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グラフの連結度の一般化について

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An (a,b)-n-fan means a union of n internally disjoint a-b paths. Menger's theorem is one of the most fundamental theorems in graph theory. Its vertex version states that a (di)graph G has an (a,b)-n-fan if and only if G is n-connected between a and b, and its edge version states that a (di)graph G has n edge-disjoint a-b paths if and only if G is n-edge-connected between a and b. As a common generalization of those two versions, Egawa, Kaneko and Matsumoto [2] proved the following theorem.

Theorem 1. Let G be a multi(di)graph of order at least two, let a and b be distinct vertices of G, and let λ and n be positive integers. Then, there exist λ edge-disjoint (a,b)-n-fans in G if and only if for any k with $0 \leqq k \leqq \min\{n-1, |V(G)|-2\}$ and for any subset X of $V(G)-\{a,b\}$ with cardinality k, $G-X$ is $\lambda(n-k)$-edge-connected between a and b.

A pair (t,s) of nonnegative integers is said to be a connectivity pair for distinct vertices x and y of a graph G if it satisfies the following conditions which were introduced by Beineke and Harary [1]:

(1) For any subset $T \subseteq V(G)-\{x,y\}$ and any subset $S \subseteq E(G)$ with $|T| \leqq t$, $|S| \leqq s$ and $|T|+|S| < t+s$, $G-(T \cup S)$ still contains an x-y path,

(2) there exist a subset $T' \subseteq V(G)-\{x,y\}$ and a subset $S' \subseteq E(G)$ with $|T'| = t$ and $|S'| = s$, $G-(T' \cup S')$ contains no x-y path.

Using the above-mentioned mixed version of Menger's Theorem, Enomoto and Kaneko [3] proved the following.

Theorem 2. Let q, r, s and t be integers with $t \geqq 0$ and $s \geqq 1$ such that $t+s = q(t+1)+r$, $1 \leqq r \leqq t+1$, and let x and y be distinct vertices of a graph G. If $q+r > t$ holds, and if a pair (t, s) is a connectivity pair for x and y, then G contains $t+s$ edge-disjoint x–y paths $P_1, P_2, \cdots, P_{t+s}$ such that $P_1, P_2, \cdots, P_{t+1}$ are openly disjoint x–y paths.

In [4], Kaneko and Ota investigated the graphs having this type of connectivity as their global connectivity. They obtained the following results.

A graph G is said to be (n, λ)-connected if it satisfies the following conditions:

(1) $|V(G)| \geqq n+1$,

(2) for any subset $S \subseteq V(G)$ and any subset $L \subseteq E(G)$ with $\lambda|S| + |L| < n\lambda$, $G-S-L$ is connected.

The (n, λ)-connectivity is a common extension of both the vertex-connectivity and the edge-connectivity, because the $(n, 1)$-connectivity is identical with the n-(vertex)-connectivity and the $(1, \lambda)$-connectivity is identical with the λ-edge-connectivity. An (n, λ)-connected graph G is said to be *minimally* (n, λ)-connected if for any edge e in $E(G)$, $G-e$ is not (n, λ)-connected. Let G be a minimally (n, λ)-connected graph and let W be the set of its vertices of degree more than $n\lambda$. Then they first proved that for any subset W' of W, the minimum degree of the subgraph of G induced by the vertex set W' is less than or equal to λ. This result is an extension of a theorem of Mader, which states that the subgraph of a minimally n-connecetd graph induced by the vertices of degree more than n is a forest. By using their result, they showed that if G is a minimally (n, λ)-connected graph, then

(1) $|E(G)| \leqq \dfrac{\lambda(|V(G)|+n)^2}{8}$ for $n+1 \leqq |V(G)| \leqq 3n-2$

(2) $|E(G)| \leqq n\lambda(|V(G)|-n)$ for $|V(G)| \geqq 3n-1$.

Furthermore, they studied the number of vertices of degree $n\lambda$ in a minimally $n\lambda$-connecetd graph.

References

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[2] Y. Egawa, A. Kaneko and M. Matsumoto, A mixed version of Menger's theorem, to appear in *Combinatorica*.

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