数理解析研究所講究録
第 770 巻 1991 年 1-20

Keio university, 1991

A Free Boundary Problem for Minimal Surface Equation

Yoshihiko YAMAURA * (山浦義彦)

1. Introduction

In this note we introduced a free boundary problem for minimal surface equation.

The following variational problem had been treated by H.W.Alt, L.A.Caffarelli and A.Freedman [1],[2]:

$$\begin{cases} \int_{\Omega}\Big(F(|\nabla u|^2) + Q^2\chi_{u>0}\Big)dL^n \longrightarrow \min. \\ u \in K \equiv \Big\{ w \in L^2_{\mathrm{loc}}(\Omega) \,\Big|\, \nabla w \in L^2(\Omega),\ w = u^0 \text{ on } S \Big\}, \end{cases} \tag{1.1}$$

where Ω is a connected Lipshitz domain contained in n-dimensional Euclidean space R^n. $F = F(t)$ is a function belonging to $C^{2,1}[0,\infty)$ with $F(0) = 0$, $0 < c \leq \partial_t F \leq C < \infty$ and $0 \leq \frac{1}{1+t}\partial_t^2 F \leq C < \infty$. Q is a given measurable function with $0 < Q_{\min} \leq Q(x) \leq Q_{\max} < \infty$ for all $x \in \Omega$, and $\chi_{u>0}$ denotes the characteristic function of domain $\Omega(u > 0) \overset{\mathrm{def}}{\equiv} \{x \in \Omega \,|\, u(x) > 0\}$. dL^n denotes the integration by n-dimenisonal Lebesgue measure. u^0 is a given non-negative function belonging to $L^2_{\mathrm{loc}}(\Omega)$ with $\nabla u^0 \in L^2(\Omega)$ and S is a subset of $\partial\Omega$ with positive H^{n-1}-measure, where H^{n-1} is the $(n-1)$-dimensional Hausdorff measure. They obtained the next three main results:

(1) The existence of a minimum:

A minimum is attained in function space K.

(2) The global regularity of the minimum:

Let u be the minimum for (1.1). *Then $u \in C^{0,1}(\Omega)$.*

(3) The regularity of a free boundary for the minimum:

Let u be the minimum for (1.1). *Then if $Q \in C^{\alpha}(\Omega)$ $\big({}^{\exists}\alpha \in (0,1)\big)$, free boundary of u: $\partial\Omega(u > 0) \overset{\mathrm{def}}{\equiv} \Omega \cap \partial\,[\Omega(u > 0)]$ is an $(n-1)$-dimensional $C^{1,\beta}$-surface $\big({}^{\exists}\beta \in (0,1)\big)$ near the point where the free boundary is sufficiently flat in some sense (Pricisely see [1], [2].)*

On the other hand, S.Omata [10] and S.Omata & Y.Yamaura [11] proved the same results for the non-linear version of (1.1) when $n = 2$:

$$\begin{cases} \int_{\Omega}\Big(a^{ij}(u)D_iuD_ju + Q^2\chi_{u>0}\Big)dL^n \longrightarrow \min. \\ u \in K \equiv \Big\{ w \in L^2_{\mathrm{loc}}(\Omega) \,\Big|\, \nabla w \in L^2(\Omega),\ w = u^0 \text{ on } S \Big\}, \end{cases} \tag{1.2}$$

where $a^{ij}(z)$ is a smooth function with the following property: there exist positive numbers λ and Λ independent of z such that $0 < \lambda|\xi|^2 \leq a^{ij}(z)\xi_i\xi_j \leq \Lambda|\xi|^2 < \infty$ (${}^{\forall}z \in R^1$) for all $\xi \in R^n\backslash\{0\}$, and matrix $[\dot{a}^{ij}(z)]$ is positive definite: $0 \leq \dot{a}^{ij}(z)\xi_i\xi_j$ (${}^{\forall}z \in R^1$) for all $\xi \in R^n$.

Now in this note we would like to treat the following variational problem:

$$\begin{cases} \int_{\Omega}\Big(\sqrt{1+|\nabla u|^2} + Q^2\chi_{u>0}\Big)dL^n \longrightarrow \min. \\ u \in \widetilde{K} \equiv \Big\{ w \in W^{1,1}(\Omega) \,\Big|\, w = u^0 \text{ on } S \Big\}. \end{cases} \tag{1.3}$$

* Post Graduate of KEIO university

The first term of energy in (1.3) denotes the area of the graph of u in Ω and then we naturally assume Ω is bounded in addition to the conditions above mentioned. u^0 is a given function belonging to $W^{1,1}(\Omega)$, and the other notations are as above.

In the special case $Q^2 \equiv 1$ in Ω and $S = \partial\Omega$, the positive part of the graph of the minimum for problem (1.3): $\text{graph}^+ u = \{(x, u(x)) \,|\, x \in \Omega\,(u > 0)\}$ describes the soap film, which is constructed by the following physical experiment: Prepare a connected framework and a sufficiently large container filled with soap liquid. We lift up the framework, which has been completely flooded under soap liquid at the beginning. When the whole of the framework rises above the surface of the soap liquid, we get a soap film with the edge consisting of both the given framework and a free boundary on the surface of the soap liquid.

Now unlike (1.1) or (1.2) the minimum in (1.3) is not necessarily attained in $\widetilde{K}$, because the function space $W^{1,1}(\Omega)$ does not have L^1-compactness. Thus we must extend admissible function space $W^{1,1}(\Omega)$ to $BV(\Omega)$ and generalize the problem itself in the same way as [7]:

$$(P) \qquad \begin{cases} J(u) = \displaystyle\int_\Omega \sqrt{1+|\nabla u|^2} + \int_\Omega Q^2 \chi_{u>0}\, dL^n + \int_S |u - u^0| dH^{n-1} \;\to \min. \\ u \in BV(\Omega). \end{cases}$$

$BV(\Omega)$ is the space of functions, whose distributional derivatives are Radon measures of locally total variation, and the first term of $J(u)$ is well-defined as Radon measure:

$$\int_\Omega \sqrt{1+|\nabla u|^2} \;\overset{\text{def}}{\equiv}\; \sup_{\substack{g \in C_0^1(\Omega, R^{n+1}) \\ |g| \le 1}} \int_\Omega \left(g^{n+1} + u \text{ div } \widehat{g} \right) dL^n.$$

Moreover it is well-known that BV-function has a L^1-trace on the Lipshitz bondary, then for a given non-negative BV-function u^0, the third term of $J(u)$ is also well-defined. Other notations, which will also be used throughout this note are as follows:

Ω : bounded, connected Lipshitz subdomain of R^n,

Q^2 : given positive L^1-function in Ω,

$\chi_{u>0}$: the characteristic function of $\Omega\,(u > 0)$,

S : non-empty connected open subset of $\partial\Omega$.

It is problem (P) that we will treat in this note. We now remark that even if a minimum for problem (P) exists, the trace of the minimum on S does not necessarily coninside with u^0. Taking it and the results for problem (1.1) into account, the following four questions arise

(1′) Is a minimum of J attained in function space $BV(\Omega)$?

(2′) If the minimum exists, how global regularity does it have?

(3′) If the minimum exists, does it have boundary regularity: $u = u^0$ on S?

(4′) If the minimum exists, how regularity does a free boundary $\partial\Omega\,(u > 0)$ have?

We study (1') and (2') in this note. We will see the affirmative answer for (1′) in section 2. Moreover in section 2 we show the maximum principle for the minimum. In section 3 we obtain the first variation formula, which tells us the information about the gradient on the free boundary. For question (2′), We expect the following results: If $\partial\Omega\,(u > 0) \neq \phi$, then

$$\begin{cases} \text{(a) } u \in C^{0,1}(\Omega) & \text{when } Q^2(x) \le Q^2_{\max} < 1 \text{ in } \Omega. \\ \text{(b) } u \in C^{0,\alpha}(\Omega) \ (^\exists \alpha \in (0,1)) & \text{when } Q^2(x) \le 1 \text{ in } \Omega. \\ \text{(c) } u \in BV(\Omega) \backslash C^0(\Omega) & \text{when } 1 < Q^2_{\min} \le Q^2(x) \le Q^2_{\max} < \infty \text{ in } \Omega. \end{cases} \tag{1.4}$$

To study the behavior of the graph u, in section 4,5 we deal with the parametric argument, which is created by De Giorgi. Using the result there, we construct a solution for radially symmetric free boundary problem in $n = 2$ (Section 6). In particular when $Q^2 > 1$, we will obtain the example of minimum, which does not have even $W^{1,1}$-regularity in the whole of the domain Ω.

2. Existence Theorem

THEOREM 2.1 (Existence) *There exists a function* $u \in BV(\Omega)$ *such that*

$$J(u) = \inf_{BV(\Omega)} J.$$

Proof. There exists a bounded Lipshitz domain V such that $V \cap \partial\Omega = S$. For given function $Q^2 \in L^1(\Omega)$ we set function $\widehat{Q}^2$ defined in extended domain $\Omega \cup V$ as follows:

$$\widehat{Q}^2 = \begin{cases} Q^2 & \text{in } \Omega, \\ 0 & \text{in } V\backslash\overline{\Omega}. \end{cases}$$

Moreover we define a function w belonging to $BV(V\backslash\overline{\Omega})$ such that

$$\begin{cases} w^{\mathrm{tr}-} = (u^0)^{\mathrm{tr}+} & \text{on } S, \\ w = 0 & \text{in } V\backslash\overline{\Omega_\varepsilon} \end{cases} \qquad \Big(\Omega_\varepsilon = \{\, x \in R^n \mid \mathrm{dis}(x, \Omega) < \varepsilon \,\}\Big),$$

where tr+, tr− denote the inner, outer trace operator on S respectively, and ε is a sufficiently small positive number such that $V\backslash\overline{\Omega_\varepsilon} \neq \phi$. To prove the assertion of theorem it is sufficient to show the existence of a minimum for the next variational problem:

$$(\widehat{P}) \qquad \begin{cases} \widehat{J}(u) = \displaystyle\int_{\Omega\cup V} \sqrt{1+|\nabla u|^2} + \int_{\Omega\cup V} \widehat{Q}^2 \chi_{u>0}\, dL^n \;\; \to \min. \\ u \in X(\Omega \cup V) \equiv \Big\{ v \in BV(\Omega\cup V) \;\Big|\; v = w \text{ in } V\backslash\overline{\Omega} \Big\}. \end{cases}$$

In fact when we denote the extension of $u \in BV(\Omega)$ to the domain $\Omega \cup V$ by w as $\widehat{u} \in X(\Omega \cup V)$, mapping $u \to \widehat{u}$ gives the byjection from $BV(\Omega)$ to $X(\Omega \cup V)$, and

$$\widehat{J}(\widehat{u}) = J(u) + \int_{V\backslash\overline{\Omega}} \sqrt{1+|\nabla w|^2} \qquad \text{for } {}^{\forall}u \in BV(\Omega). \tag{2.1}$$

The second term of the right hand side of (2.1) is a constant independent of $u \in BV(\Omega)$, and thus the minimum for problem (P) is obtained by restricting the minimum for problem $(\widehat{P})$ to Ω, if the latter exists.

We now show the existence of a minimum for problem $(\widehat{P})$. We take a minimizing sequence $\{u_j\}_{j=1}^{\infty} \subset X(\Omega \cup V)$: $\lim_{j\to\infty} \widehat{J}(u_j) = \inf_X \widehat{J}$, then obviously

$$\int_{\Omega\cup V} |\nabla u_j| \;\leq\; M_1 \qquad \text{for } {}^{\forall}j \in \mathcal{N}. \tag{2.2}$$

In order to estimate the L^1-norm of trace of u_j on $\partial(\Omega\cup V)$, we deform domain $\Omega\cup V$ to $\mathcal{B}_R(0)\times(0,R)$ (Here and subsequently we denote $(n-1)$-dimensional ball by $\mathcal{B}_R$.) for some $R > 0$ by Lipshitz homeomorphism Φ such that

$$\begin{cases} \Phi(V\backslash\overline{\Omega_\varepsilon}) = D_\delta^R \overset{\mathrm{def}}{\equiv} [\mathcal{B}_R(0)\backslash\mathcal{B}_{R-\delta}(0)] \times (0,R) \cup \mathcal{B}_R(0) \times (R-\delta, R), \\ \Phi\big(\partial(\Omega \cup V) \cap \Omega_\varepsilon\big) = \mathcal{B}_{R-\delta}(0) \times \{0\} \end{cases}$$

for some positive number $\delta << R$. Moreover we define $\widetilde{u}_j$ by

$$\widetilde{u}_j(x) \overset{\mathrm{def}}{\equiv} u_j(\Phi^{-1}(x)) \qquad x \in \mathcal{B}_R(0) \times (0,R).$$

Since $\widetilde{u}_j \in BV(\mathcal{B}_R(0)\times(0,R))$, for some $t_j \in (R-\delta,R)$ and for L^1-a.a. $\varepsilon \in (0,R)$

$$\int_{\mathcal{B}_R(0)} \left| \widetilde{u}_j(\overline{x},\varepsilon) - \widetilde{u}_j(\overline{x},t_j) \right| d\overline{x} \ \le\ \int_{\mathcal{B}_R(0)\times[\varepsilon,t_j]} |\nabla \widetilde{u}_j|.$$

Recalling that $\widetilde{u}_j = 0$ in D_δ^R for all $j \in \mathcal{N}$, this inequality implies

$$\int_{\mathcal{B}_R(0)} \left| \widetilde{u}_j(\overline{x},\varepsilon) \right| d\overline{x} \ \le\ \int_{\mathcal{B}_R(0)\times(0,R)} |\nabla \widetilde{u}_j| \ \le\ \widetilde{M}_1.$$

Letting $\varepsilon \downarrow 0$, by the definition of the trace,

$$\int_{\mathcal{B}_R(0)} \left| (\widetilde{u}_j)^{\mathrm{tr}} \right| dH^{n-1} \ \le\ \widetilde{M}_1 \qquad \text{for } {}^\forall j \in \mathcal{N}.$$

In this way we get

$$\int_{\partial(\Omega\cup V)} |u_j|\, dH^{n-1} \ \le\ M_2 \qquad \text{for } {}^\forall j \in \mathcal{N}. \tag{2.3}$$

From (2.2) and (2.3)

$$\int_{R^n} |\nabla \overline{u}_j| \ =\ \int_{\Omega\cup V} |\nabla u_j| + \int_{\partial(\Omega\cup V)} |u_j|\, dH^{n-1} \ \le\ M_3 \qquad \text{for } {}^\forall j \in \mathcal{N},$$

where

$$\overline{u}_j = \begin{cases} u_j & \text{in } \Omega\cup V, \\ 0 & \text{otherwise.} \end{cases}$$

Using BV-version Sobolev's imbedding theorem $BV_0(R^n) \hookrightarrow L^{\frac{n}{n-1}}(R^n)$, Hölder inequality and the fact that spt $\overline{u}_j \subset \overline{\Omega\cup V}$ for all $j \in \mathcal{N}$,

$$\int_{R^n} |\overline{u}_j|\, dL^n \ \le\ \left\{ \int_{R^n} |\overline{u}_j|^{\frac{n}{n-1}} \right\}^{\frac{n-1}{n}} L^n\left(\overline{\Omega\cup V}\right)^{\frac{1}{n}} \ \le\ c(n) \int_{R^n} |\nabla \overline{u}_j| \ \le\ M_4,$$

and therefore

$$\int_{\Omega\cup V} |u_j|\, dL^n \ \le\ M_4 \qquad \text{for } {}^\forall j \in \mathcal{N}. \tag{2.4}$$

Since $\Omega\cup V$ is the bounded Lipshitz domain, (2.2) and (2.4) enable us to apply BV-version Rellich's compactness theorem $BV(\Omega) \to L^1(\Omega)$: there exists a subsequence $\{u_k\} \subset \{u_j\}$ and a function u_∞ belonging to $L^1(\Omega\cup V)$ such that

$$u_k \longrightarrow u_\infty \qquad \text{in } L^1(\Omega\cup V).$$

By the lower-semi-continuity (0.4) we can easily check that $u_\infty \in X(\Omega\cup V)$.

Now sequence $\{\chi_{u_k>0}\}_{k=1}^\infty \subset L^\infty(\Omega\cup V)$ is uniformly bounded with respect to norm $\|\cdot\|_\infty$, then there exists a subsequence $\{u_l\} \subset \{u_k\}$ and a bounded function γ such that

$$\chi_{u_l>0} \longrightarrow \gamma \qquad \text{in weakly} * L^\infty(\Omega\cup V).$$

We can readily show that

$$\begin{cases} 0 \le \gamma(x) \le 1 & \text{for } L^n\text{-a.a. } x \in \Omega\cup V, \\ \gamma(x) = 1 & \text{for } L^n\text{-a.a. } x \in \left\{ \xi \in \Omega\cup V \;\middle|\; u_\infty(\xi) > 0 \right\}. \end{cases}$$

Thus we get

$$\begin{aligned} \widehat{J}(u_\infty) &\le \int_{\Omega\cup V} \sqrt{1+|\nabla u_\infty|^2} + \int_{\Omega\cup V} \widehat{Q}^2 \gamma\, dL^n \\ &\le \liminf_{l\to\infty} \int_{\Omega\cup V} \sqrt{1+|\nabla u_l|^2} + \lim_{l\to\infty} \int_{\Omega\cup V} \widehat{Q}^2 \chi_{u_l>0}\, dL^n = \inf_X \widehat{J}. \end{aligned}$$

Q.E.D.

THEOREM 2.2 (**Maximum principle**) *Let u be the minimum for (P). Then $0 \le u \le \sup_S u^0$ in Ω.*

Proof. We show $0 \le u$ in Ω.

(a) Reduction 2.

It is sufficient to prove that for $u^- = \min(0, u)$

$$\int_\Omega |\nabla u^-| = 0. \tag{2.5}$$

In fact if (2.5) holds, then by the definition of the variation measure,

$$\int_\Omega u^- \operatorname{div} g \, dL^n = 0 \qquad \text{for } {}^\forall g \in C_0^1(\Omega, R^n).$$

Then we have $\nabla u^- = 0$ in Ω in the sense of weak derivative, and hence $u^- \equiv C$ in Ω for some non-positive constant C. Especially it holds that $C = 0$, because if $C < 0$, then using the assumption $u^0 \ge 0$ on S,

$$\begin{aligned} J(u) &= L^n(\Omega) + \int_S |u^0 - C| \, dH^{n-1} \\ &= L^n(\Omega) + \int_S |u^0| \, dH^{n-1} + \int_S |C| \, dH^{n-1} \\ &> L^n(\Omega) + \int_S |u^0| \, dH^{n-1} = J(0), \end{aligned}$$

which contradicts to the minimality of u. Thus $u^- \equiv 0$ in Ω, and hence we obtain $u \ge 0$.

(b) Reduction 2.

Equality (2.5) follows from the next fact:

$$\text{If } \int_\Omega |\nabla u^-| > 0, \quad \text{then } \int_\Omega \sqrt{1 + |\nabla u^+|^2} < \int_\Omega \sqrt{1 + |\nabla u|^2}, \tag{2.6}$$

because if (2.5) does not hold, using (2.6) we deduce $J(u^+) < J(u)$, which is a contradiction.

The fact we have to show is (2.6), but using the approximation argument it is easy to see that

$$\int_\Omega \sqrt{1 + |\nabla u|^2} - \int_\Omega \sqrt{1 + |\nabla u^+|^2} \ \ge \ \int_\Omega \sqrt{1 + |\nabla u^-|^2} - L^n(\Omega),$$

then instead of (2.6) we prove the following:

$$\text{If } \int_\Omega |\nabla u^-| > 0, \quad \text{then } \int_\Omega \sqrt{1 + |\nabla u^-|^2} > L^n(\Omega). \tag{2.7}$$

(c) Proof of (2.7).

Define a positive number δ as follows:

$$\delta \equiv \min\left(\frac{1}{2} \int_\Omega |\nabla u^-|, \ 4L^n(\Omega) \right).$$

Then there exists vector valued function $g_0 \in C_0^1(\Omega, R^n)$ such that

$$\begin{cases} \text{(a)} & |g_0|_\infty \le 1, \\ \text{(b)} & \displaystyle\int_\Omega u^- \operatorname{div} g_0 \, dL^n > \delta. \end{cases} \tag{2.8}$$

Now for an arbitrarily fixed number $M \geq 1$, we choose a function $g^{n+1} \in C_0^1(\Omega)$ satisfying the next two conditions:

$$\begin{cases} \text{(a)} \quad |g^{n+1}|_\infty \leq \dfrac{\sqrt{M^2-1}}{M}, \\ \text{(a)} \quad \displaystyle\int_\Omega g^{n+1}\, dL^n > \left(2\frac{\sqrt{M^2-1}}{M} - 1\right) L^n(\Omega). \end{cases} \tag{2.9}$$

(2.8-a) and (2.9-a) imply

$$\left|\left(\frac{g_0}{M}, g^{n+1}\right)\right|_\infty^2 \leq \frac{|g_0|_\infty^2}{M^2} + |g^{n+1}|_\infty^2 \leq 1,$$

and hence using (2.8-b) and (2.9-b),

$$\begin{aligned} \int_\Omega \sqrt{1+|\nabla u^-|^2} &= \sup_{\substack{f \in C_0^1(\Omega, R^{n+1}) \\ |f|\leq 1}} \int_\Omega \left(f^{n+1} + u^- \operatorname{div} \hat{f}\right) dL^n \\ &\geq \int_\Omega \left(g^{n+1} + u^- \operatorname{div} \frac{g_0}{M}\right) dL^n \\ &> \left(2\frac{\sqrt{M^2-1}}{M} - 1\right) L^n(\Omega) + \frac{\delta}{M}. \end{aligned}$$

We now choose

$$M = \frac{2L^n(\Omega)}{\delta} + \frac{\delta}{8L^n(\Omega)},$$

then we get

$$\int_\Omega \sqrt{1+|\nabla u^-|^2} - L^n(\Omega) \geq \frac{4\delta^2 L^n(\Omega)}{16L^n(\Omega)^2 + \delta^2} > 0.$$

In a similar way, we obtain $u \leq \sup_S u^0$ in Ω.

Q.E.D.

3. The first variation formula

THEOREM 3.1 (The first variation formula) *Let $u \in BV(\Omega)$ be the minimum for (P) with $Q^2 \in W^{1,1}(\Omega)$. Assume $u \in C^0(\Omega)$, then*

$$\lim_{\substack{\delta\to 0 \\ \delta \in L}} \int_{\partial\Omega(u>\delta)} \left(Q^2 - \left(1 - \frac{1}{\sqrt{1+|\nabla u|^2}}\right)\right) \langle \eta, \nu_\delta \rangle \, dH^{n-1} = 0 \tag{3.1}$$

for all $\eta \in C_0^1(\Omega, R^n)$, and for some $L \subset (0, \sup_\Omega u)$ with $L^1((0,\sup_\Omega u)\backslash L) = 0$ where ν_δ is the unit outer normal for the boundary of domain $\Omega(u > \delta) = \{x \in \Omega \mid u(x) > \delta\}$. Moreover $\lim_{\delta\to 0}$ is uniform for any $\eta \in B_M(\Omega) \equiv \{\varphi \in C_0^1(\Omega, R^n) \mid |\varphi| + |\nabla\varphi| \leq M \text{ in } \Omega\}$, where M is an arbitrarily fixed positive number.

Proof. We first note that by the assumption $u \in C^0(\Omega)$ it holds that $u \in C^\infty(\Omega(u > 0))$ and u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \Omega(u > 0) \tag{3.2}$$

in the classical sense. Moreover we can easily obtain $u \in W^{1,1}(\Omega)$.

Now let $\tau_\varepsilon(x) = x + \varepsilon\,\eta(x)$ and $u_\varepsilon(x) = u \circ \tau_\varepsilon^{-1}(x)$. Since $u^{tr} = u_\varepsilon^{tr}$ on S for sufficiently small $\varepsilon > 0$,

$$J(u_\varepsilon) = \int_\Omega \left(\sqrt{1+|\nabla u_\varepsilon|^2} + Q^2\chi_{u_\varepsilon>0}\right) dL^n + \int_S |u - u^0| \, dH^{n-1}.$$

Calculating the first integration-term of $J(u_\varepsilon)$,

$$\begin{aligned}
&\int_\Omega \left(\sqrt{1+|\nabla u_\varepsilon|^2} + Q^2 \chi_{u_\varepsilon>0} \right) dL^n \\
&= \int_\Omega \left(\sqrt{1+\left|\nabla u(\tau_\varepsilon^{-1}(x))\, D\tau_\varepsilon^{-1}(x)\right|^2} + Q^2(x)\cdot \chi_{u>0}(\tau_\varepsilon^{-1}(x)) \right) dL^n(x) \\
&= \int_\Omega \left(\sqrt{1+\left|\nabla u\,(D\tau_\varepsilon)^{-1}\right|^2} + Q^2 \circ \tau_\varepsilon \cdot \chi_{u>0} \right) \left|\det D\tau_\varepsilon\right| dL^n.
\end{aligned}$$

Using $(D\tau_\varepsilon)^{-1} = I - \varepsilon\, D\eta + O(\varepsilon^2)$, $\left|\det D\tau_\varepsilon\right| = 1 + \varepsilon \operatorname{div} \eta + O(\varepsilon^2)$. (Here and subsequently in this proof we omit the $O(\varepsilon^2)$-term.).

$$\begin{aligned}
&\int_\Omega \left(\sqrt{1+|\nabla u_\varepsilon|^2} + Q^2 \chi_{u_\varepsilon>0} \right) dL^n \\
&\quad = \int_\Omega \left(\sqrt{1+\left|\nabla u - \varepsilon\, \nabla u \cdot D\eta\right|^2} + Q^2 \circ \tau_\varepsilon \cdot \chi_{u>0} \right) (1+\varepsilon \operatorname{div}\eta)\, dL^n \\
&\quad = \int_\Omega \Bigg(\sqrt{1+|\nabla u|^2} + \varepsilon \operatorname{div}\eta \cdot \sqrt{1+|\nabla u|^2} - \varepsilon\, \frac{\nabla u\,(\nabla u\cdot D\eta)}{\sqrt{1+|\nabla u|^2}} \\
&\qquad\qquad + Q^2 \circ \tau_\varepsilon \cdot \chi_{u>0} + \varepsilon\, Q^2 \circ \tau_\varepsilon \cdot \chi_{u>0} \operatorname{div}\eta \Bigg) dL^n.
\end{aligned}$$

Then

$$\begin{aligned}
\left[J(u_\varepsilon) - J(u) \right] &= \varepsilon \int_{\Omega(u=0)} \operatorname{div}\eta\, dL^n \\
&\quad + \varepsilon \int_{\Omega(u>0)} \left(\operatorname{div}\eta \cdot \sqrt{1+|\nabla u|^2} - \frac{\nabla u\,(\nabla u\cdot D\eta)}{\sqrt{1+|\nabla u|^2}} \right) dL^n \\
&\quad + \int_{\Omega(u>0)} \left((Q^2 \circ \tau_\varepsilon - Q^2) + \varepsilon\, (Q^2 \circ \tau_\varepsilon) \operatorname{div}\eta \right) dL^n \\
&= \varepsilon \int_{\Omega(u>0)} \left\{ \operatorname{div}\eta \cdot \sqrt{1+|\nabla u|^2} - \frac{\nabla u\,(\nabla u\cdot D\eta)}{\sqrt{1+|\nabla u|^2}} + \operatorname{div}\left((Q^2-1)\eta\right) \right\} dL^n, \qquad (3.3)
\end{aligned}$$

We thus get

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left[J(u_\varepsilon) - J(u) \right] \\
&= \int_{\Omega(u>0)} \left\{ \operatorname{div}\eta \cdot \sqrt{1+|\nabla u|^2} - \frac{\nabla u\,(\nabla u\cdot D\eta)}{\sqrt{1+|\nabla u|^2}} + \operatorname{div}\left((Q^2-1)\eta\right) \right\} dL^n.
\end{aligned}$$

Here we remember the Sard theorem, then $\partial\Omega(u>\delta)$ is smooth $(n-1)$-dimensional curve for L^1-a.a. $\delta \in (0, \sup_\Omega u)$. Moreover by the Co-area formula for BV-function ([5],[7],[12]):

$$\int_0^{\sup_\Omega u} d\delta \int_\Omega \left|\nabla \chi_{\Omega(u>\delta)}\right| = \int_\Omega |\nabla u| < \infty,$$

$H^{n-1}(\partial\Omega(u>\delta)) < \infty$ for L^1-a.a. $\delta \in (0, \sup_\Omega u)$.

Thus there exists a set $L \subset (0, \sup_\Omega u)$ with $L^1((0,\sup_\Omega u)\backslash L) = 0$ such that

$\partial\Omega(u>\delta)$ is smooth and has H^{n-1}-finite measure for all $\delta \in L$.

Now

$$\lim_{\substack{\delta\to 0\\ \delta\in L}} \int_{\Omega(u>\delta)} \left\{ \operatorname{div}\eta \cdot \sqrt{1+|\nabla u|^2} - \frac{\nabla u\,(\nabla u\cdot D\eta)}{\sqrt{1+|\nabla u|^2}} + \operatorname{div}\left((Q^2-1)\eta\right) \right\} dL^n = 0,$$

where we can easily check that for any $\eta \in B_M(\Omega) \equiv \{\varphi \in C_0^1(\Omega, R^n) \mid |\varphi| + |\nabla\varphi| \le M\}$. Using the minimal surface equation (3.2),

$$
\begin{aligned}
0 &= \lim_{\substack{\delta\to 0\\ \delta\in L}} \int_{\Omega(u>\delta)} \operatorname{div}\left(\eta\sqrt{1+|\nabla u|^2} - \frac{\nabla u\,\langle\nabla u, \eta\rangle}{\sqrt{1+|\nabla u|^2}} + (Q^2-1)\eta\right) dL^n \\
&= \lim_{\substack{\delta\to 0\\ \delta\in L}} \int_{\partial\Omega(u>\delta)} \left\langle \eta\sqrt{1+|\nabla u|^2} - \frac{\nabla u\,\langle\nabla u, \eta\rangle}{\sqrt{1+|\nabla u|^2}} + (Q^2-1)\eta\ ,\ \nu_\delta\right\rangle dH^{n-1} \\
&= \lim_{\substack{\delta\to 0\\ \delta\in L}} \int_{\partial\Omega(u>\delta)} \left(Q^2 - \left(1 - \frac{1}{\sqrt{1+|\nabla u|^2}}\right)\right)\langle\eta, \nu_\delta\rangle\, dH^{n-1},
\end{aligned}
$$

where we use Green's formula (noting that $\nu_\delta = -\nabla u/|\nabla u|$, on $\partial\Omega(u > \delta)$ for $\delta \in L$.).

Q.E.D.

4. A construction of the Radon measure

We would like to get a local estimate for the perimeter of D which is the subgraph of minimum u for (P) (See Lemma 5.6). To do that we need the parametric representation for $Q^2\chi_{u>0}$-term. Thus our aim of this section is to construct the Radon measure corresponding to $Q^2\chi_{w>0}$, for general BV-function w, and to obtain the parametric representation of $J(w)$.

Before that we recall some defintions: Borel set $E \subset R^{n+1}$ is called a Caccioppoli set when

$$\int_D |\nabla\chi_E| \overset{\text{def}}{\equiv} \sup_{\substack{g\in C_0^1(D,R^{n+1})\\ |g|\le 1}} \int_D \chi_E \operatorname{div} g\, dL^{n+1} \tag{4.1}$$

is finite for each bounded open set $D \subset R^{n+1}$. We call the value defined by (4.1) a perimeter of E in domain D. Moreover Caccioppoli set E is called a minimal set in some bounded domain D if

$$\int_D |\nabla\chi_E| \le \int_D |\nabla\chi_F| \tag{4.2}$$

for every Borel set F with $\operatorname{spt}(\chi_E - \chi_F) \subset D$.

In this and later chapters we assume $Q^2 \in L^\infty(\Omega)$. We first show the following Lemma:

LEMMA 4.1 *Define $Q^2 \in L^\infty(\Omega \times R^1)$ as follows:*

$$Q^2(x,t) \overset{\text{def}}{\equiv} Q^2(x) \qquad \text{for } {}^\forall t \in R^1.$$

Then for arbitrarily fixed $v \in BV(\Omega \times R^1)$ and open set $D \subset \Omega \times R^1$, the following functional is bounded:

$$g \longmapsto \int_D v\,Q^2\partial_t g\, dL^{n+1} \qquad (g \in C_0^1(D)). \tag{4.3}$$

Proof. Let g be a function belonging to $C_0^1(D)$ with $|g|_\infty \le 1$. Noting that Q^2 is constant with respect to t-variable, $Q^2\partial_t g = \partial_t(Q^2 g)$, and then

$$\int_D v\,Q^2\partial_t g\, dL^{n+1} = \int_D v\,\partial_t(Q^2 g)\, dL^{n+1}. \tag{4.4}$$

Now let $(Q^2 g)_\varepsilon$ be the mollified function of $Q^2 g$, then

$$
\begin{aligned}
\int_D v\,\partial_t\left[(Q^2 g)_\varepsilon\right] dL^{n+1} &= Q^2_{\max}\int_D v\,\partial_t\left[\frac{1}{Q^2_{\max}}(Q^2 g)_\varepsilon\right] dL^{n+1} \\
&\le Q^2_{\max}\int_D |\nabla v|.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we get

$$\int_D v\,\partial_t(Q^2 g)\,dL^{n+1} \le Q_{\max}^2 \int_D |\nabla v| < \infty.$$

Combining (4.4) and this inequality, we obtain the conclusion.

Q.E.D.

DEFINITION 4.2 *We denote the Radon measure, which is uniquely determined for functional (4.3) as follows (See [12]):*

$$\int_D Q^2 |\partial_t v|.$$

We are now in a position to construct the Radon measure. Let $w \in BV(\Omega)$, and W be the subgraph of w:

$$W = \{(x,t) \in \Omega \times R^1 \mid w(x) > t\}.$$

It is well known that $\chi_W \in BV(\Omega \times R^1)$ (See [7]). Then the following Radon measure is well defined:

$$\int_D Q^2 |\partial_t \chi_W| \qquad \text{for } {}^\forall D \subset \Omega \times R,$$

We remark that by the construction of Radon measure (See[12]) it holds that

$$\int_D Q^2 |\partial_t \chi_W| = \sup_{\substack{g \in C_0^1(D) \\ |g| \le 1}} \int_D Q^2 \chi_W\, \partial_t g\, dL^{n+1} \tag{4.5}$$

for all open set $D \subset \Omega \times R_+^1$. We show that this measure corresponds to the second term of J. We begin with the smooth case, though that will not be used later.

PROPOSITION 4.3 *Let $w \in C^2 \cap BV(\Omega)$, then*

$$\int_\Omega Q^2 \chi_{w>0}\, dL^n = \int_{\Omega \times R_+^1} Q^2 |\partial_t \chi_W|.$$

Proof. The next equality is proved by using the Green's formula in the same way as [7]:

$$\int_{\Omega \times R_+^1} Q^2 |\partial_t \chi_W| = \int_{\partial W \cap [\Omega \times R_+^1]} Q^2 |\nu_t|\, dH^n,$$

where ν_t is a t-compornent of unit outer normal for ∂W. Thus

$$\begin{aligned}
\int_{\partial W \cap [\Omega \times R_+^1]} Q^2 |\nu_t|\, dH^n &= \int_{\partial W \cap [\Omega(u>0) \times R^1]} Q^2 |\nu_t|\, dH^n \\
&= \int_{\Omega(w>0)} Q^2 \frac{1}{\sqrt{1+|\nabla w|^2}} \sqrt{1+|\nabla w|^2}\, dL^n \\
&= \int_\Omega Q^2 \chi_{w>0}\, dL^n.
\end{aligned}$$

Q.E.D.

Next the general case is proved by taking an analogous testing function to Lemma 14.6 ([7]).

THEOREM 4.4 *Let $w \in BV(\Omega)$, then*

$$\int_\Omega Q^2 \chi_{w>0}\, dL^n = \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\mathrm{w}}|.$$

Proof. We first suppose that u is bounded. We define a testing function $\eta_\varepsilon(t)$ as follows:

$$\eta_\varepsilon \in C^1_0(R^1_+) \text{ with}$$

$$\eta_\varepsilon = \begin{cases} 0 & \text{in } (0, \frac{\varepsilon}{2}) \cup (\sup_\Omega w + 1, \infty), \\ 1 & \text{in } (\varepsilon, \sup_\Omega w). \end{cases}$$

Then for each fixed $g \in C^1_0(\Omega)$ with $|g|_\infty \le 1$,

$$\begin{aligned} \int_{\Omega\times R^1_+} Q^2|\partial_t\chi_{\mathrm{w}}| &\ge \int_{\Omega\times R^1_+} Q^2(x,t)\,\chi_{\mathrm{w}}(x,t)\,\partial_t\Big[g(x)\,\eta_\varepsilon(t)\Big]\,dxdt \\ &= \int_{\Omega(w>0)} Q^2(x)\,dx \int_0^{w(x)} \partial_t\Big[g(x)\,\eta_\varepsilon(t)\Big]\,dt \\ &= \int_{\Omega(w>0)} Q^2(x)\,g(x)\,dx \int_0^{w(x)} \eta'_\varepsilon(t)\,dt \\ &= \int_{\Omega(w>0)} Q^2(x)\,g(x)\,\eta_\varepsilon(\,w(x)\,)\,dx. \end{aligned}$$

Letting $\varepsilon \to 0$, $\eta_\varepsilon(\,w(x)\,) \to 1$ for each $x \in \Omega$, and so

$$\int_{\Omega\times R^1_+} Q^2|\partial_t\chi_{\mathrm{w}}| \ge \int_{\Omega(w>0)} Q^2 g\, dL^n,$$

and hence

$$\int_{\Omega\times R^1_+} Q^2|\partial_t\chi_{\mathrm{w}}| \ge \int_\Omega Q^2\chi_{w>0}\, dL^n.$$

On the other hand, for all $g \in C^1_0(\Omega)$ with $|g|_\infty \le 1$,

$$\begin{aligned} \int_{\Omega\times R^1_+} Q^2\chi_{\mathrm{w}}\,\partial_t g\, dL^{n+1} &= \int_{\Omega(w>0)} Q^2(x)\,dx \int_0^{w(x)} (\partial_t g)(x,t)\,dt \\ &= \int_{\Omega(w>0)} Q^2(x)\Big[g(x, w(x)\,)\Big]\,dx \\ &\le \int_\Omega Q^2\chi_{w>0}\, dL^n. \end{aligned}$$

From (4.5) we deduce that

$$\int_{\Omega\times R^1_+} Q^2|\partial_t\chi_{\mathrm{w}}| \le \int_\Omega Q^2\chi_{w>0}\, dL^n.$$

Now when w is unbounded, we first apply the above argument to $w_M = \min(w, M)$, and then letting $M \to \infty$, we obtain the conclusion.

Q.E.D.

Combinig Lemma 14.6 ([7]) and the preceding theorem, we get the parametric representation of $J(w)$, which is the aim of this section:

COROLLARY 4.5 *Let $w \in BV(\Omega)$, then*

$$\int_\Omega \sqrt{1+|\nabla w|^2} + \int_\Omega Q^2 \chi_{w>0}\, dL^n = \int_{\Omega\times R^1} |\nabla \chi_W| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_W|.$$

5. Local estimate for perimeter of the minimum

Our aim of this section is to obtain a local estimate for measure $|\nabla \chi_U|$, where U is the subgraph of the minimum for (P). To do that we need the next theorem:

THEOREM 5.1 *Let u be the minimum for (P), and U be the subgraph of u, and let D be a bounded subdomain of $\Omega \times R^1$. Then*

$$\int_D |\nabla \chi_U| + \int_{D\cap(\Omega\times R^1_+)} Q^2 |\partial_t \chi_U| \le \int_D |\nabla \chi_F| + \int_{D\cap(\Omega\times R^1_+)} Q^2 |\partial_t \chi_F|$$

for all measurable sets F with $\mathrm{spt}(\chi_F - \chi_U) \subset D$.

We shall show some lemmata to prove this theorem.

LEMMA 5.2 *Let F be a measurable set with $\Omega\times(-\infty,0) \subset F \subset \Omega\times(-\infty,T)$, where T is a positive number. We define function w_F as follows:*

$$w_F(x) \overset{\text{def}}{\equiv} \int_0^T \chi_F(x,t)\, dt \qquad \text{for } {}^\forall x \in \Omega.$$

Then

$$\int_\Omega Q^2 \chi_{w_F>0}\, dL^n \le \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_F|. \tag{5.1}$$

Proof. For simplicity we assume $T = 1$. Define

$$\eta(x,t) = \begin{cases} \displaystyle\int_0^t \frac{\chi_F(x,\tau)}{w_F(x)}\, d\tau & \text{for } (x,t) \in \Omega(w_F>0)\times(0,1), \\ 2-t & \text{for } (x,t)\in \Omega(w_F>0)\times[1,2], \\ 0 & \text{otherwise in } \Omega\times R^1_+. \end{cases}$$

We have $0 \le \eta \le 1$ in $\Omega \times R^1_+$, and for each $x \in \Omega$, η is absolutely continuous for t-variable, and $\partial_t\eta$ is as follows:

(a) $x_0 \in \Omega(w_F > 0)$

$$\partial_t \eta(x_0,t) = \begin{cases} \dfrac{\chi_F(x_0,t)}{w_F(x_0)} & \text{for } t\in(0,1), \\ -1 & \text{for } t \in (1,2), \\ 0 & \text{for } t \in (-\infty,0)\cup(2,\infty). \end{cases}$$

(b) otherwise

$$\partial_t\eta(x_0,t) \equiv 0.$$

Moreover $\partial_t \eta$ belongs to $L^1(\Omega \times R^1_+)$, because

$$\begin{aligned}\int_{\Omega\times R^1_+} |\partial_t \eta(x,t)|\, dL^{n+1} &= \int_{\Omega(w_F>0)\times R^1_+} |\partial_t \eta|\, dL^{n+1} \\ &= \int_{\Omega(w_F>0)} dx \int_0^\infty |\partial_t \eta(x,t)|\, dt \\ &= \int_{\Omega(w_F>0)} dx \left[\int_0^1 \frac{\chi_F(x,t)}{w_F(x)}\, dt + \int_1^2 1\, dt \right] \\ &= 2L^n(\Omega(w_F>0)) < \infty.\end{aligned}$$

Now let $g \in C^1_0(\Omega), |g| \le 1$, then from the property of η stated above we can take $g(x)\eta(x,t)$ as a testing function, and therefore

$$\begin{aligned}\int_{\Omega\times R^1_+} Q^2|\partial_t \chi_F| &\ge \int_{\Omega\times R^1_+} \chi_F(x,t)Q^2(x)\, \partial_t \Big[g(x)\,\eta(x,t)\Big]\, dxdt \\ &= \int_{\Omega(w_F>0)\times(0,1)} \chi_F(x,t)Q^2(x)g(x)\partial_t\eta(x,t)\, dxdt \\ &= \int_{\Omega(w_F>0)} Q^2(x)\,g(x)\, dx \int_0^1 \Big(\chi_E(x,t)\cdot\partial_t\eta(x,t)\Big)\, dt \\ &= \int_{\Omega(w_F>0)} Q^2(x)\,g(x)\, dx \int_0^1 \left(\chi_F(x,t)^2\cdot\frac{1}{w_F(x)}\right)\, dt \\ &= \int_{\Omega(w_F>0)} Q^2(x)\,g(x)\, dx\cdot\frac{1}{w_F(x)}\int_0^1 \chi_F(x,t)^2\, dt \\ &= \int_\Omega Q^2\chi_{w_F>0}\, g\, dL^n.\end{aligned}$$

Thus (5.1) follows on taking the supremum over all such g.

Q.E.D.

Combining Lemma 14.7 ([7]) and the last lemma, we obtain the following:

COROLLARY 5.3 *Let F, w_F be as defined in Lemma 5.2. Then*

$$\int_\Omega \sqrt{1+|\nabla w_F|^2} + \int_\Omega Q^2\chi_{w_F>0}\, dL^n \le \int_{\Omega\times R^1} |\nabla\chi_F| + \int_{\Omega\times R^1_+} Q^2|\partial_t\chi_F|.$$

LEMMA 5.4 *Let u be the minimum for (P), U be the subgraph of u, and let D be a bounded subdomain contained in $\Omega\times R^1$. Then*

$$\int_{\Omega\times R^1} |\nabla\chi_U| + \int_{\Omega\times R^1_+} Q^2|\partial_t\chi_U| \le \int_{\Omega\times R^1} |\nabla\chi_F| + \int_{\Omega\times R^1_+} Q^2|\partial_t\chi_F|$$

for all measurable set F with

$$\begin{cases} F \supset \Omega\times(-\infty,0), \\ \operatorname{spt}(\chi_F-\chi_U) \subset D. \end{cases}$$

Proof. Using Corollary 5.3,

$$\int_{\Omega\times R^1} |\nabla\chi_F| + \int_{\Omega\times R^1_+} Q^2|\partial_t\chi_F| \ge \int_\Omega \sqrt{1+|\nabla w_F|^2} + \int_\Omega Q^2\chi_{w_F>0}\, dL^n.$$

Since $w_F^{\rm tr} = u^{\rm tr}$ on S,

$$\int_\Omega \sqrt{1+|\nabla w_F|^2} + \int_\Omega Q^2 \chi_{w_F>0}\, dL^n \ \geq \ \int_\Omega \sqrt{1+|\nabla u|^2} + \int_\Omega Q^2 \chi_{u>0}\, dL^n$$
$$= \int_{\Omega\times R^1} |\nabla \chi_{\rm U}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm U}|,$$

where in the last equality we use Corollary 5.5.

Q.E.D.

LEMMA 5.5 *Let u be the minimum for (P), U be the subgraph of u, and let D be a bounded subdomain contained in $\Omega \times R^1$. Then*

$$\int_{\Omega\times R^1} |\nabla \chi_{\rm U}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm U}| \ \leq \ \int_{\Omega\times R^1} |\nabla \chi_{\rm F}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm F}|$$

for all measurable set F with $\text{spt}(\chi_{\rm F} - \chi_{\rm U}) \subset D$.

Proof. Suppose the lemma is not true, then there exists measurable set F with $\text{spt}(\chi_{\rm F} - \chi_{\rm U}) \subset D$ such that

$$\int_{\Omega\times R^1} |\nabla \chi_{\rm U}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm U}| \ > \ \int_{\Omega\times R^1} |\nabla \chi_{\rm F}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm F}|.$$

Now let $H = \{\, (x,t) \in R^{n+1} \,|\, t < 0\,\}$, obviously

$$\int_{\Omega\times R^1} |\nabla \chi_{\rm F}| \ \geq \ \int_{\Omega\times R^1} |\nabla \chi_{F\cup H}|$$

and

$$\int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm F}| \ = \ \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{F\cup H}|.$$

Therefore

$$\int_{\Omega\times R^1} |\nabla \chi_{\rm U}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{\rm U}| \ > \ \int_{\Omega\times R^1} |\nabla \chi_{F\cup H}| + \int_{\Omega\times R^1_+} Q^2 |\partial_t \chi_{F\cup H}|.$$

But $F \cup H$ satisfies $F \cup H \supset \Omega \times (-\infty, 0)$, then the last inequality contradicts to Lemma 5.4.

Q.E.D.

Theorem 5.1 is immediately followed by the last lemma.

Now using Theorem 5.1 we can get a local estimate for a perimeter. Before that we recall the follow fact: Let D be a connected domain in R^{n+1} and let E be a minimal set in D in the sense of (5.2). Then it is well-known that (See[7]):

$$\int_{B_\rho} |\nabla \chi_{\rm E}| \ \leq \ \frac{1}{2}(n+1)\,\omega_{n+1}\rho^n \qquad \text{for } {}^\forall B_\rho \subset D, \tag{5.2}$$

where ω_n is the measure of the unit ball in R^n.

We can now state a local estimate for a perimeter:

THEOREM 5.6 *Let u be the minimum for (P), and U be the subgraph of u. Then*

$$\int_{B_\rho} |\nabla \chi_{\rm U}| \ \leq \ \frac{1}{2}(1+Q_{\max})(n+1)\,\omega_{n+1}\rho^n \qquad \text{for } {}^\forall B_\rho \subset\subset \Omega \times R^1.$$

Proof. Let E be a minimal set in B_ρ with

$$E = U \qquad \text{in } B_\rho^c.$$

Then from Theorem 5.1

$$\int_{B_\rho} |\nabla \chi_U| + \int_{B_\rho \cap (\Omega \times R^1_+)} Q^2 |\partial_t \chi_U| \ \le \ \int_{B_\rho} |\nabla \chi_E| + \int_{B_\rho \cap (\Omega \times R^1_+)} Q^2 |\partial_t \chi_E|,$$

and so

$$\int_{B_\rho} |\nabla \chi_U| \ \le \ (1 + Q_{\max}) \int_{B_\rho} |\nabla \chi_E|.$$

Using (5.2), we obtain the result.

Q.E.D.

6. Radially symmetric free boundary problem

In this section we treat the free boundary problem in the radially symmetric situation ($n \ge 2$), and finally we will construct a solution in 2-dimensional case. We set

$$\begin{aligned} &\Omega = B_R = B_R(0) \subset R^n : \text{n-dimensional ball}, \\ &Q^2 = \text{Const.} > 0, \\ &S = \partial\Omega = \partial B_R, \\ &u^0 \equiv h = \text{Const.} > 0 \quad \text{in } B_R, \end{aligned}$$

that is, we consider the following problem:

$$(P_S) \qquad \begin{cases} J_S(u) = \displaystyle\int_{B_R} \sqrt{1 + |\nabla u|^2} + \int_{B_R} Q^2 \chi_{u>0}\, dL^n + \int_{\partial B_R} |u^{tr} - h|\, dH^{n-1} \ \to \min. \\ u \in BV(B_R). \end{cases}$$

Define function space $BVS(B_R)$ as follows:

$$BVS(B_R) \ \stackrel{\text{def}}{\equiv} \ \{ u \in BV(B_R) \mid u(x) = {}^{\exists}\phi(|x|) \},$$

then we first assart

PROPOSITION 6.1

$$\inf_{BV(B_R)} J_S \ = \ \inf_{BVS(B_R)} J_S.$$

Proof. We shall only prove

$$\inf_{BV(B_R)} J_S \ \ge \ \inf_{BVS(B_R)} J_S, \tag{6.1}$$

because the reverse inequality is trivial. Let u be a minimum for problem (P_S), then to prove (6.1) it is sufficient to show that there exists function $v \in BVS(B_R)$ such that

$$J_S(u) \ \ge \ J_S(v). \tag{6.2}$$

We define

$$\tilde{u} = \begin{cases} u & \text{in } B_R, \\ h & \text{in } B_{2R} \backslash \overline{B_R}. \end{cases}$$

Then $\widetilde{u}$ is the minimum for the next problem:

$$\begin{cases} \widetilde{J_S}(w) = \int_{B_{2R}} \sqrt{1+|\nabla w|^2} + \int_{B_{2R}} Q^2 \chi_{w>0}\, dL^n \quad \to \text{min.} \\ w \in \{\zeta \in BV(B_{2R}) \mid \zeta = h \text{ in } B_{2R}\backslash B_R\}. \end{cases}$$

Let $\widetilde{U}$ be the subgraph of $\widetilde{u}$: $\widetilde{U} = \{(x,t) \in B_{2R} \times R^1 \mid \widetilde{u}(x) > t\}$. From Corollary 5.5

$$\widetilde{J_S}(\widetilde{u}) = \int_{B_{2R}\times R^1} |\nabla \chi_{\widetilde{U}}| + \int_{B_{2R}\times R^1_+} Q^2 |\partial_t \chi_{\widetilde{U}}|.$$

On the other hand, by the maximum principle it holds that

$$0 \leq \widetilde{u} \leq h \qquad \text{in } B_{2R}.$$

Therefore

$$\begin{aligned} \widetilde{J_S}(\widetilde{u}) &= \int_{B_{2R}\times[0,h]} |\nabla \chi_{\widetilde{U}^c}| + \int_{B_{2R}\times(0,h]} Q^2 |\partial_t \chi_{\widetilde{U}^c}| \\ &= \int_{R^n\times[0,h]} |\nabla \chi_{\widetilde{U}^c}| + \int_{R^n\times(0,h]} Q^2 |\partial_t \chi_{\widetilde{U}^c}|. \end{aligned}$$

Let $(\widetilde{U}^c)_s$ be a symmetrized set for $\widetilde{U}^c$ (See [8]), that is,

$$(\widetilde{U}^c)_s \overset{\text{def}}{\equiv} \{(x,t) \in R^{n+1} \mid |x| < \rho(t)\},$$

where

$$\rho(t) = \left(\frac{1}{\omega_n}\int_{R^n} \chi_{\widetilde{U}^c}(\cdot,t)\, dL^n\right)^{\frac{1}{n}}.$$

Since Q^2 is constant, we can apply [8] and so

$$\widetilde{J_S}(\widetilde{u}) \geq \int_{R^n\times[0,h]} |\nabla \chi_{(\widetilde{U}^c)_s}| + \int_{R^n\times(0,h]} Q^2 |\partial_t \chi_{(\widetilde{U}^c)_s}|. \tag{6.3}$$

By the method of symmetrization we readily deduce

$$B_{2R} \times (h,\infty) \subset (\widetilde{U}^c)_s \subset B_{2R} \times [0,\infty),$$

and so

$$\begin{aligned} &\int_{R^n\times[0,h]} |\nabla \chi_{(\widetilde{U}^c)_s}| + \int_{R^n\times(0,h]} Q^2 |\partial_t \chi_{(\widetilde{U}^c)_s}| \\ &\qquad = \int_{B_{2R}\times R^1} |\nabla \chi_{(\widetilde{U}^c)_s}| + \int_{B_{2R}\times R^1_+} Q^2 |\partial_t \chi_{(\widetilde{U}^c)_s}|. \end{aligned} \tag{6.4}$$

Moreover by the definition of Radon measures

$$\begin{aligned} &\int_{B_{2R}\times R^1} |\nabla \chi_{(\widetilde{U}^c)_s}| + \int_{B_{2R}\times R^1_+} Q^2 |\partial_t \chi_{(\widetilde{U}^c)_s}| \\ &\qquad = \int_{B_{2R}\times R^1} |\nabla \chi_{\widetilde{U}^s}| + \int_{B_{2R}\times R^1_+} Q^2 |\partial_t \chi_{\widetilde{U}^s}|, \end{aligned} \tag{6.5}$$

where we denote $\widetilde{U}^s = (B_{2R} \times R^1)\backslash(\widetilde{U}^c)_s$. Combinig (6.3),(6.4) and (6.5),

$$\widetilde{J_S}(\widetilde{u}) \geq \int_{B_{2R}\times R^1} |\nabla \chi_{\widetilde{U}^s}| + \int_{B_{2R}\times R^1_+} Q^2 |\partial_t \chi_{\widetilde{U}^s}|. \tag{6.6}$$

Now define function $\widetilde{u}^s$ as follows:

$$\widetilde{u}^s(x) = \int_0^h \chi_{\widetilde{U}^s}(x,t)\, d\tau.$$

Then $\widetilde{u}^s$ is radially symmetric, because so is $\widetilde{U}^s$, and furthermore by Lemma 14.7 ([7]), $\widetilde{u}^s \in BV(B_{2R})$. Thus $\widetilde{u}^s \in BVS(B_{2R})$. Using Corollary 6.4, we can estimate the right hand side of (6.6),

$$\widetilde{J_S}(\widetilde{u}) \geq \int_{B_{2R}} \sqrt{1+|\nabla \widetilde{u}^s|^2} + \int_{B_{2R}} Q^2 \chi_{\widetilde{u}^s>0}\, dL^n = \widetilde{J_S}(\widetilde{u}^s).$$

It holds that $\widetilde{u}^s \equiv h$ in $B_{2R}\backslash\overline{B_r}$ by the construction of $\widetilde{u}^s$, and therefore

$$J_S(u) \geq J_S(u^s),$$

where $u^s = \widetilde{u}^s|_{B_R}$. We establish (6.2) taking $u^s \in BVS(B_r)$ as v.

Q.E.D.

COROLLARY 6.2 *If* $\inf_{BVS(B_R)} J_S$ *is attained nuiquely in* $BVS(B_R)$, *then the function, which attains* $\inf_{BV(B_R)} J_S$ *in* $BV(B_R)$ *is also unique.*

Proof. If there exists function u belonging to $BV(B_R)\backslash BVS(B_R)$ such that

$$J_S(u) = \inf_{BV(B_R)} J_S,$$

then by [8] we get the following strong inequality:

$$J_S(u) > J_S(u^s),$$

where $u^s \in BVS(B_R)$ is constructed as shown in the proof of Proposition 6.1. This is the contradiction.

Q.E.D.

From Proposition 6.1 to construct a solution for (P_S) it is sufficient to do that for the next problem:

$$(P'_S) \qquad \begin{cases} J_S(u) = \int_{B_R} \sqrt{1+|\nabla u|^2} + \int_{B_R} Q^2 \chi_{u>0}\, dL^n + \int_{\partial B_R} |u^{\mathrm{tr}} - h|\, dH^{n-1} \to \min. \\ u \in BVS(B_R). \end{cases}$$

Since the admissible function space is $BVS(B_R)$, by a slight variation of the proof of Proposition 4.4 we can obtain the same result:

The minimum for (P'_S) is uniquely determined as any of the following three type functions:

$$\begin{aligned} &(\mathrm{a})\ u > 0 \quad \text{a.e. in } B_R, \\ &(\mathrm{b})\ u \begin{cases} = 0 & \text{a.e. in } B_\rho \\ > 0 & \text{a.e. in } B_R\backslash\overline{B_\rho}, \end{cases} \quad (\exists \rho \in (0,R)), \\ &(\mathrm{c})\ u = 0 \quad \text{a.e. in } B_R. \end{aligned} \tag{6.7}$$

Especially in case of (6.7-a) it is trivial that u must be identically h by the form of energy J_S. Furthermore in case of (6.7-b) u is a minimum for the next area minimizing problem in $B_R\backslash\overline{B_\rho}$:

$$\begin{cases} \int_{B_R\backslash\overline{B_\rho}} \sqrt{1+|\nabla v|^2} + \int_{\partial B_\rho} |v|\, dH^{n-1} + \int_{\partial B_R} |v-h|\, dH^{n-1} \to \min. \\ v \in BV(B_R\backslash\overline{B_\rho}). \end{cases} \tag{6.8}$$

Thus we can rewrite (6.7-a) $\sim$ (6.7-c) as follows:

The minimum for (P_S') is uniquely determined as any of the following three type functions:

$$\begin{aligned}
&(\mathrm{a})\ u \equiv h \qquad \text{in } B_R,\\
&(\mathrm{b})\ u = \begin{cases} 0 & \text{in } B_\rho \\ v & \text{in } B_R\backslash\overline{B_\rho}, \end{cases} \qquad ({}^{\exists}\rho \in (0,R)),\\
&(\mathrm{c})\ u \equiv 0 \qquad \text{in } B_R,
\end{aligned} \tag{6.9}$$

where v is the minmum for (6.8).

We now want to express (6.9-b)-type function by a concrete function. To do that we study the minimum for (6.8). First the interior regularity of minimal surface (See [7]) tells us that v is a classical solution of the minimal surface equation in the interior of $B_R\backslash\overline{B_\rho}$:

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0 \qquad \text{in } B_R\backslash\overline{B_\rho}.$$

From the uniqueness of minimal surface v is radially symmetric function. For simplicity we use the same notation v to represent 1-dimensional function:

$$v:\ r \longmapsto v(x) \qquad (\,|x| = r\,).$$

Then v satisfies the next ordinary differential equation:

$$v''(r) + \frac{n-1}{r}v'(r) + \frac{n-1}{r}(\,v'(r)\,)^3 = 0 \qquad \text{in } (\rho, R).$$

In particular when $n = 2$, v can be written by the elementary function:

$$v(r) = c_1\ \log\left(\,r + \sqrt{r^2 - c_1^2}\,\right) + c_2 \qquad (\,c_1, c_2 : \text{constants}).$$

We next study the relation between h and the boundary regularity of v. In [7] it is well-known $v^{\mathrm{tr}} = h$ on ∂B_R, and it is not known $v^{\mathrm{tr}} = 0$ on ∂B_ρ generally. However, to study the boundary regularity on ∂B_ρ we assume $v(\rho) = 0$. Then we ontain

$$v(r) = c_1\ \log\frac{r + \sqrt{r^2 - c_1^2}}{\rho + \sqrt{\rho^2 - c_1^2}} \qquad \text{in } (\rho, R).$$

Now we consider function $[\,v(R)\,] : c_1 \longmapsto v(R)\ (\,c_1 \in [\,0,\rho\,]\,)$. It is easy to see that

$$[\,v(R)\,]'(c_1) > 0 \qquad \text{for } {}^{\forall}c_1 \in [\,0,\rho\,],$$

and for the range of value, we get

$$0\ \le\ [v(R)](c_1)\ \le\ \rho\log\frac{R + \sqrt{R^2-\rho^2}}{\rho} \qquad \text{for } {}^{\forall}c_1 \in [\,0,\rho\,]. \tag{6.10}$$

Here the maximum is attained when $c_1 = \rho$:

$$[v(R)](\rho)\ =\ \rho\log\frac{R + \sqrt{R^2-\rho^2}}{\rho}.$$

From these facts it is known that there are following two cases:

(1) $0\ \le\ h\ \le\ \rho\log\dfrac{R + \sqrt{R^2-\rho^2}}{\rho}$:

Minimum v satisfies $v = h$ on ∂B_R and $v = 0$ on ∂B_ρ.

(6.11)

(2) $\rho \log \dfrac{R + \sqrt{R^2 - \rho^2}}{\rho} < h$:

Minimum v satisfies $v = h$ on ∂B_R, but cannot satisfy $v = 0$ on ∂B_ρ.

In the former case (6.11-1) we define function u_ρ^h as follows:

DEFINITION 6.3 *When* $0 \le h \le \rho \log \dfrac{R + \sqrt{R^2 - \rho^2}}{\rho}$, *we define* u_ρ^h *as the function satisfying*

$$(1)\ u_\rho^h \equiv 0 \quad \textit{in } B_\rho,$$

$$(2)\ u_\rho^h = \begin{cases} 0 & \textit{on } \partial B_\rho, \\ h & \textit{on } \partial B_R \end{cases}$$

$$(3)\ \operatorname{div}\left(\frac{\nabla u_\rho^h}{\sqrt{1 + |\nabla u_\rho^h|^2}}\right) = 0 \quad \textit{in } B_R \backslash \overline{B_\rho}.$$

Let's consider the latter case (6.11-2) precisely. The minimum v is radially symmetric, and therefore

$$v^{\mathrm{tr}} \equiv c > 0 \qquad \text{on } \partial B_\rho.$$

for some positive constant c. But v is regular in the interior of $B_R \backslash \overline{B_\rho}$, and so from (6.10)

$$c \in \left[h - \rho \log \frac{R + \sqrt{R^2 - \rho^2}}{\rho},\ h \right] \stackrel{\text{def}}{\equiv} L.$$

Let u_ρ^c be the function satisfying

$$(1)\ u_\rho^c \equiv 0 \quad \text{in } B_\rho,$$

$$(2)\ u_\rho^c = \begin{cases} c & \text{on } \partial B_\rho, \\ h & \text{on } \partial B_R \end{cases}$$

$$(3)\ \operatorname{div}\left(\frac{\nabla u_\rho^c}{\sqrt{1 + |\nabla u_\rho^c|^2}}\right) = 0 \quad \text{in } B_R \backslash \overline{B_\rho}.$$

Direct calculation tells us that

$$\tilde{J}(u_\rho^{c(h)}) \le \tilde{J}(u_\rho^c) \qquad \text{for all } c \in L,$$

where $c(h) = h - \rho \log \dfrac{R + \sqrt{R^2 - \rho^2}}{\rho}$. Thus for given ρ, in the latter case (6.11-2) the following function can minimize J_S in Problem(P_S') among (6.9-b)-type functions:

DEFINITION 6.4 *When* $h > \rho \log \dfrac{R + \sqrt{R^2 - \rho^2}}{\rho}$, *we define* u_ρ^h *as the function satisfying*

$$(1)\ u_\rho^h \equiv 0 \quad \textit{in } B_\rho,$$

$$(2)\ u_\rho^h = \begin{cases} h - \rho \log \dfrac{R + \sqrt{R^2 - \rho^2}}{\rho}. & \textit{on } \partial B_\rho, \\ h & \textit{on } \partial B_R \end{cases}$$

$$(3)\ \operatorname{div}\left(\frac{\nabla u_\rho^h}{\sqrt{1 + |\nabla u_\rho^h|^2}}\right) = 0 \quad \textit{in } B_R \backslash \overline{B_\rho}.$$

In conclusion we need only to consider u_ρ^h defined in Definition 6.3 and Definition 6.4 as (6.9-b)-type functions. In this way to construct a solution for (P_S') it is sufficiently to consider $\{u_\rho^h, 0, h\}_{0<\rho<R}$ as admissible function space instead of $BVS(B_R)$.

REMARK 6.5

$$\begin{cases} (1)\ u_\rho^h \in C^0(B_R) & \text{when } 0 \le h \le \rho \log \dfrac{R+\sqrt{R^2-\rho^2}}{\rho}, \\ (2)\ u_\rho^h \in BV(B_R)\backslash C^0(B_R) & \text{when } h > \rho \log \dfrac{R+\sqrt{R^2-\rho^2}}{\rho}. \end{cases} \tag{6.12}$$

Now we are in a position to state the result, which is obtained by the direct calculation of energy J_S in Problem (P_S'):

RESULT 6.6 (**The solution for** (P_S))

Case 1 $(0 < Q^2 \le 1)$ *When Q^2 is a constant with $0 < Q^2 \le 1$, the minimum for (P_S) is uniquely determined as follows:*

$$\begin{cases} u_{\rho(h)}^h & \text{when } \ h \le h(Q^2)R, \\ h & \text{when } \ h(Q^2)R \le h, \end{cases}$$

where

$h(Q^2)$ is a solution of the next equation:

$$1 + h^2 + 2\,(2-Q^2)\log h = 0,$$

$\rho(h)$ is a larger solution of the next equation:

$$\sqrt{2Q^2 - Q^4}\ \rho \log \frac{R + \sqrt{R^2 - (2Q^2 - Q^4)\rho^2}}{(2-Q^2)\,\rho} = h.$$

Case 2 $(1 < Q^2 < 2)$ *When Q^2 is a constant with $1 < Q^2 < 2$, the minimum for (P_S) is uniquely determined as follows:*

$$\begin{cases} 0 & \text{when } \ h \le (Q^2-1)R, \\ u_{\rho(h)}^h & \text{when } \ (Q^2-1)R \le h \le h(Q^2)R, \\ h & \text{when } \ h(Q^2)R \le h, \end{cases}$$

where

$h(Q^2)$ is a larger solution of the next equation:

$$1 + \frac{2}{1+h^2}\Big(\log h + (1+Q^2)\Big) = 0,$$

$\rho(h)$ is a larger solution of the next equation:

$$\rho \log \frac{R+\sqrt{R^2-\rho^2}}{\rho} - (1-Q^2)\rho = h.$$

Case 3 $(2 \le Q^2)$ *When Q^2 is a constant with $Q^2 \ge 2$, the minimum for (P_S) is uniquely determined as follows:*

$$\begin{cases} 0 & \text{when } \ h \le \frac{1}{2} Q^2 R, \\ h & \text{when } \ h \ge \frac{1}{2} Q^2 R. \end{cases}$$

REMARK 6.7 *The former case in* **Case 1** *is contained in case* (6.12-1), *and the second case in* **Case 2** *is contained in case* (6.12-2).

REFERENCE

[1] ALT, H.W. & L.A.CAFFARELLI : Existence and regularity for a minimum problem with free boundary, J.Reine Angew. Math. **325** (1981) 105-144.

[2] ALT, H.W., L.A.CAFFARELLI & A.FRIEDMAN: A Free Boundary Problem for Quasi linear elliptic equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984) 1-44.

[3] BEMERMANS, J., & U.DIERKES,: On a singular variational integral with linear growth. *I*: Existence and regularity of minimizers, Arch. Rat. Mech. Anal. **100** (1987),83-103.

[4] DE GIORGI, E.: Suula properieta isoperimetricà dell'ipersfera, nelle classi degli insiemi aventi frontiera orientata di misura finita, Memorie Accad. Naz. Lincei, serie 8, 5 (1958).

[5] FEDERER, H.: Geometric Measure Theory, New York, 1969.

[6] GILBARG, D. & N.S.TRUDINGER : Equation Elliptic Partial Differential Equations of Second Order, 2nd edition Springer-Verlag.

[7] GIUSTI, E.: Minimal Surfaces and functions of bounded variation, Basel,1984.

[8] GONZALEZ, E.: Sul Problema della goccia appoggiata, Rend. Sem. Mat. Univ. Padova, **58** (1977).

[9] MASSARI, U.: Esistenza e regolarita delle ipersurfici di curvatura media assegnata in R^n, Rend. Sem. Mat. Univ. Padova, **55** (1974).

[10] OMATA, S.: A Lipshitz Estimate for a Free Boundary Problem for Non-linear Elliptic Equations, preprint.

[11] OMATA, S. & Y.YAMAURA : A Free Boundary Problem for Non-Linear Elliptic Equations, preprint.

[12] SIMON, L.: Lecture on Geometric Measure Theory, Canberra 1983.