set of functions:

$$H(t_\infty, n) = \mathrm{span}\{(T_n\varphi_i)e_j;\ 1 \le i \le N(\alpha, d, \ell),\ 3 \le j \le N\} \subset E. \tag{3.7}$$

By (3.3) and (3.6), we can see for sufficiently large n that

$$I''_{\epsilon_n}(q_n)(h,h) < 0 \quad \text{for all } h \in H(t_\infty, n) \setminus \{0\}. \tag{3.8}$$

We remark

$$\dim H(t_\infty, n) = (N-2)N(\alpha, d, \ell).$$

Finally we set

$$i(\alpha) = \sup_{\ell > 0} \min_{d \in [0,(2-\alpha)/2]} N(\alpha, d, \ell). \tag{3.9}$$

Choosing $\ell > 0$ sufficiently large, we may assume

$$\dim H(t_\infty, n) \ge (N-2)i(\alpha). \tag{3.10}$$

In Section 4, we will give a representation (0.7) of $i(\alpha)$.

Proposition 3.1. *Assume (V2) and (W1)–(W3) and suppose $(q_n(t))_{n=1}^\infty \subset \Lambda$ satisfies (2.1)–(2.4). Let ν be the number of times $q_\infty(t)$ enters the singularity 0:*

$$\nu = \#\{t \in (0,T];\ q_\infty(t) = 0\}.$$

Then we have

$$\liminf_{n\to\infty} \mathrm{index}\, I''_{\epsilon_n}(q_n) \ge (N-2)i(\alpha)\nu. \tag{3.11}$$

Proof. Suppose $\nu < \infty$ and

$$\{t_{\infty,1}, t_{\infty,2}, \cdots, t_{\infty,\nu}\} = \{t \in (0,T];\ q_\infty(t) = 0\}.$$

For any given subsequence $n_m \to \infty$, we can extract a subsequence — we still denote it by n_m — such that Proposition 2.3 holds for each $t_{\infty,k}$ for suitable orthonormal basis $e_1^{(k)}$, $e_2^{(k)}$, $\cdots$, $e_N^{(k)}$ and $d^{(k)} \in [0, \frac{2-\alpha}{2}]$. Thus we can construct subspaces $H(t_{\infty,k}, n_m) \subset E$ for each $t_{\infty,k}$ ($k = 1, 2, \cdots, \nu$) as in (3.7). From the construction, we have

$$\dim H(t_{\infty,k}, n_m) \ge (N-2)i(\alpha) \quad \text{for all } k.$$

For any $\delta > 0$, we find a constant $m_0(\delta) \in \mathbf{N}$ such that

$$\mathrm{supp}\, h(t) \subset [t_{\infty,k} - \delta, t_{\infty,k} + \delta]$$

for all $h(t) \in H(t_{\infty,k}, n_m)$ and $m \geq m_0(\delta)$. Thus we get

$$H(t_{\infty,i}, n_m) \cap H(t_{\infty,j}, n_m) = \{0\} \quad (i \neq j)$$

for sufficiently large n. Set

$$H_{n_m} = H(t_{\infty,1}, n_m) \oplus H(t_{\infty,2}, n_m) \oplus \cdots \oplus H(t_{\infty,\nu}, n_m).$$

Choosing sufficiently large $\ell > 0$, we obtain from (3.8) and (3.10) that

$$\dim H_{n_m} \geq (N-2)i(\alpha)\nu,$$

$$I''_{\epsilon_n}(q_{n_m})(h,h) < 0 \quad \text{for } h \in H_{n_m} \setminus \{0\}$$

for sufficiently large m.
Therefore we get (3.11). In case $\nu = \infty$, for any $k \in \mathbf{N}$ we can see in a similar way that

$$\liminf_{n\to\infty} \operatorname{index} I''_{\epsilon_n}(q_n) \geq (N-2)i(\alpha)k.$$

Thus we conclude

$$\liminf_{n\to\infty} \operatorname{index} I''_{\epsilon_n}(q_n) = \infty.$$

∎

4. Representation of the number $i(\alpha)$ and proof of Proposition 0.3

The aim of this section is to give a representation (0.7) of the number $i(\alpha)$, that is, to prove

Proposition 4.1. *Let $i(\alpha) \in \mathbf{N}$ be the number defined in (3.4)–(3.9). Then for any $\alpha \in (0,2)$ the number $i(\alpha)$ can be represented as*

$$i(\alpha) = \max\{m \in \mathbf{N};\, m < \frac{2}{2-\alpha}\}.$$

We remark

$$i(\alpha) = 1 \quad \text{for } 0 < \alpha \leq 1, \tag{4.1}$$

$$i(\alpha) \geq 2 \quad \text{for } 1 < \alpha < 2, \tag{4.2}$$

$$i(\alpha) \to \infty \quad \text{as } \alpha \to 2. \tag{4.3}$$

First, we consider the solution $y_{\alpha,d}(s)$ of (2.22)–(2.24).

Lemma 4.2. *For any $0 < \alpha < 2$, $d \in [0, \frac{2-\alpha}{2}]$, the equation (2.22)–(2.24) has a global solution $y_{\alpha,d}(s)$. Moreover, $y_{\alpha,d}(s)$ satisfies*

$$1 \leq \mid y_{\alpha,d}(s) \mid \leq \mid y_{\alpha,0}(s) \mid \tag{4.4}$$

for all $d \in [0, \frac{2-\alpha}{2}]$ and $s \in \mathbf{R}$.

Proof. First we remark that $y_{\alpha,d}(s)$ satisfies

$$\frac{1}{2} \mid \dot{y}_{\alpha,d}(s) \mid^2 - \frac{1}{\mid y_{\alpha,d} \mid^\alpha} - \frac{d}{\mid y_{\alpha,d} \mid^4} = 0 \qquad \text{for } s \in \mathbf{R}. \tag{4.5}$$

We fix here $\alpha \in (0, 2)$ and set $R_d(s) = \mid y_{\alpha,d}(s) \mid^2$. Using (2.22) and (4.5), we get

$$\begin{aligned} \ddot{R}_d &= 2(\ddot{y}_{\alpha,d}, y_{\alpha,d}) + 2 \mid \dot{y}_{\alpha,d} \mid^2 \\ &= 2(2-\alpha)\frac{1}{R_d^{\alpha/2}} - 4d\frac{1}{R_d^2}, \end{aligned} \tag{4.6}$$

$$R_d(0) = 1, \tag{4.7}$$

$$\dot{R}_d(0) = 0. \tag{4.8}$$

We can easily see from (4.6)–(4.8) that $R_{(2-\alpha)/2}(s) \equiv 1$ and for $d \in [0, \frac{2-\alpha}{2})$

$$\ddot{R}_d(0) = 2(2-\alpha) - 4d > 0$$

and

$$\ddot{R}_d(s) \geq 2(2-\alpha)(\frac{1}{R_d^{\alpha/2}} - \frac{1}{R_d^2}) > 0 \quad \text{if } R_d(s) > 1.$$

Thus we get for $d \in [0, \frac{2-\alpha}{2})$

$$\begin{aligned} R_d(s) > 1 &\quad \text{for all } s \neq 0, \\ s\,\dot{R}_d(s) > 0 &\quad \text{for all } s \neq 0. \end{aligned} \tag{4.9}$$

Next we fix $d \in (0, \frac{2-\alpha}{2})$ and prove $R_d(s) < R_0(s)$ for all s. Since $\ddot{R}_d(0) = 2(2-\alpha) - 4d < 2(2-\alpha) = \ddot{R}_0(0)$ for $d \in (0, \frac{2-\alpha}{2})$, we have

$$R_d(s) < R_0(s) \quad \text{for sufficiently small } s > 0.$$

Suppose there is an $s_1 > 0$ such that $R_d(s_1) = R_0(s_1)$. Then there is an $s_0 > 0$ such that

$$\begin{aligned} R_d(s) < R_0(s) &\quad \text{for } s \in (0, s_0), \\ R_d(s_0) = R_0(s_0). & \end{aligned} \tag{4.10}$$

Since $R_d(s)$ satisfies (4.6)–(4.8), we have

$$\overset{\bullet}{R}_d(s)^2 - 4R_d^{(2-\alpha)/2} - \frac{4d}{R_d} = -4(1+d).$$

Thus we get

$$\overset{\bullet}{R}_d(s_0)^2 - \overset{\bullet}{R}_0(s_0)^2 = 4d(\frac{1}{R_d(s_0)} - 1) < 0.$$

By (4.9), we get $\overset{\bullet}{R}_d(s_0) < \overset{\bullet}{R}_0(s_0)$. But this contradicts with (4.10). Therefore we have

$$R_d(s) < R_0(s) \quad \text{for } s > 0.$$

Similarly we get $R_d(s) < R_0(s)$ for $s < 0$. ∎

Corollary 4.3. *For any $0 < \alpha < 2$, $d \in [0, \frac{2-\alpha}{2}]$ and $\ell > 0$,*

$$N(\alpha, 0, \ell) \leq N(\alpha, d, \ell),$$

i.e.,

$$i(\alpha) = \sup_{\ell > 0} N(\alpha, 0, \ell). \tag{4.11}$$

Proof. By (4.4), we have

$$J_{\alpha,d,\ell}(\varphi) \leq J_{\alpha,0,\ell}(\varphi) \quad \text{for all } \varphi \in H_0^1(-\ell, \ell; \mathbf{R}).$$

Thus we get the desired result from the definition of $N(\alpha, d, \ell)$ and $i(\alpha)$. ∎

By (4.11), from now on, we deal with only the case $d = 0$. The following lemma is a consequence of Sturm Comparison Theorem.

Lemma 4.4. *The number $i(\alpha) + 1$ is equal to the maximal number of zeros of nontrivial solutions $u(s)$ of*

$$-\ddot{u} - \frac{\alpha}{|\, y_{\alpha,0}(s)\,|^{\alpha+2}} u = 0 \quad \textit{in } \mathbf{R}. \tag{4.12}$$

That is,

$$i(\alpha) + 1 = \max\{\, \#\{s \in \mathbf{R};\ u(s) = 0\};\ u(s) \textit{ is a nontrivial solution of (4.12)}\,\}.$$

Proof. Suppose $i(\alpha) = k$ and let $\ell > 0$ be sufficiently large so that $N(\alpha, 0, \ell) = k$. Then k-th eigenvalue λ_k of (3.5) is negative, that is, there is an eigenfunction $u_k(s)$ of

$$-\ddot{u}_k - \frac{\alpha}{|\, y_{\alpha,0}(s)\,|^{\alpha+2}} u_k = \lambda_k u_k \quad \text{in } (-\ell, \ell),$$
$$u_k(\pm\ell) = 0,$$

which has exactly $(k+1)$ zeros in $[-\ell,\ell]$. Consider initial value problem (4.12) with initial data $u(-\ell)=0$ and $\dot{u}(-\ell)=1$, then by Sturm Comparison Theorem, $u(s)$ has at least $(k+1)$ zeros in $[-\ell,\ell]$.

Conversely, suppose (4.12) has a nontrivial solution with $(k+1)$ zeros $t=t_1<t_2<\cdots<t_{k+1}$ and consider the eigenvalue problem:

$$-\ddot{u}-\frac{\alpha}{|\, y_{\alpha,0}(s)\,|^{\alpha+2}}u=\lambda u \quad \text{in } (t_1,t_{k+1}),$$
$$u(t_1)=u(t_{k+1})=0.$$

Then we can see that the k-th eigenvalue λ_k equals to 0. Choosing $\ell>0$ such that $[t_1,t_{k+1}]\subset(-\ell,\ell)$, we have $N(\alpha,0,\ell)\geq k$. ∎

Therefore we will consider the number of zeros of nontrivial solutions $u(s)$ of (4.12). We write $y_i^{(\alpha)}(s)=(y_{\alpha,0}(s),e_i):\ \mathbf{R}\to\mathbf{R}$ $(i=1,2)$. Then $\{y_1^{(\alpha)},y_2^{(\alpha)}\}$ are linearly independent solutions of (4.12). Thus any solution $u(s)$ of (4.12) can be represented by their linear combinations. That is, we can write

$$u(s)=\sin\beta\, y_1^{(\alpha)}(s)+\cos\beta\, y_2^{(\alpha)}(s) \qquad (\beta\in\mathbf{R})$$

up to multiplicative constants. Using polar coordinate (r_α,θ_α), we write

$$(y_1^{(\alpha)}(s),y_2^{(\alpha)}(s))=(r_\alpha(s)\cos\theta_\alpha(s),r_\alpha(s)\sin\theta_\alpha(s)) \tag{4.13}$$

where $r_\alpha(s)>0$ and $\theta_\alpha(s)\in\mathbf{R}$ with $\theta_\alpha(0)=0$. Then any solution $u(s)$ of (4.12) can be written (up to multiplicative constants) as

$$u(s)=r_\alpha(s)\sin(\theta_\alpha(s)+\beta) \qquad (\beta\in\mathbf{R}). \tag{4.14}$$

From (4.14), we can easily see

Lemma 4.5. *The maximal number of zeros of nontrivial solutions of (4.12) is equal to the number*

$$\max\{m\in\mathbf{Z};\ m<\frac{\theta_\alpha^+-\theta_\alpha^-}{\pi}\}+1. \tag{4.15}$$

Here $\theta_\alpha^\pm$ is defined by

$$\theta_\alpha^\pm=\lim_{s\to\pm\infty}\theta_\alpha(s).$$

∎

Remark 4.6. The number (4.15) describes twice of the number of times the point $(y_1^{(\alpha)}(s),y_2^{(\alpha)}(s))$ turns around the singularity 0 while $-\infty<s<\infty$.

Proof. For $u_\beta(s) = r_\alpha(s)\sin(\theta_\alpha(s)+\beta)$, we can easily see

$$u_\beta(s) = 0 \quad \text{if and only if} \quad \theta_\alpha(s)+\beta = m\pi \text{ for some } m \in \mathbf{Z}.$$

Thus we can see the maximal number of zeros of nontrivial solutions of (4.12) is equal to the number (4.15), that is,

$$\max\{\,\#\{s\in\mathbf{R};\ u_\beta(s)=0\};\ \beta\in(0,2\pi]\,\} = \max\{m\in\mathbf{Z};\ m<\frac{\theta_\alpha^+-\theta_\alpha^-}{\pi}\}+1. \qquad \blacksquare$$

Proof of Proposition 4.1. Since $y(s) = y_{\alpha,0}(s) = (r_\alpha(s)\cos\theta_\alpha(s), r_\alpha(s)\sin\theta_\alpha(s))$ satisfies (2.22)–(2.24) with $d=0$, we have

$$r_\alpha(s)^2\,\dot{\theta}_\alpha(s) = \sqrt{2} \qquad \text{for all } s\in\mathbf{R} \tag{4.16}$$

(conservation of the angular momentum). Thus we can make a change of independent variables $s\to\theta=\theta_\alpha$. We set $\rho_\alpha = \rho_\alpha(\theta) = \frac{1}{r_\alpha(\theta)}$. Then $\rho_\alpha(\theta)$ satisfies

$$(\rho_\alpha)_{\theta\theta}+\rho_\alpha-\frac{\alpha}{2}(\rho_\alpha)^{\alpha-1}=0, \tag{4.17}$$

$$\rho_\alpha(0)=1, \tag{4.18}$$

$$(\rho_\alpha)_\theta(0)=0, \tag{4.19}$$

and $\theta_\alpha^\pm$ can be characterized as

$$\theta_\alpha^\pm = \pm\sup\{\theta>0;\ \rho_\alpha(\tau) \text{ exists and is positive for all } \tau\in[0,\theta).\} \tag{4.20}$$

By (4.16)–(4.19), we have

$$(\rho_\alpha)_\theta(\theta)^2+\rho_\alpha(\theta)^2-\rho_\alpha(\theta)^\alpha=0 \quad \text{for all } \theta\in(\theta_\alpha^-,\theta_\alpha^+).$$

Since $(\rho_\alpha)_\theta(\theta)<0$ for all $\theta>0$ (it follows from (4.9)), we have

$$\frac{-(\rho_\alpha)_\theta(\theta)}{\sqrt{\rho_\alpha(\theta)^\alpha-\rho(\theta)^2}}=1.$$

Integrating over $[0,\theta]$, we get

$$\int_{\rho_\alpha(\theta)}^1\frac{d\rho}{\sqrt{\rho^\alpha-\rho^2}}=\theta \quad \text{for all } \theta\in[0,\theta_\alpha^+).$$

By (4.20), we can see

$$\theta_\alpha^\pm = \pm \int_0^1 \frac{d\rho}{\sqrt{\rho^\alpha - \rho^2}} = \pm \frac{\pi}{2-\alpha}. \tag{4.21}$$

Thus by Lemmas 4.4, 4.5 and (4.21), we obtain Proposition 4.1. ∎

Proof of Proposition 0.3. We can easily deduce Proposition 0.3 from Propositions 3.1 and 4.1. ∎

5. Proofs of Theorems 0.1 and 0.2

Now we can deduce Theorems 0.1 and 0.2 from Propositions 1.3, 0.3 and (4.1)–(4.2).

Proof of Theorem 0.1. Let $(q_\epsilon(t))_{\epsilon\in(0,1]}$ be a sequence of critical points given in Proposition 1.3. By Proposition 1.5, we can extract a subsequence $\epsilon_n \to \infty$ such that $q_n(t) = q_{\epsilon_n}(t)$ satisfies the assumptions of Proposition 0.3. Since $i(\alpha) \geq 2$ for $\alpha \in (1,2)$, we have from Proposition 0.3

$$\liminf_{n\to\infty} \operatorname{index} I''_{\epsilon_n}(q_n) \geq 2(N-2)\nu.$$

Comparing with (1.18), we can see

$$\nu = 0.$$

That is, $q_\infty(t)$ does not enter the singularity 0 and $q_\infty(t)$ is a non-collision T-periodic solution of (HS). ∎

Proof of Theorem 0.2. Proof of Theorem 0.2 can be done in a similar way to the proof of Theorem 0.1. However, by (4.1), $i(\alpha) = 1$ for $\alpha \in (0,1]$. Thus

$$\liminf_{n\to\infty} \operatorname{index} I''_{\epsilon_n}(q_n) \geq (N-2)\nu.$$

Comparing with (1.18), we get

$$\nu \leq 1.$$

This is the desired result. ∎

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