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ON THE SUBSCHEMES OF THE JOHNSON SCHEME J(v,d)

by Daiyu Uchida Faculty of Science
Kyushu University

Abstract.

For two association schemes χ and χ', defined on the same set, we call χ' a subscheme of χ if each relation of χ' is a union of some relations of χ. In this paper we prove that the Johnson scheme J(v,d) has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$. This slightly improves the earlier result of Muzichuk that the conclusion holds if $v\geq 3d+4$.

1. Introduction

Let $\chi=(X,\{R_i\}_{0\leq i\leq d})$ and $\chi'=(X,\{R'_i\}_{0\leq i\leq d'})$ be two association schemes defined on the same set X. We say that χ' is a subscheme of χ if each relation R'_i is a union of some relations R_i's.
The purpose of this paper is to prove that the Johnson scheme J(v,d) have no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$ .For a subscheme χ' of χ, we have a partition $\tau=\{T_0=\{0\},T_1\cdots,T_{d'}\}$ of the index set $\{0,1,\cdots,d\}$ such that

$$R'_i=\cup_{\alpha\in T_i} R_\alpha \quad (\text{ using the adjacency matrices, } A'_i=\sum_{\alpha\in T_i} A_\alpha) .$$

Clearly $\chi_0=(X,\{R_0,\cup_{i>0} R_i\})$ and χ are subschemes of χ. We call these subschemes trivial subschemes. Clearly if the association scheme χ is commutative then it's subscheme χ' is commutative, and if χ is symmetric then χ' is symmetric,too.

Throughout the whole paper we shall consider only symmetric association schemes. If we look at the Bose-Mesner algebra $\mathfrak{a}$ and $\mathfrak{a}'$, it immediately holds that χ' is a subscheme of χ if and only if $\mathfrak{a}' \subset \mathfrak{a}$. Then we also have the partition $\Pi=\{\pi_0=\{0\},\pi_1,\cdots,\pi_{d'}\}$ of the index set $\{0,1,\cdots,d\}$ such that

$$E'_j=\sum_{\beta\in\pi_j} E_\beta .$$

Now we calculate the entries of the first and second eigenmatrices of χ' . (For the notation of association scheme, see [2],[4])

$$|X|E'_j=\sum_{0\le i\le d'} q'_j(i)A'_i . \qquad \cdots (1,1)$$

The LHS of (1,1) $=|X|\sum_{\beta\in\pi_j} E_\beta$

$=\sum_{0\le\alpha\le d} (\sum_{\beta\in\pi_j} q_\beta(\alpha))A_\alpha .$

The RHS of (1,1) $=\sum_{0\le i\le d'}\sum_{\alpha\in T_i} q'_j(i)A_\alpha .$

Therefore compairing the both sides, we get $q'_j(i)=\sum_{\beta\in\pi_j} q_\beta(\alpha) \quad (\alpha\in T_i).$

Dually we get $p'_i(j)=\sum_{\alpha\in T_i} p_\alpha(\beta) \quad (\beta\in\pi_j).$

With the above notation ,we get the following lemma.

Lemma 1. Let χ' be a subscheme of χ, and τ,Π be the partitions of χ'. The indices i,j are glued in τ (namely R_i and R_j are in a same realtion R' of χ') (dually in Π) if and only if for each $0\le k\le d'$

$$\sum_{\beta\in\pi_k} q_\beta(i)=\sum_{\beta\in\pi_k} q_\beta(j) \text{ (dually } \sum_{\alpha\in T_k} p_\alpha(i)=\sum_{\alpha\in T_k} p_\alpha(j) \text{) .}$$

(cf. [1],[4])

Using this lemma, we get the following corollary.

Corollary. Let $\chi=(X,\{R_i\}_{0\le i\le d})$ be an association scheme and

$\chi'=(X,\{R'_i\}_{0\le i\le d'})$ be a non-trivial subscheme of χ .
If there exist $0< i_0,j_0 \le d$ such that
$q_j(i_0)>q_j(i)$ $(0<i\ne i_0 ,0<j\ne j_0)$,then
$R_{i_0}\in\{R'_i\}_{0\le i\le d'}$.

Proof Since $\chi'$is non-trivial, there exists a part π $(\ne\{0\})$ of the partition Π which dose not contain j_0 . Then

$$\sum_{j\in\pi} q_j(i_0)> \sum_{j\in\pi} q_j(i) \qquad (0<i\ne i_0) .$$

With Lemma 1 , we have $R_{i_0}\in\{R'_i\}_{0\le i\le d'}$. (Q.E.D.)

We now state our main result. Before that, we mention a brief history of this probrem (according to Muzichuk [4]) . Let J(v,d) be the Johnson scheme of class d (see [2],[3],for the details about the Johnson scheme) . As for the study of subscheme of J(v,d) or on the enumeration of subschemes of the Johnson scheme, first L.A.Kaluznin M.H.Klin proved that there exists a function f(d) such that Jonhson scheme J(v,d) does not have non-trivial subscheme for $v\ge f(d)$. M.E.Muzichuk proved that the same conclusion is true if $v\ge 3d+4$. The purpose of this paper is to prove the following Theorem.

Theorem A The Johnson scheme J(v,d) has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$.

This slightly improves the result of Muzichuk [4], because our condition becomes $v\ge 3d$ for $d\ge 6$.

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2. Proof of the Theorem A

For the Johnson scheme J(v,d), we have

$$p_i(j)=\sum_{0\le\nu\le j}(-1)^{\nu}\binom{j}{\nu}\binom{d-j}{i-\nu}\binom{v-d-j}{i-\nu}\ ,\ k_i=\binom{d}{i}\binom{v-d}{i}\ .\quad (\text{ see [2],[3],[4]})$$

In order to prove Theorem A , we need the following lemma.

Lemma 2. For J(v,d) with $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$, we have

$$q_j(1)>|q_j(i)| \qquad (1<i\le d,0<j<d)\ .$$

First we show that Thorem A is easily obtained for Lemma 2 .

Proof of Theorem A from Lemma 2.

Let $\chi'=(X,\{R_i\}_{0\le i\le d'})$ be a non-trivial subscheme of J(v,d) .

Then by Corollary, we have $R_1\in\{R_i'\}_{0\le i\le d'}$. i.e. we have $\{1\}\in\tau$.

If the indices i,j are glued in π, with Lemma1 , we have

$p_1(i)= p_1(j)$. Since

$p_1(i)= i^2-(v+1)i+d(v-d)$, $p_1(j)= j^2-(v+1)j+d(v-d)$, we have

$i= j$. (Q.E.D.)

Now we prove Lemma 2 with the inductive formula of the Johnson scheme:

$$p_i^{v,d}(j)=\begin{cases} p_i^{v-2,d-1}(j-1)-p_{i-1}^{v-2,d-1}(j-1) & (0<i<d) \\ -p_{d-1}^{v-2,d-1}(j-1) & (i=d)\ . \end{cases}$$

From now on, if necessary, we let $p_i^{v,d}(j)$ and $k_i(v,d)$ be the entries of the first eigenmatrix P and the valencies of J(v,d) respectively. Concerning Lemma 2 , we check at the special value of j= d-1 by the following Proposition.

Proposition. For J(v,d) with $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$, we have

$$q_{d-1}(1)>|q_{d-1}(i)| \qquad (\ 1<i\le d\ ) .$$

Proof Since $q_j(i)/m_j=p_i(j)/k_i$, we only have to show that

$$|p_i(d-1)|/k_i<p_1(d-1)/k_1 \qquad (\ 1<i\le d\ ) .$$

With the direct calculation, we have

$$p_i(d-1)/k_i=(-1)^{i-1}\ \frac{(v-2d+2)i-d}{d\binom{v-d}{i}} \ .\ \text{Then}$$

$$d(v-d-i)\binom{v-d}{i}(|p_i(d-1)|/k_i-|p_{i+1}(d-1)|/k_{i+1})$$

$$= -2(v-2d+2)i^2+(v^2-3dv+2d^2+4d-4)i-(d+1)v+d^2+3d-2 \ . \quad \cdots(*)$$

This reaches it's minimum when i=1 or d-1, i.e.

$$(*)\ge \min\{v^2-(4d+3)v+3d^2+11d-10,(d-1)(v-2d+1)\{v-(3d+\frac{2}{d-1})\}\}$$

Since $v^2-(4d+3)v+3d^2+11d-10>0$ is equal to $v>2d+\frac{3}{2}+\sqrt{(d-\frac{5}{2})^2+6}$,

we have

1)If $v\ge 3d+\frac{2}{d-1}$ (i.e. $v\ge 3d+1$) ,then we have $(*)\ge 0$. Then

$$p_1(d-1)/k_1>|p_2(d-1)|/k_2\ge \cdots \ \ge|p_d(d-1)|/k_d \ .$$

2)In the case of v=3d ($d\ge 6$), we have

$$|p_i(d-1)|/k_i\le \max\{\ p_1(d-1)/k_1\ ,\ |p_d(d-1)|/k_d\ \}$$

$$= \max\{\ 1/d^2,\ (d+1)/\binom{2d}{d}\ \} \ .\quad \text{Since}$$

$$\frac{d+1}{\binom{2d}{d}}d^2= \frac{d^2(d+1)!}{2d(2d-1)\cdots(d+1)}$$

$$< \frac{5\times 4\times 3}{(2d-1)(2d-2)} \le \frac{6}{11}\ ,\ \text{we have}$$

$$|p_i(d-1)|/k_i< p_1(d-1)/k_1 \qquad (\ i>1\ ) \ . \qquad (\text{ Q.E.D.})$$

Proof of Lemma 2

By Proposition, we only have to show that

$$|p_i(j)|/k_i< p_1(j)/k_1 \qquad (\ 1<i\le d,\ 0<j<d-1\ ) \ .$$

1)d=3

With the direct calculation, we have

$$p_1(1)/k_1= \frac{2v-9}{3(v-3)}\ ,\ p_2(1)/k_2= \frac{v-9}{3(v-3)}\ ,\ p_3(1)/k_3= \frac{-3}{v-3} \ .$$

Since $v>10$, we have

$$|p_i(1)|/k_i < |p_1(1)|/k_1 \qquad (\ i=2,3\ ) .$$

2) $d>3$

We consider the following two cases, Case 2.1; $1<i<d$, $0<j<d-1$ and Case 2.2; $i=d$, $0<j<d-1$,separately.

Case 2.1 $1<i<d$, $0<j<d-1$

With the inductive formula, we have

$$p_i^{v,d}(j) = p_i^{v-2,d-1}(j-1) - p_{i-1}^{v-2,d-1}(j-1) .$$

Since $v-2>2(d-1)+\frac{3}{2}+\sqrt{\{(d-1)-\frac{5}{2}\}^2+6}$, we have

$$|p_i^{v-2,d-1}(j-1)| \le p_1^{v-2,d-1}(j-1)\frac{k_i(v-2,d-1)}{k_1(v-2,d-1)} ,$$

$$|p_{i-1}^{v-2,d-1}(j-1)| \le p_1^{v-2,d-1}(j-1)\frac{k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)} , \text{ and}$$

$$|p_i^{v,d}(j)|/k_i(v,d) \le \frac{k_i(v-2,d-1)+k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)} p_1^{v-2,d-1}(j-1) .$$

We show that

$$\frac{k_i(v-2,d-1)+k_{i-1}(v-2,d-1)}{k_1(v-2,d-1)k_i(v,d)} p_1^{v-2,d-1}(j-1)k_1(v,d)/p_1^{v,d}(j)<1 \quad (*)$$

With the direct calculation, we have

$$\text{The LHS of } (*) = \frac{2i^2-vi+d(v-d)}{(d-1)(v-d-1)}\left(1+\frac{1}{j^2-(v+1)j+d(v-d)}\right) .$$

This reaches its maxmum when $i=2$, $j=d-2$. Therefore

$$\text{The LHS of}(*) \le \left(1-\frac{v-7}{(d-1)(v-d-1)}\right)\left(1+\frac{1}{2v-5d+6}\right) .$$

Since $\frac{1}{2v-5d+6} \le \frac{1}{d+6} < \frac{v-7}{(d-1)(v-d-1)}$, we have the LHS of $(*)<1$.

Case 2.2 $i=d$, $0<j<d-1$

Since $p_d^{v,d}(j) = -p_{d-1}^{v-2,d-1}(j-1)$, we have

$$|p_d^{v,d}(j)|/k_d(v,d) \le \frac{k_{d-1}(v-2,d-1)}{k_1(v-2,d-1)k_d(v,d)} p_1^{v-2,d-1}(j-1) .$$

Now we get

$$\frac{k_{d-1}(v-2,d-1)}{k_1(v-2,d-1)k_d(v,d)}p_1^{v-2,d-1}(j-1)k_1(v,d)/p_1^{v,d}(j)$$
$$= \frac{d^2}{(d-1)(v-d-1)}\left(1+\frac{1}{j^2-(v+1)j+d(v-d)}\right)$$
$$\leq \left(1-\frac{v-7}{(d-1)(v-d-1)}\right)\left(1+\frac{1}{2v-5d+6}\right) < 1 \ .$$

This completes the proof of Lemma 2, hence of Theorem A.

Added in proof

Recently we improve the Theorem A in this paper as follows;

The Johnson scheme $J(v,d)$ has no non-trivial subscheme if $v>2d+\frac{3}{2}+\sqrt{(d-\frac{7}{2})^2+6}$.

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