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Embeddings of Geometries and related Codes

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Abstract

Results are described on embeddings of primitive regular near polygons and flag-transitive classical locally polar geometries of rank 3 defined over $\mathbf{F}_2$, together with a simple idea to generalize the notion of apartments.

1. Introduction.

The aim of this exposition is to give a survey of results concerning embeddings of incidence geometries (see 3.1), including my recent contributions, together with a generalization of the notion of apartments to general geometries (see 2.5). I will treat these two unrelated-looking subjects uniformly from view of coding theory. Though we restrict ourselves to embeddings of geometries of rank 2, the notion of embeddings can be treated more generally in the context of homology groups of simplicial complexes with locally constant presheaves. (The usual homology groups in 2.2 are nothing more than those for the constant presheaf of the tivial module. See [11] Ex.1 p.328). As for this interesting and promissing setting, see papers by Mark Ronan and Steve Smith [11] [14] [15].

The organization is as follows: In §2, we first summarize necessary terminology for incidence geometries and then introduce a code in the space with basis the maximal flags. Proposition 2.4 shows that minimum supports of this code are candidates for generalization of apartments to general geometries. The sporadic A_7-geometry (with a diagram of type C_3) will be treated as an interesting example. (Proposition 2.4 and Example 2.6 are worked out by the author in the summer 1988 under communication with Steve Smith, but not yet published. [26])

In the reminder of this exposition are devoted to embeddings of geometries of rank 2. Fundamental facts about embedding are reviwed in §3 together with an elementary lemma which motivates the investigation of minimum supports of some codes. They appeared in [10] in a slightly different form (in terms of cohomologies). In §4, we briefly review properties of some important families of geometries of rank 2: regular near polygons, generalized polygons (buildings of rank 2) and plane-line truncations of flag-transitive classical locally polar geometries of rank 3. Answers for problems proposed in 3.7 will be summarized in §5 to these geometries defined over $\mathbf{F}_2$. Some of them are calculated earlier in the different setting, but at least the results for the following geometries seem to be new: thin generalized polygons, the near n-gon $H(n,3)$ of Hamming graph for any n, the near hexagons $B_3(2)$ and ${}^2A_5(2)$ for dual polar spaces, the sporadic near polygons on 729 points and 315 points, and the truncations of flag-transitive classical locally polar spaces of rank 3 over $\mathbf{F}_2$.

2. Homology Groups for Incidence Geometries.

2.1 Incidence Geometries. We consider an ordered sequence $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{r-1}; *)$ of r (≥ 2) pairwise disjoint nonempty sets $\mathcal{G}_i$ ($i = 0, \ldots, r-1$) together with a reflexive and symmetric relation $*$ (called *incidence*) on their union $V\mathcal{G} := \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_{r-1}$. A nonempty subset F of $V\mathcal{G}$ is called a *flag* if x is *incident* with y (that is, $x*y$) for any $x, y \in F$. The sequence $\mathcal{G}$ is called an *incidence geometry* defined on $I = \{0, \ldots, r-1\}$, if any flag of $\mathcal{G}$ is contained in a maximal flag F with $|F \cap \mathcal{G}_i| = 1$ for all $i = 0, \ldots, r-1$. The number r is called the *rank* of $\mathcal{G}$. Elements of $\mathcal{G}_0$ and $\mathcal{G}_1$ are usually called *points* and *lines*, respectively.

For a flag $F = (x_{j_0}, \ldots, x_{j_i})$, where $0 \le j_0 < \cdots < j_i \le r-1$ and $x_{j_m} \in \mathcal{G}_{j_m}$ $(m = 0, \ldots, i)$, the *type* of F is defined to be the subset $\{j_0, \ldots, j_i\}$ of I and denoted by $Type(F)$. The number $|Type(F)|$ is called the *rank* of F. If $i \neq 0$ in the above, the j_m-th *face* of F is defined to be a flag obtained from F by deleting x_{j_m} $(m = 0, \ldots, i)$. If there are exactly $s_i + 1$ maximal flags containing each flag of type $I - \{i\}$ and any $i \in I$, $\mathcal{G}$ is called of *order* $(s_0, \ldots, s_{r-1})$. If s_0 is a power of a prime, say q, $\mathcal{G}$ is said to be *defined over* $\mathbf{F}_q$.

The full automorphism group $Aut(\mathcal{G})$ is the group of all bijections g on $V\mathcal{G}$ such that $x^g \in \mathcal{G}_i$ iff $x \in \mathcal{G}_i$ $(i \in I)$ and $x * y$ iff $x^g * y^g$ for any $x, y \in V\mathcal{G}$. A pair $(\mathcal{G}, G)$ of an incidence geometry $\mathcal{G}$ and a subgroup G of $Aut(\mathcal{G})$ is called *flag-transitive*, if G acts transitively on the set of maximal flags of $\mathcal{G}$. If $(\mathcal{G}, G)$ is flag-transitive, there is an element $g \in G$ with $F^g = F'$ for any flag F, F' with $Type(F) = Type(F')$. An incidence geometry $\mathcal{G}$ is called *flag-transitive*, if $(\mathcal{G}, Aut(\mathcal{G}))$ is flag-transitive.

The *incidence graph* of $\mathcal{G}$ is defined to be a graph with $V\mathcal{G}$ as its set of vertices by declaring that two vertices form an edge whenever they are incident. Two points of $\mathcal{G}_0$ are called *collinear* if they are incident with a line of $\mathcal{G}_1$ in common. The graph defined on $\mathcal{G}_0$ whose edges are the pairs of collinear points is called the *collinearity graph* of $\mathcal{G}$. A geometry $(\mathcal{G}_i, \mathcal{G}_j; *)$ of rank 2 obtained from $\mathcal{G}$ by taking vertices of type i is called the (i,j)-*truncation* of $\mathcal{G}$.

As for examples of geometries of rank 2, see §4.

2.2 Homology groups. Let $\mathcal{G}$ be an incidence geometry defined on $I = \{0, \ldots, r-1\}$. The set $\Delta = \Delta(\mathcal{G})$ of all flags of $\mathcal{G}$ can be recognized as an abstract simplicial complex with its set of vertices $V\mathcal{G}$. Thus we may associate with an incidence geometry $\mathcal{G}$ several standard modules in algebraic topology.

For $i \in I$, we denote by $C_i = C_i(\Delta)$ the group of *i-chains*, that is, the free $\mathbf{Z}$-module with basis Δ_i, where Δ_i denotes the set of flags of $\mathcal{G}$ of rank $i+1$. The i-th *boundary* map ∂_i from C_i to C_{i-1} $(i \neq 0)$ is defined by $\partial_i(F) = \sum_{m=0}^{i} F_{j_m}$ for a flag F with type $\{j_0, \ldots, j_m\}$ and by extending linearly, where F_{j_m} means the j_m-face of F $(m = 0, \ldots, i)$. The map ∂_0 from C_0 to $C_{-1} := \mathbf{Z}$ is defined to be the zero map. We denote by the image and the kernel of ∂_i by $B_i(\Delta) = B_i$ $(\subseteq C_{i-1})$ and $Z_i(\Delta) = Z_i$ $(\subseteq C_i)$, respectively $(i \in I)$. We also set $H_i(\Delta) = H_i := Z_i / B_{i+1}$ $(i \in I)$. The $\mathbf{Z}$-modules B_i, Z_i and H_i are called the i-th group of *boundaries*, *cycles* and i-th *homology* group, respectively. By tensoring with any field K over its prime field, we have K-vector spaces $C_i(K) := C_i \otimes K$, $B_i(K) := B_i \otimes K$, $Z_i(K) := Z_i \otimes K$ and $H_i(K) := H_i \otimes K$.

If $(\mathcal{G}, G)$ is flag-transitive, the group G permutes Δ_i and so acts on C_i $(i \in I)$. The module C_{-1} is a trivial G-module. Since G preserves the types of vertices, we may verify that the action of G is compatible with the boundary maps: $(F^g)\partial_i = ((F)\partial_i)^g$, $g \in G$, $i \in I$, $F \in \Delta_i$. Thus B_i, Z_i and H_i are considered as G-modules, and therefore, we have representations $B_i(K)$, $Z_i(K)$ and $H_i(K)$ of G over any field K $(i \in I)$.

2.3 $H_{r-1}(K)$ as a code in $C_{r-1}(K)$. Since $B_{r-1} = \{0\}$, the top dimensional homology group $H_{r-1}(K) = Z_{r-1}(K)$ is a subspace of the vector space $C_{r-1}(K)$ with the specified basis $C := \Delta_{r-1}$, the set of maximal flags of $\mathcal{G}$. Thus we may consider $H_{r-1}(K)$ as a linear *code* in $C_{r-1}(K)$ for any field K. From view of coding theory, it is natural and fundamental to ask "what is the dimension, the minimum weight and the corresponding support of this code?", where the *support* of a vector $x \in H_{r-1}(K)$ (denoted by $supp(x)$) is defined to be the subset $supp(x) := \{F \in \Delta_{r-1} \mid x = \sum_{F\in\Delta_{r-1}} \alpha_F F,\ \alpha_F \neq 0 \in K\}$ of Δ_{r-1} and the *weight* of x is $|supp(x)|$. That is, the minimum number w of the weights for all nonzero vectors of $H_{r-1}(K)$ and the subsets $supp(x)$ of Δ_{r-1} with $wt(x) = w$ are main concern in coding theory, together with $\dim H_{r-1}(K)$.

If the geometry $\mathcal{G}$ is a natural finite geometry associated with a finite group of Lie type (called a *building*), it is rather easy to answer these questions.

2.4 Proposition. *Let $\mathcal{G}$ be a finite building for a finite group G of Lie type defined over $K = \mathbf{F}_{p^e}$ and W the Weyl group of G. Then the code $H_{r-1}(K)$ in $C_{r-1}(K)$ affords the Steinberg module of G, and so its dimension is equal to the highest power of p dividing $|G|$. The minimum weight is given by $|W|$ and for $x \in H_{r-1}(K)$ we have $wt(x) = |W|$ if and only if $supp(x)$ forms the set of maximal flags in an apartment of $\mathcal{G}$.*

The above proposition suggests that the minimum supports of the code $H_{r-1}(K)$ for suitablly chosen field K are considered as "apartments" of a general geometry $\mathcal{G}$. Thus it is natural to propose the following problem.

2.5 Problem. For a general incidence geometry $\mathcal{G}$, determine the minimum weight and the corresponding supports of the code $H_{r-1}(\Delta(\mathcal{G}), K)$ in $C_{r-1}(\Delta(\mathcal{G}), K)$.

2.6 Example. The sporadic A_7-geometry. (see e.g. [3] p.392) This geometry $(\mathcal{P}, \mathcal{L}, \mathcal{Q}; *)$ is known to be the unique example of geometries belonging to diagrams those for finite buildings, but not a building or its quotient. The sets $\mathcal{P}$ and $\mathcal{L}$ are the set $\{1, \dots, 7\}$ and the set of triples of points, respectively. There are in total 30 ways to choose a subset $\mathcal{T}$ of $\mathcal{L}$ so that $(\mathcal{P}, \mathcal{T})$ forms a projective plane of order 2 with incidence by natural inclusion, and they are divided into two orbits of length 15 under the action of the alternating group A_7 on $\mathcal{P}$. We define $\mathcal{Q}$ to be one of these two orbits and define $*$ by natural inclusion. Thus $\mathcal{G}$ belongs to the *diagram* C_3, that is, the residues at points (resp. lines and planes) are generalized quadrangles (resp. digons and triangles) (see 4.2). Note that the building admitting $O_7(2) \cong Sp_6(2)$ or $U_6(2)$ also belongs to the same diagram.

We choose $F = \mathbf{F}_2$ as the coefficient field of homologies of $\Delta(\mathcal{G})$. The space C_2 of 2-chains of $\mathcal{G}$ has a basis indexed by $315 = 15 \cdot 7 \cdot 3$ maximal flags. The code H_2 is a subspace of C_2 of dimension 56, which is a multiple of the highest power of 2 dividing $|A_7|$. As an A_7-module, H_2 is projective, but not irreducible. (This follows from [17] 3.1 and the fact that $\Delta(\mathcal{G})$ is *Cohen-Macaulay*, that is, the reduced homology group $\tilde{H}_i$ of $\mathcal{G}$ and its residues are trivial except those for top dimension.) The minimum supports are of weight 36 and form one conjugacy class under A_7 with stabilizers S_3. Note that the minimum support of the code $H_2(\Delta(\mathcal{B}), \mathbf{F}_q)$ is $|W| = |2^3 S_3| = 48$ for the buildings $\mathcal{B}$ belonging to the diagram of type C_3. For the detail, see [26].

3. Embeddings and Codes for Geometries.

In this section, we consider realizations of incidence geometries as configurations of subspaces in projective spaces, in which incidences are defined by natural inclusion. We formalize this notion as follows. For simplicity, we restrict ourselves only to the case of rank 2 and try to define any objects as simple as possible.

3.1 Embeddings. Assume that a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of rank 2 is defined over $\mathbf{F}_q$. A pair (V, ρ) of a vector space V over $\mathbf{F}_q$ with $\dim V \geq 2$ and a map $\rho : \mathcal{P} \to V$ is called a (point-line) *embedding* of $\mathcal{G}$, if they satisfy the following conditions:
(1) For each $\alpha \in \mathcal{P}$, $\rho(C)$ is a subspace of V of dimension 1.
(2) For any $l \in \mathcal{L}$ and $q+1$ points α_i $(i = 0, \ldots, q)$ through l, the subspace $\rho(l)$ of V spanned by $\rho(\alpha_i)$ $(i = 1, \ldots, q+1)$ is of dimension 2. Furthermore, $\{\rho(\alpha_i) | i = 0, \ldots, q\}$ coincides with the set of all subspaces of dimension 1 of the 2-dimensional subspace $\rho(l)$.
(3) The vector space V is spanned by all $\rho(\alpha)$ for $\alpha \in \mathcal{P}$.

The subspace V is called an *ambient* space. For two embeddings (ρ, V) and (ρ', V') of $\mathcal{G}$, a *morphism* of (ρ, V) to (ρ', V') means an $\mathbf{F}_q$-linear map f of V to V' for which $\rho'(\alpha) = f(\rho(\alpha))$ for any $\alpha \in \mathcal{P}$. If such a morphism f is a bijection, (ρ, V) and (ρ', V') are called *isomorphic.*

3.2 Universal embeddings. It is known that for any embedding (ρ, V) of $\mathcal{G}$ there is a unique (up to isomorphism) embedding $(\tilde{\rho}, \tilde{V})$ and a morphism $\tilde{p}$ of $(\tilde{\rho}, \tilde{V})$ to (ρ, V) satisfying the following universal property: [10] For any morphism g of (ρ', V') to (ρ, V), there is a morphism $\tilde{g}$ of $(\tilde{\rho}, \tilde{V})$ to (ρ', V') such that $g\tilde{g} = \tilde{p}$. Such an embedding $(\tilde{\rho}, \tilde{V})$ is called a *universal* embedding of the embedding (ρ, V). In general, the universal embeddings of two non-isomorphic embeddings of $\mathcal{G}$ may or may not be isomorphic. However, the situation is quite simple if $\mathcal{G}$ is defined over $\mathbf{F}_2$ (see 2.1). Here we need some terminology about codes.

3.3 Incidence codes for geometries. For a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of rank 2 and a field F, we denote by $F\mathcal{P}$ the vector space over F with basis e_α $(\alpha \in \mathcal{P})$ indexed by $\mathcal{P}$. The *incidence matrix* N of $\mathcal{G}$ is a matrix with rows and columns indexed by $\mathcal{L}$ and $\mathcal{P}$ respectively, in which the (l, α)-th entry $(l \in \mathcal{L},\ \alpha \in \mathcal{P})$ is 1 if $l * \alpha$ and 0 otherwise. Each row $(x_\alpha)_{\alpha \in \mathcal{P}}$ of N can be identified with the vector $\sum_{\alpha \in \mathcal{P}} x_\alpha \mathrm{e}_\alpha$ of $F\mathcal{P}$, and the subspace of $F\mathcal{P}$ spanned by all rows of N is denoted by $C_F(\mathcal{G})$ (or simply $C(\mathcal{G})$). The *dual* code of $C(\mathcal{G})$ is defined to be the subspace $C(\mathcal{G})^\perp := \{\mathrm{x} \in F\mathcal{P} | \mathrm{x} \cdot \mathrm{y} = 0 (\forall \mathrm{y} \in C(\mathcal{G}))\}$ of $F\mathcal{P}$, where $\mathrm{x} \cdot \mathrm{y} = \sum_{\alpha \in \mathcal{P}} x_\alpha y_\alpha$ for $\mathrm{x} = \sum_{\alpha \in \mathcal{P}} x_\alpha \mathrm{e}_\alpha$ and $\mathrm{y} = \sum_{\alpha \in \mathcal{P}} y_\alpha \mathrm{e}_\alpha$.

These codes have been discussed by several authors (for example, [5]). From geometric point of view, these codes are most interesting if $\mathcal{G}$ is defined over $\mathbf{F}_2$ and $F = \mathbf{F}_2$.

3.4 Lemma. *For a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of rank 2 defined over $\mathbf{F}_2$, the map from $\mathcal{P}$ to $V(\mathcal{G}) := \mathbf{F}_2\mathcal{P}/C(\mathcal{G})$ given by $\rho(\alpha) = \mathrm{e}_\alpha + C(\mathcal{G})$ for $\alpha \in \mathcal{P}$ affords the universal embedding of $\mathcal{G}$, if $V(\mathcal{G}) \neq \{0\}$.* We will call $V(\mathcal{G})$ the *universal* embedding of $\mathcal{G}$ (see [10], Prop. 3).

3.5 Geometric hyperplanes and embeddings. In general, a geometry $\mathcal{G}$ defined over $\mathbf{F}_q$ may not have any embedding. However, there is a nice criterion for the existence of embeddings of a geometry defined over $\mathbf{F}_2$ (see [10], §3, Cor.2,4), in terms of a geometric object: A non-empty proper subset H of $\mathcal{P}$ for a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of rank 2 is called a *geometric hyperplane* of $\mathcal{G}$, if H contains either all the points on l or exactly one point on l for any line $l \in \mathcal{L}$.

3.6 Lemma. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ be a geometry of rank 2 defined over $\mathbf{F}_2$. Then the map associating $\mathcal{P} - supp(\mathrm{x})$ with a vector $\mathrm{x} = \sum_{\alpha \in \mathcal{P}} k_\alpha \mathrm{e}_\alpha$ $(k_\alpha \in \mathbf{F}_2)$ gives a bijection from $C(\mathcal{G})^\perp$ onto the set of all geometric hyperplanes of $\mathcal{G}$ and $\mathcal{P}$, where $supp(\mathrm{x}) := \{\alpha \in \mathcal{P} | k_\alpha \neq 0\}$ is the support of x. In particular, there is a geometric hyperplane H of $\mathcal{G}$ if and only if there is an embedding of $\mathcal{G}$.*

The above lemma shows that the support of a non-zero vector $\mathbf{x}$ of $C^{\perp}$ with minimum weight is a complement in $\mathcal{P}$ of a geometric hyperplane of $\mathcal{G}$ of maximum size. Thus it is interesting from both coding-theoretic and geometric point of view to solve the following problems.

3.7 Problems. For a geometry $\mathcal{G}$ defined over $\mathbf{F}_2$, determine the dimension of $C^{\perp}$ (= the dimension of the ambient space of the universal embedding of $\mathcal{G}$) and the minimum supports of $C^{\perp}$.

3.8 Remark. The dimensions, minimum weights and supports of the codes $C_F(\mathcal{G})$ and $C_F(\mathcal{G})^{\perp}$ for finite field F have been investigated by Bagchi and Sastry [5] when $\mathcal{G}$ is a generalized $2d$-gon of order (s,t) $(s,t \geq 2)$ (see 4.2). However, similar arguments to them (see [5], Theorem 3.6) do not apply to $F = \mathbf{F}_2$.

4. Generalized Polygons, Regular Near Polygons and truncated Locally polar spaces of rank 3 over $\mathbf{F}_2$.

We consider Problems 3.7 for several important geometries of rank 2 defined over $\mathbf{F}_2$; that is, regular near polygons, generalized polygons and the line-plane truncations of locally polar spaces of rank 3. In this section, these geometries are briefly reviewed. As for general terminology for geometries, see 2.1. For regular near polygons and generalized polygons, see e.g. [3] 6.4, and for locally polar geometries, see [9], [23] (in which they are called $c.C_n$-geometries).

As we shall see below, the locally polar geometries of rank 3 defined over $\mathbf{F}_2$ (exept for that admitting flag-transitive group $3O_6^-(3)$) are subgeometries of suitable dual polar spaces of rank 3, in which point-line truncations form regular near hexagons. (This article seems to be the first literature referring the fact that the points of the locally polar geometry admitting $O_6(3)$ form a geometric hyperplane of the dual polar space admitting $U_6(2)$.) Thus solutions for Problem 3.7 for these geometries are closely related.

4.1 Regular near polygons. (See [3],6.4 p.198 or [16] p.2. Note that, for example, we exclude generalized quadrangles from regular near hexagons.) An incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of order (s,t) is called a *near $2d$-gon* (resp. $(2d+1)$-*gon*) for a natural number d, if the diameter of the collinearity graph of $\mathcal{G}$ is d (resp. the diameter of the incidence graph $I\Gamma$ of $\mathcal{G}$ is $2d+1$) and for any integer i with $0 \leq i \leq d-1$ and any point α and any line l at distance $2i+1$ (in $I\Gamma$), there is a unique point β on l with $\delta(\alpha,\beta) = 2i$ (resp. furthermore, all points on a line l at distance (in $I\Gamma$) $2d+1$ from a point α are at distance $2d$ from α). A near polygon ($2d$ or $(2d+1)$-gon) is called *regular*, if its collinearity graph is distance-regular.

4.2 Generalized polygons. (As for generalized quadrangles, see [8].) A geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of rank 2 of order (s,t) is called a *generalized n-gon*, if the incidence graph of $\mathcal{G}$ is of diameter n and of girth $2n$. A $(1,t)$-*sub $2d$ gon* of a generalized $2d$-gon $\mathcal{G}$ is a subset X of $\mathcal{P}$ such that the geometry $(X, Y; *)$ is a generalized $2d$-gon of order $(1,t)$, where Y is the set of lines incident with at least two distinct points of X.

A generalized $2d$-gon $\mathcal{G}$ of order (s,t) is a regular near $2d$-gon and generalized 3-gons (generalized *triangles*) of order (s,t) are nothing more than projective planes of order (s,t). In particular, $s = t$ for generalized triangles. Generalized 4-, 6- and 8-gons are called generalized *quadrangles, hexagons* and *octagons*, and denoted by GQ, GH and GO, respectively. Finite generalized n-gons are (weak) buildings of rank 2, admitting flag-transitive actions of associated groups of Lie type.

There are six known infinite families of flag-transitive GQ of order (s,t) with s or $t \geq 2$: they are $Q^+(3,q)$, $Q(4,q)$, $Q^-(5,q)$, $H(3,q^2)$, $H(4,q^2)$ and $W(q)$, admitting flag-transitive actions of finite groups $O_4^+(q)$, $O_5(q)$, $O_6^-(q)$, $U_4(q^2)$, $U_5(q^2)$ and $S_4(q)$ of Lie type, respectively. They (sometimes omitting $Q^+(q)$ of order $(q,1)$ as in [24]) are called *classical* GQ (see [8] 3.1 pp.36–37). Note that except $O_4^+(q) \cong L_2(q) \times L_2(q)$, all groups above are simple.

4.3 Generalized polygons defined over $\mathbf{F}_2$. We now consider the isomorphism classes of generalized polygons of order $(2,t)$. Except GOs of order $(2,4)$, the classification is completed, as we will described below. First, by the remarkable theorem of Feit-Higman (e.g. [3] 6.5.1 p.210), one of the following holds for a generalized n-gon $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of order $(2,t)$.
(1) $t = 1$ and $n = 6, 8$ or 12. There is a generalized $n/2$-gon $\mathcal{G}_0$ of order $(2,2)$ such that $\mathcal{P}$ and $\mathcal{L}$ the sets of vertices and edges of the incidence graph of $\mathcal{G}_0$ and $*$ is the natural inclusion.
(2) $n = 3$ and $t = 2$. (3) $n = 4$ and $1 \leq t \leq 4$.
(4) $n = 6$ and $t = 2$ or 8. (5) $n = 8$ and $t = 4$.

In the case (2), $\mathcal{G}$ is isomorphic to the projective plane $\mathbf{PG}(2,\mathbf{F}_2)$ with 7 points and lines. In the case (3), it follows from [PT] 6.1 pp.122–123 that $\mathcal{G}$ is isomorphic to the classical generalized quadrangle $Q^+(3,2)$, $Q(4,2) \cong W(2)$ or $Q^-(5,2)$ for $t = 1, 2$ or 4, respectively (we do not have $t = 3$). In the case (4), it follows from theorems 1 and 2 in [CT] that $\mathcal{G}$ is isomorphic to the GH $\mathcal{H}(2)$ with the full automorphism group $G_2(2)$ and point-stabilizers $4S_4 : 2$, the dual $\mathcal{H}(2)^*$ of $\mathcal{H}(2)$ with the full automorphism group $G_2(2)$ and point-stabilizers $4^2 : D_{12}$, or the GH $\mathcal{D}$ with the full automorphism group ${}^3D_4(2).3$ and point-stabilizers $2_+^{1+8} : L_2(8) : 3$.

There is a unique known GO of order $(2,4)$, that is, the GO $\mathcal{O}(2)$ with the full automorphism group ${}^2F_4(2)$ and point-stabilizers $2.[2^8] : 5 : 4$. Though it seems to be believed that any GO of order $(2,4)$ is isomorphic to $\mathcal{O}(2)$, no reference of a proof is available, as far as the autor knows.

4.4 Other families of regular near polygons defined over $\mathbf{F}_2$. In 4.4–5, we assume that regular polygons are not generalized polygons. There are three known families $H(n,3)$, $C_d(2) \cong B_d(2)$ and ${}^2A_{2d-1}(2)$ of regular near polygons defined over $\mathbf{F}_2$. (See [3], Table 6.6 p.206, noticing that a near polygon is defined over $\mathbf{F}_2$ iff $\lambda = 1$. Furthermore, note that there are misprints in Table 6.6 ! We should erase the number d appearing the column for k and rows (R4) (R5) (R6), and then shift the numbers in the colums for λ and c_i into the columns for k and λ, respectively. As for the column for c_i, the quantities in the rows (R4),(R5) and (R6) should be $c_i = i$ ($i < d$), $c_d = \gamma d$ (for (R4)), $c_i = i$ (for (R5)) and $c_i =$ the q-binomial coefficient for i choose 1 (for (R6)).)

The geometry $H(n,3)$ is a geometry whose incidence graph is the Hamming graph on the n-dimensional space $\mathbf{F}_3^n$ over $\mathbf{F}_3$, that is, the set of points and lines consist of vectors of $\mathbf{F}_3^n$ and the translations of the sets $\{0, \mathrm{e}_i, -\mathrm{e}_i\}$ ($i = 1, \ldots, n$) by vectors of $\mathbf{F}_3^n$, respectively, where we denote by e_i the ith natural base of $\mathbf{F}_3^n$.

The other two families are members of dual polar spaces (more precisely, truncations of them) $B_d(2) = C_d(2)$ and ${}^2A_{2d-1}(2)$ [3] 9.4. The points of $B_d(2)$ (resp. ${}^2A_{2d-1}(2)$) are totally singular (resp. isotropic) subspaces of dimension d of a $2d+1$-(resp. $2d$-)dimensional vector space over $\mathbf{F}_2$ (resp. $\mathbf{F}_4$) with a non-degenerate orthogonal (resp. unitary) form. A triple of points containing a totally singular (resp. isotopic) subspace of dimension $d-1$ in common is called a line. The incidence is defined by natural inclusion. By taking all singular subspaces as vertices, we get a geometry of rank d, which is called a *dual polar space.* Thus our geometries are truncations (see 2.1) of dual polar spaces on maximal and submaximal singular spaces.

4.5 Three sporadic examples of primitive regular near n-gons. Besides the families in 4.4, the following three sporadic examples are known to be regular near $2d$-gons defined over $\mathbf{F}_2$, admitting primitive actions of automorphism groups on the sets of points. (See [3], Table 6.7 p.207. Remark that the full automorphism group of (N2) should be $3^6 : (2M_{12})$, not 3^6M_{12}.) In the below, we denote by $\Gamma_i(\omega)$ the set of points at distance i from a point ω in the collinearity graph Γ. The characterizations of these near polygons by parameters are established (see [3], 11.4.1, 11.3.1, 11.6.1 and [19]).

(1) A regular near hexagon $\mathcal{M}$ on 759 points of order $(2,14)$ admitting $Aut(\mathcal{M}) \cong M_{24}$, in which the orbits of the stabilizer of a point ω on the points are $\Gamma_i(\omega)$ $(i = 0,1,2,3)$ of lengths $1,30,280,448$, respectively.

(2) A regular near hexagon $\mathcal{N}$ on 729 points of order $(2,11)$ admitting $Aut(\mathcal{N}) \cong 3^6 : 2M_{12}$, in which the orbits of the stabilizer of a point ω on the points are $\Gamma_i(\omega)$ $(i = 0,1,2,3)$ of lengths $1,24,264,440$, respectively.

(3) A regular near octagon $\mathcal{J}$ on 315 points of order $(2,4)$ admitting $Aut(\mathcal{J}) \cong J_2.2$, in which the orbits of the stabilizer of a point ω on the points are $\Gamma_i(\omega)$ $(i = 0,1,2,3,4)$ of lengths $1,10,80,160,64$, respectively.

4.6 Flag-transitive classical locally polar spaces of rank 3 defined over $\mathbf{F}_2$. [18][23][9][24] A geometry $(\mathcal{P},\mathcal{L},\mathcal{Q};*)$ of rank 3 is called a *classical locally polar geometry* if the residue at each $p \in \mathcal{P}$ (resp. $l \in \mathcal{L}$ and $\pi \in \mathcal{Q}$) is a classical GQ of order (s,t) with s and $t \geq 2$ (see 4.2) (resp. a generalized 2-gon and the geometry of vertices and edges of a complete graph). In this article, we call them FECQs (flag-transitive extended classical quadrangles). They are now completely classified (see e.g. [9],[18],[23]). We consider the plane-line (that is, $(2,1)$-) truncations (see 2.1) of them.

There are six FECQs with $(2,1)$-truncations defined over $\mathbf{F}_2$; The FECQ $\mathcal{A}$ with 32 points admitting $2^5O_5(2).2$ and its quotient $\overline{\mathcal{A}}$ with 16 points, the FECQ $\mathcal{S}$ with 28 points admitting $O_6^+(2).2$, a FECQ $\mathcal{F}$ with 36 points admitting $O_6^-(2).2$, and the FECQ $\mathcal{O}$ with 378 points admitting $3O_6^-(3)$ and its quotient $\overline{\mathcal{O}}$ with 126 points. The residues at points are isomorphic to $W(2) \cong Q(4,2)$ for FECQs $\mathcal{A}$, $\overline{\mathcal{A}}$, $\mathcal{S}$, $\mathcal{F}$ and to $H(3,2^2)$ for FECQs $\mathcal{O}$ and $\overline{\mathcal{O}}$.

We will observe that, except $\mathcal{O}$, they are subgeometries of the dual polar spaces for near hexagons $B_3(2)$ and ${}^2A_5(2)$. Let $\mathcal{G} = (\mathcal{Q},\mathcal{L},\mathcal{P};*)$ be the dual polar space for $B_3(2)$, that is, $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{P}$ are the sets of 135 singular planes, 315 lines and 63 points of a 7-dimensional orthogonal space V. For any hyperplane H of V and $X = \mathcal{P}$, $\mathcal{L}$ or $\mathcal{Q}$, we denote by $X \cap H$ the set of members of X lying completely in H, and set $X' = X - X \cap H$. Since singular subspaces containing $p \in \mathcal{P}'$ does not lie in H, the residue at p of the subgeometry $\mathcal{G}' = (\mathcal{P}',\mathcal{L}',\mathcal{Q}')$ coincides with that of $\mathcal{G}$, and so it is a classical GQ $Q(4,2)$. Then we may verify that $\mathcal{G}'$ is a FECQ. We let v the dimension of maximal singualr subspaces contained in H. Then $v = 6$, 3 or 2. If $v = 6$, $H = p^{\perp}$ for some $p \in \mathcal{P}$, and so $|\mathcal{Q} \cap H| = |\mathcal{Q}(p)| = 15$, $|\mathcal{L} \cap H| = 75$ and $|\mathcal{P} \cap H| = |\{p,q \in \mathcal{P} | q \in l \in \mathcal{L}(p)\}| = 31$, and $\mathcal{G}'$ is isomorphic to $\mathcal{A}$. If $v = 3$, $(\mathcal{Q} \cap H, \mathcal{L} \cap H, \mathcal{P} \cap H)$ is the dual polar space for $O^+(6,2)$ and $\mathcal{G}'$ is isomorphic to $\mathcal{S}$. Since any $l \in \mathcal{L} \cap H$ is contained in two planes of $\mathcal{Q} \cap H$, $\mathcal{Q}'$ is a geometric hyperplane of the near hexagon $(\mathcal{Q},\mathcal{L})$. If $v = 2$, $\mathcal{Q} \cap H = \emptyset$ and $(\mathcal{P} \cap H, \mathcal{L} \cap H)$ is the classical GQ $Q^-(5,2)$, and so $\mathcal{G}'$ is isomorphic to $\mathcal{F}$.

We may also establish that the the $(2,1)$-truncation of the FECQ $\overline{\mathcal{O}}$ is a subgeometry of the near hexagon ${}^2A_5(2)$, and that the set of planes forms a geometric hyperplane by an elementary counting argument. This accounts for the existence of maximal subgroups $U_4(3)$ in $U_6(2)$, and also gives an embedding of $\overline{\mathcal{O}}$ of dimension 21 via the embedding of ${}^2A_5(2)$ into the Leech lattice modulo 2. For the detail, see [22] [25].

5. Summary of Results.

In this section (see Tables 1–3 in the last page), we summarize the answers for Problem 3.7 to regular near polygons, generalized polygons and the plane-line truncations of flag-transitive locally polar geometries of rank 3 defined over $\mathbf{F}_2$ described in §4. As far as the author knows, the results for the following geometries are new: generalized polygons of order $(2,1)$, the near hexagons $H(3,n)$ for any n, $B_3(2)$, ${}^2A_5(2)$, the sporadic near polygons admitting $3^6 2M_{12}$, $J_2.2$ and the truncated FECQ. For some geometries, the answers are left open. (I do not claim that the remaining problems are difficult, especially for minimum weights.) Apparently, it is most interesting to get answers for families of geometries $B_d(2)$ and ${}^2A_{2d-1}(2)$ (and related subgeometries). I simply worked with the smallest cases so far, but suspect that the equalities always hold in the inequalities in Table 3.

In Table 1 and 3 in the last page, the columns for dim and Min.Wt. show the dimension of the ambient space of the universal embedding of $\mathcal{G}$ and the minimum weight of the code $C(\mathcal{G})^{\perp}$, respectively. The supports of minimum weight are briefly described in the last column of Table 1, if it is a sub n-gon. For the other cases, see below. The generalized hexagon (6) and (7) in Table 1 denote the hexagons $\mathcal{H}(2)$ and $\mathcal{H}(2)^*$ in 4.3, respectively. In Table 2, the weight enumerator of the code $C(\mathcal{G})^{\perp}$ (in $\mathbf{F}_2\mathcal{P}$) means a formal sum $\sum_{n=0} A_n z^n$, where A_n is the number of vectors of $C(\mathcal{G})^{\perp}$ of weight n.

It should be mentioned that several values in the tables below have already calculated. Amongst them, the work by Buekenhout and Lefèvre [4] determined the dimensions of embeddings for generalized quadrangles with $s \geq 2$ and $t \geq 2$, Mark Ronan and Steve Smith calculated the dimension of the universal embedding of generalized hexagons for $G_2(2)$ ([11] (3.3), [10] Examples 3,4) and the weight enumerators for the geometries (4),(6),(7) can be read from the works by Brooke (See [1] the column for (4), [2] the columns $14l$ in p.391 and in p.398 for (6) and (7), respectively. These enumerators can also be seen in [13] pp.308–309). The structure of the universal embedding of the near haxagon for M_{24} on 579 points is described in [10] Example 2, [15] pp.536–537 as well as an example of its geometric hyper planes.

Now I describe the method. First, we consider the problem to determine the dimensions of the universal embeddings. The collinearity graphs of regular near polygons and generalized polygons are distance regular graphs, and therefore we may calculate the eigenvalues and their multiplicities of the adjacency matrices A of these graphs ([3] 4.1(B)). Since ${}^tNN = A + (t+1)I$ for the incidence matrix N of the corresponding geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ of order $(2,t)$, the $\mathbf{Q}$-rank of N (the $\mathbf{Q}$-dimension of the subspace of $\mathbf{Q}\mathcal{P}$ (see 3.3) spanned by the rows of N) is given by $|\mathcal{P}| - m$, where m is the multiplicity of the eigenvalue $-(t+1)$ of A. Since the $\mathbf{F}_2$-rank is smaller than the $\mathbf{Q}$-rank in general, we have $\dim V(\mathcal{G}) \geq m$ by Lemma 3.4. (Up to here, exactly the same argument has developed in the proof of Theorem 3.6 in [5].)

Thus it is crusial to get a nice upper bound of $\dim V(\mathcal{G})$. This problem is treated by so called "geometric spanning argument", that is, to examine the *geometric span* of suitable chosen points. This argument has adopted by several people (see e.g. [11] pp.340–341, [7] for embeddings of locally Petersen geometries). Here, the *geometric span* of a subset X of points means the set of points obtained as the inductive limit of X_i $(i = 0, 1, \ldots,)$ with $X_0 = X$, where X_{i+1} consists of all the points lying on lines which contain at least two distint points of X_i. This part is of strong geometric flavor, and requires certain amount of works depending on detailed information on each geometry. For the regular near octagon admitting $J_2.2$, we consider a graph on suitable imprimitivity blocks for the subgroup $L_3(2)$ of J_2 (see [25]).

For the regular near hexagon $B_3(2)$ admitting $O_7(2)$, we may argue as follows: To each singular point p of the associated orthogonal space, the subspace $V(p)$ of the universal embedding $V(\mathcal{G})$

spanned by one-dimensional spaces for isotropic planes containing p admits an embedding of the generalized quadrangle $W(2) = Q(4,2)$ (the residue at p in the dual polar space for $\mathcal{G}$). Thus either $\dim V(p) = 4$ or $\dim V(p) = 5$ and $V(p)$ contains the non-zero vector w_p fixed by the stabilizer of p in $O_7(2)$. Thus the quotient space $V(B_3(2))/W$ by the subspace W of $V(B_3(2))$ spanned by w_p for all singular points p (we take $w_p = 0$ if the former case happens) admitts an embedding of the dual polar space for $\mathcal{G}$ in which singular points, lines and planes correspond to subspaces of dimensions 4, 2 and 1, respectively. Thus this affords the fixed-point presheaf [11] for the spin module for $O_7(2) = Sp_6(2)$, and we have $\dim V(\mathcal{G})/W \leq 2^3 = 8$ by [11] Theorem 4.1. As for the subspace W, there are one-dimensional subspaces $\langle w_p \rangle$ for singular points p. We may verify that the subspaces $W(l)$ and $W(\pi)$ spanned by w_p for all p contained in singular lines l and planes π are of dimension 2 and 3, respectively. Thus W affords the fixed-point presheaf of the polar space for $S_6(2) = O_7(2)$ for the natural module for $S_6(2)$, and therefore we have $\dim W \leq 7$ by [11] Theorem 4.1. Hence we have $\dim V(B_3(2)) = 15$, as 15 is the above multiplicity giving the upper bound.

Since FECQ defined over $\mathbf{F}_2$ are subgeometries of the above dual polar space for $B_3(2)$, we may determine the dimensions of embeddings of these geometries. In particular, we may verify that the locally polar geometry $\mathcal{A}$ for $O_6^+(2) \cong A_8$ gives a geometric hyperplane of $B_3(2)$. Thus we may give an another proof for the fact $\dim V(B_3(2)) = 15$ by first directly showing that $\dim V(\mathcal{A}) = 14$ using geometric spanning argument, which is not so difficult to establish [22]. The similar situation occurs when we try to determine $\dim V(^2A_5(2))$ and $\dim V(\overline{\mathcal{O}})$, since the locally polar geometry $\overline{\mathcal{O}}$ admitting $O_6^-(3)$ gives a geometric hyperplane of $^2A_5(2)$.

As for minimum weights and supports, it is immediate to observe that a sub $(1,t)$-gon (see 4.2) of a generalized polygon $\mathcal{G}$ of order $(2,t)$ affords a minimum support, provided it exists (see [5] Lemma 2.4). However, there is no such sub n-gon in the geometries (1),(4),(7),(8),(10). The minimum supports for (1) are non-degenerated quadrangles in $\mathbf{PG}(2,\mathbf{F}_2)$, those for (4) are $\Gamma_2(p) \oplus \Gamma_2(q)$ (symmetric difference) for non-adjacent distinct points p, q in the collinearity graph Γ. On the other hand, the similar questions for the remaining regular near polygons and locally polar geometries are left open except for the near octagon on 315 points, in which minimum supports are $\Gamma_4(p)$ for points p (in the collinearity graph Γ) and the vectors of minimum weight span $C(\mathcal{G})^\perp$ [25].

Finally, we list the dimensions of universal embeddings for FECQs defined over $\mathbf{F}_2$: 14, 9, 14, 15, 21, $12+21$ for the FECQ $\mathcal{A}$, $\overline{\mathcal{A}}$, $\mathcal{S}$, $\mathcal{F}$, $\overline{\mathcal{O}}$ and $\mathcal{O}$, respectively. See [22], for the detail and the dimensions of "group-admissible" embeddings of FECQs not defined over $\mathbf{F}_2$.

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Table 1: Generalized n-gons of order $(2, t)$ $(t \geq 1)$.

	n	t	Full Aut.	# Pts.	dim	Min.Wt.	Min.Sup.
(1)	3	2	$L_3(2)$	7	3	4	
(2)	4	1	$(S_3 \times S_3)2$	9	4	4	(1, 1) sub 4-gon
(3)	4	2	$O_5(2).2$	15	5	6	(1, 2) sub 4-gon
(4)	4	4	$O_6^-(2).2$	27	6	12	
(5)	6	1	$L_3(2)$	21	8	6	(1, 1) sub 6-gon
(6)	6	2	$G_2(2)$	63	14	16	
(7)	6	2	$G_2(2)$	63	14	14	(1, 2) sub 6-gon
(8)	6	8	${}^3D_4(3).3$	819	≥ 26		
(9)	8	1	$O_5(2).2$	45	16	8	(1, 1) sub 8-gon
(10)	8	4	${}^2F_4(2)$	1755	≥ 78		
(11)	12	1	$G_2(2)$	189	64	12	(1, 1) sub 12-gon
(12)	12	1	$G_2(2)$	189	64	12	(1, 1) sub 12-gon

Table 2: Weight Enumerators of $C(\mathcal{G})^\perp$ for generalized n-gons (1)–(7) in Table 1.

	Enumerator
(1)	$1 + 7z^4$
(2)	$1 + 9z^4 + 6z^6$
(3)	$1 + 10z^6 + 15z^8 + 6z^{10}$
(4)	$1 + 36z^{12} + 27z^{16}$
(5)	$1 + 28z^6 + 21z^8 + 84z^{10} + 98z^{12} + 24z^{14}$
(6)	$1 + 126z^{16} + 1596z^{24} + 2880z^{28} + 7497z^{32} + 4032z^{36} + 252z^{40}$
(7)	$1 + 36z^{14} + 56z^{18} + 252z^{20} + 378z^{24} + 1764z^{26} + 1800z^{28} + 1764z^{30}$ $+3591z^{32} + 4032z^{34} + 2044z^{36} + 504z^{38} + 126z^{40} + 36z^{42}$

Table 3: Known primitive near n-gons (not gen.polygons) of order $(2, t)$.

	n	t	Full Aut.	# Pts.	dim	Min.Wt.
(1.n)	n	$n-1$	$S_3 \wr S_n$	3^n	2^n	2^n
(2.d)	$2d$	$2^d - 2$	$P\Sigma O_{2d+1}(2)$	$\Pi_{i=1}^{d}(2^i + 1)$	$\geq (2^{d-1} + 1)(2^d + 1)/3$	
(2.3)	6	6	$O_7(2)$	135	15	
(3.d)	$2d$	$(4^d - 1)/3 - 1$	$P\Sigma U_{2d}(2)$	$\Pi_{i=1}^{d}(2^{2i-1} + 1)$	$\geq 2(2^{2d-1} + 1)/3$	
(3.3)	6	20	$U_6(2).2$	891	22	
(4)	6	14	M_{24}	759	23	≤ 448
(5)	6	11	$3^6(2M_{12})$	729	24	≤ 440
(6)	8	4	$J_2.2$	315	28	64