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Singular Sets of Energy Minimizing Maps

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0. Introduction

Let (M,γ) be a smooth, compact Riemannian manifold with smooth, possibly empty, boundary ∂M, and let (N,g) be a smooth, compact Riemannian manifold without boundary. The energy of a map $u : U \to N$ is defined by

$$\mathcal{E}(u) = \int_M e(u) \tag{0.1}$$

Here the energy density in local coordinate system $(x_1,\ldots,x_n)$ of M is given by

$$e(u) \equiv \gamma^{ij}(x)\, g_{k\ell}(u)\, u^k_{x_i} u^\ell_{x_j} \sqrt{\det(\gamma_{ij}(x))}\, dx\ . \tag{0.2}$$

As usual, $H^1(M,N)$ denotes all whose maps $v : M \to N$ such that $\mathcal{E}(v) < \infty$.

A map $u \in H^1(M,N)$ is called almost-minimizing if there is an $R > 0$ such that

$$\int_{B_r(a)} e(u) \le \left(1 + cr^\beta\right) \int_{B_r(a)} e(v) \tag{0.3}$$

for all $a \in M$, $r \in (0,R)$ and $v \in H^1(B_r(a),N)$ with $v = u$ on $\partial B_r(a)$, where c and β are positive constants, and, where $B_r(a)$ is the geodesic ball of radius r and centered at $a \in M$.

Next we define

$$\operatorname{sing}(u) = \{x \in M : u \text{ is not continuous at } x\}\ . \tag{0.4}$$

Following the theory of Schoen-Uhlenbeck [SU] and Giaquinta-Giusti [G], one has

Theorem A. *If $u \in H^1(M,N)$ is almost-minimizing, then* $\operatorname{sing}(u)$ *is of Hausdorff dimension* $\le n-3$.

On the other hand, we remark that there are smooth maps $g_N : \mathbb{S}^{n-1} \to \mathbb{S}^2$ such that solutions of

$$(0.5) \qquad \min\Big\{ \int_{B^n} |\nabla v|^2 \, dx : v \in H^1(B^n, \mathbb{S}^2) \, , \ v = g_N \text{ on } \mathbb{S}^{n-1} \Big\}$$

are not smooth. Moreover, the singular sets have at least N disconnected components of positive $(n-3)$-dimensional Hausdorff measure, for any $N \in \mathbb{Z}^+$ given (cf. [HL]).

Here we have the following

Theorem B. *Let $u : M \to \mathbb{S}^2$ be an almost-minimizing map. Then* $\operatorname{sing}(u)$ *consists of a union of a finite set and a finite family of $C^{0,\alpha}$ closed curves with finitely many crossings provided that either $\partial M = \emptyset$ or $u\big|_{\partial M}$ is smooth when $\partial M = \emptyset$.*

In fact, the above theorem is valid when the target manifold N is homeomorphic to $\mathbb{S}^2$ and when the metric g on N is sufficiently clsoe to the Euclidean metric on $\mathbb{S}^2$ in the $\mathbb{C}^1$-norm.

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1. A list of known facts.

It was first shown by C. B. Morrey that energy minimizing maps from a 2-dimensional domain are smooth. The same is true when maps are stationary (see [S]). Recently, F. Hélein [H] proved that a weak harmonic map (i.e., it is a weak solution of Euler-Lagrange equations for harmonic maps) from a 2-dimensional domain to spheres is smooth. Using a similar idea as that of [H] and the blow-up argument, L. C. Evans [E] shows that a stationary harmonic map from an n-manifold to spheres is smooth away from a relatively closed subset of Hausdorff $(n-2)$-dimensional measure zero.

Now we consider the special case that M is a ball in $\mathbb{R}^3$ and N is the standard sphere, and let $u : B^3 \to \mathbb{S}^2$ be an energy minimizing map. Then

(a) Schoen-Uhlenbeck [SU]:

sing(u) consists of isolated points.

(b) L. Simon [SL] (uniqueness of tangent maps).

If $a \in$ sing(u), there is a unique smooth harmonic map $\phi : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$|u(x) - \phi(\frac{x-a}{|x-a|})| \to 0 \quad \text{as} \quad x \to a \, . \tag{1.1}$$

(c) L. Simon and Gulliver-White [GW] (asymptotic behavior).

There are positive constants c and α so that

$$|u(x) - \phi(\frac{x-a}{|x-a|})| \le c|x-a|^{\alpha} \tag{1.2}$$

(d) Brezis-Coron-Lieb [BCL] (classification of tangent maps):

If $v : B^3 \to \mathbb{S}^2$ is a minimizing tangent map, then $v(x) = \pm R \cdot \dfrac{x}{|x|}$ for some rotation R of $\mathbb{R}^3$.

(e) Hardt-Lin [HL2] (stability of singularities):

If $g : \mathbb{S}^2 \to \mathbb{S}^2$ is such that $\|g - id\|_{C^1} \le \epsilon_0$, then any energy minimizing map $u : B^3 \to \mathbb{S}^2$ with $u\big|_{\partial B^3} = g$ has a unique singular point a such that

$$\|u(x) - R_a \circ (\frac{x-a}{|x-a|})\|_{C^\alpha(B^3)} \le C\epsilon_0^{1/4} \text{ and} \tag{1.3}$$

$$|a| \le C\epsilon_0^{1/2} \ , \ \ |R_a - id| \le C\epsilon_0^{1/4} \ . \tag{1.4}$$

Here $\alpha, \epsilon_0 \in (0,1)$ and C are constants, and here R_a is a rotation of $\boldsymbol{R}^3$.

In general, if $u : B^3 \to \mathbb{S}^2$ is a minimizing map, then

$$\# \text{ of } \operatorname{sing}(u) \le C_0(\operatorname{lip} u\big|_{\partial B^3}) \ . \tag{1.5}$$

(f) Almgren-Lieb (Better bounds on singularities) [AL]:

If $u : B^3 \to \mathbb{S}^2$ is energy minimizing, then

$$\# \text{ of } \operatorname{sing}(u) \le C_0 \cdot \text{ energy of } u\big|_{\partial B^3} \tag{1.6}$$

(g) Hardt-Kinderlehrer-Lin [HKL] (Universal energy bound):

There is a positive constant C_0 so that for any energy minimizing map $u : B^3 \to \mathbb{S}^2$ satisfies

$$\frac{1}{r}\int_{B_r} |\nabla u|^2 \, dx \le C_0 \cdot \frac{1}{1-r} \ , \quad 0 < r < 1 \ . \tag{1.7}$$

In general, if $u : B_R(a) \subset M \to N$ is energy minimizing, and if N is simply connected, then

$$\frac{1}{r^{n-2}}\int_{B_r(a)} |\nabla u|^2 \, dx \le C_1 \frac{1}{R-r} \ , \quad 0 < r < R \ , \tag{1.8}$$

where $C_1 = C_1(M, N)$.

It is not hard to see that all the statements above except possibly statement (e) are valid when u is an almost-minimizing map.

2. When N is a bumpy sphere.

If the target manifold N is $(\mathbb{S}^2, g)$, i.e., a sphere endowed with some Riemannian metric g, then statements (a), (b), (c) and (g) remain valid. For statement (d) one has only partial results. In [HL3], it was shown that there is a degree 1 minimizing tangent map from B^3 to $N \equiv (\mathbb{S}^2, g)$ for any metric g on N. It uses the continuity method and monotonicity of energy ([S]). Recently, a differnt proof of this fact was given by H. Shin [Sh]. In fact, he proved that there is a unique energy minimizing degree 1 tangent map up to rotation in $\mathbb{R}^3$.

On the other hand, we have the following

Lemma. *There is a positive constant ϵ_1 such that if*

$$\|g - g_0\|_{C^1(\mathbb{S}^2)} \le \epsilon_1 \ . \tag{2.1}$$

(Here g_0 is the Euclidian metric on $\mathbb{S}^2$ and $\|\cdot\|$ also indicate norm in this metric). Then any energy minimizing tangent map from B^3 to $(\mathbb{S}^2, g)$ is of degree one.

Proof: Suppose $\phi(x) = \phi(\frac{x}{|x|})$ is an energy minimizing tangent map from B^3 to $(\mathbb{S}^2, g)$. Then by statement (g) of §1 and regularity theory for harmonic maps [SU] one has

$$\|\phi\|_{C^{1,\alpha}(\mathbb{S}^2)} \le C_1(g) \ , \qquad 0 < \alpha < 1 \ . \tag{2.2}$$

In particular, $\deg(\phi) \le C_2$.

Next one observes that there is a constant $\eta > 0$ depending only on C_1 such taht if ϕ satisfies (2.2) and, if $\deg \phi \ge 2$, then

$$\min\{\int_{B^3} |\nabla u|^2 : u = \phi \text{ on } \mathbb{S}^2\} \le \int_{B^3} |\nabla \phi|^2 - \eta \ . \tag{2.3}$$

The last inequality follows from statement (d) of §1. Now the lemma follows from (2.3). QED

Corollary. *Statements* (a) *through* (g) *of* §1 *are valid provided that* $N = (\mathbb{S}^2, g)$ *with* g *satisfying* (2.1).

We also recall the following result of [HL3, §6].

Proposition. *The family of all energy minimizing tangent maps $v : B^4 \to (\mathbb{S}^2, g)$ such that $v\big|_{\mathbb{S}^3}$ is smooth is compact in C^k for all k. Here g is a smooth Riemannian metric on $\mathbb{S}^2$.*

The above proposition implies, in particular, that the Hopf-invariant of $v : \mathbb{S}^3 \to \mathbb{S}^2$ is uniformly bounded. We remark that the homogeneous degree zero extension of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ is energy minimizing, see [C,G].

Finally, we have the following lemma concerning the behavior of minimizing tangent maps. It was proved in [HL3] for the case $N \equiv \mathbb{S}^2$ with Euclidian metric. It is easy to see the same conclusion remains when $N \equiv (\mathbb{S}^2, g)$ with g satisfying (2.1).

Lemma A. *There are positive constants D_0, N_0, d_0 and C, α so that any minimizing tangent map $v : B^4 \to (\mathbb{S}^2, g)$ with (2.1) satisfies the following:*

$$\int_{B^4} \|\nabla v\|^2 \, dx \le D_0 \ . \tag{2.4}$$

(2.5) $\mathbb{S}^3 \cap \operatorname{sing}(v)$ *consists of an even number, not exceeding N_0, of points separated by distance at least d_0.*

(2.6) *for each $a \in \mathbb{S}^3 \cap sing(v)$, there is the asymptotic estimate*

$$|v(x) - R_a \circ \frac{P_a(x-a)}{|P_a(x-a)|}| \le C|x-a|^\alpha \ ,$$

for some orthogonal projection $P_a : R^4 \to R^3$ and a rotation R_a of R^3.

Here $\|\nabla v\|$ denote length of ∇v in the g metric, and here $R_a \circ \dfrac{P_a(x-a)}{|P_a(x-a)|}$ should be replaced by $\phi\left(R_a \circ \dfrac{P_a(x-a)}{|P_a(x-a)|}\right)$ when $g \not\equiv g_0$, ϕ is a degree minimizing tangent map from B^3 to N.

3. Proof of Theorem B (sketch).

To study the structure of sing(u), we introduce the following definitions:

$$\mathrm{sing}_0(u) \equiv \{a \in \mathrm{sing}(u) : \mathbb{S}^3 \cap \mathrm{sing}(v) = \phi \text{ for some tangent map } v \text{ of } u \text{ at } a\} ,$$

$$\mathrm{sing}_1(u) \equiv \mathrm{sing}(u) \sim \mathrm{sing}_0(u) .$$

By the statement (b) of §1, one may change "some" to "every" in the definition of $\mathrm{sing}_0(u)$. Also from (b) of §1, each point of $\mathrm{sing}_0(u)$ is an isolated point of sing(u).

A technical lemma proved in [HL3] says that, *under the hypothesis of Theorem* B, $\mathrm{sing}_0(u)$ *is finite*. So our task reduces to studying nonisolated singularities $\mathrm{sing}_1(u)$.

We introduce the following

Definition. $a \in \mathrm{sing}(u)$ is called "crossing" of

$$\Theta(u,a) = \lim_{r\downarrow 0} \frac{1}{r^2} \int_{B_r(a)} |\nabla u|^2 \, dx \geq 4\pi^2 + \delta_0 .$$

Here δ_0 is a positive constant which is chosen properly according to Reifenberg's topological disk theorem. We note that $4\pi^2$ is the energy density of the map: $(x,y) \in \mathbb{R}^3 \times \mathbb{R} \simeq \mathbb{R}^4 \to \frac{x}{|x|} \in \mathbb{S}^2$. When $\mathbb{S}^2$ is replaced by $(\mathbb{S}^2, g)$, $4\pi^2$ should be replaced by $\pi \cdot$ Area of N in metric g. In this case, π Area of N is the energy density of the map: $(x,y) \to \phi(\frac{x}{|x|})$. Here $\phi : B^3 \to N \equiv (\mathbb{S}^2, g)$ is a degree one energy minimizing tangent map. Such an energy minimizing tangent map is unique up to rotation of $\mathbb{R}^3$ (see [Sh]).

Lemma B. *Under the assumption of Theorem* B, *the number of crossings of* sing(u) *is finite. The same statement is valid when* $\mathbb{S}^2$ *is replaced by* $N = (\mathbb{S}^2, g)$ *provided* g *satisfies* (2.1).

The proof of Lemma B uses Lemma A and DeGiorgi's arguments concerning "tangent maps of a tangent map".

Finally we shall prove that $\mathrm{sing}_1(u)$ away from whose finitely many "crossings" and isolated $\mathrm{sing}_0(u)$ is $C^{0,\alpha}$. To do this, one verifies that the locally compact set $\mathrm{sing}_1(u)\setminus$ "crossings" satisfying Reifenberg's $(\mathcal{E}_0, R_0)$ condition (see [M, Chapter 10]).

A locally compact set S of B^4 is said to satisfy the $(\mathcal{E}, R)$ condition if for all $a \in B^4 \cap S$, $0 < r < R$, there is a line L_r^a in $\boldsymbol{R}^4$ such that the Hausdorff distance between L_r^a and $S \cap B_r(a) \leq \mathcal{E}r$.

Reifenberg's Theorem(see [M, Chapter 10]). *A locally compact set S satisfying (ϵ, R) condition consists of $C^{0,\alpha}$ Jordan curves provided that $\epsilon \leq \epsilon_0$. Here $\alpha = \alpha(\epsilon_0) > 0$.*

It is shown in [HL3] that $\text{sing}_1(u) \backslash$ "crossings" satisfies Riefenberg's (ϵ_0, R_0) condition. As a result we have Theorem B.

4. Open problems.

We learned from L. Simon that the set $\mathrm{sing}_1(u)\backslash$ "crossings" actually consists of $C^{1,\alpha}$ curves. As a consequence of his main result, one also deduces that the one dimensional Hausdorff measure of $\mathrm{sing}(u)$ is finite.

One natural question is that: whether or not $\mathrm{sing}(u)$ is smooth or analytic?

On the other hand, one would also like to know whether one may prescribe curve singularities for harmonic maps from a 4-dimensional domain.

Here we studied only a special case that N is either standard $\mathbb{S}^2$ or $\mathbb{S}^2$ with Riemannian metrics which is close to the standard one. We do not know if one may eliminate the condition (2.1) so that the statement (d) of §1 remains valid. The validity of statement (d) of §1, in this case, will imply Theorem B. It is certainly of interest to study the structure of $\mathrm{sing}(u)$ for evergy minimizing maps u between Riemannian manifolds. But one expects that it will be very difficult.

Another interesting problem is to study $\mathrm{sing}(u)$ when u is energy minimizing harmonic map from an n-dimensional domain to $\mathbb{S}^2$. Is $H^{n-3}(\mathrm{sing}(u)) < \infty$?

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