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Simplified Morasses which capture the Δ-systems

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Abstract

We define certain types of simplified morasses which capture the Δ-systems. We have a partial answer concerning their existence and nonexistence and this provides some clue to a question raised in [3].

§0 Introduction.

For a regular cardinal κ, we investigate certain types of simplified $(\kappa, 1)$-morasses which naturally come out in the forcing construction of simplified $(\kappa, 1)$-morasses.

Roughly speaking, given a collection $\mathcal{X} \subseteq [\kappa^+]^{<\kappa}$ s.t. $|\mathcal{X}| = \kappa^+$, a complete amalgamation system in [1] captures a pair of two elements of $\mathcal{X}$ in such a manner that the two have the same origin and the origin develops splitting finitely many times along the $(\kappa, 1)$-morass in reaching the two. And as for simplified $(\kappa, 1)$-morasses constructed by forcing, we may demand that the two elements be captured in such a way that they have the same origin and the origin reaches the two along the tree structure with just one splitting.

It seemed that constructions using those types of morasses which require just one splitting is easier to comprehend than those ordinary ones which require finitely many splittings, although what important here is that finitely many splittings provide no real harm to the constructions.

It turns out that the morasses which can do the business with just one splitting may not exist at all (at least for the case $\kappa = \omega_0$ under MA_{ω_1}).

In §1, we make basic definitions and list relevant facts. In §2, we establish an existence of this type of morass. In §3, we partially answer that they may not exist. In §4, we list open questions.

For a history and application of these morasses, see [3] and [4].

§1 Nice Simplified Morasses.

(1.1) Notation. In this paper κ will always denote a regular cardinal. For sets X, Y of ordinals $X < Y$ denotes that for any $i \in X$ and any $j \in Y$, $i < j$ holds. For a set X of ordinals and an ordinal j, $X < j$ means $X < \{j\}$. Other notations should be fairly standard.

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(1.2) Definition. A collection $\mathcal{X}$ is a *nice Δ-system in* $[\kappa^+]^{<\kappa}$ if

(1) $\mathcal{X} \subset [\kappa^+]^{<\kappa}$ and $|\mathcal{X}| = \kappa^+$.

(2) $\mathcal{X}$ forms a Δ-system (i.e. there is $\Delta \in [\kappa^+]^{<\kappa}$, which we call the *root* of the system $\mathcal{X}$, s.t. for any pair (x, y) of distinct elements of $\mathcal{X}$, $x \cap y = \Delta$).

(3) For any pair (x, y) of elements of $\mathcal{X}$, the structures $(x, \in)$ and $(y, \in)$ are isomorphic.

(4) For any pair (x, y) of distinct elements of $\mathcal{X}$, either $\Delta < x - \Delta < y - \Delta$ or $\Delta < y - \Delta < x - \Delta$ holds.

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We usually enumerate $\mathcal{X}$ in one-to-one manner, say, $\mathcal{X} = \{\mathcal{X}_\xi : \xi < \kappa^+\}$ s.t. for any (ξ, ζ) with $\xi < \zeta < \kappa^+$, $X_\xi \cap X_\zeta < X_\xi - X_\zeta < X_\zeta - X_\xi$ holds.

(1.3) Proposition. $(2^{<\kappa} = \kappa)$ For any $\mathcal{X}$ s.t. $\mathcal{X} \subseteq [\kappa^+]^{<\kappa}$ and $|\mathcal{X}| = \kappa^+$, there is a nice Δ-system Y with $Y \subseteq X$.

Proof. We may use the Fodor's lemma. The rest is by using $2^{<\kappa} = \kappa$.

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From (1.4) through (1.7), we follow [2] with an eye to forcing constructions of simplified $(\kappa, 1)$-morasses.

(1.4) Definition. A pair of sequences $(\langle\theta_\alpha\rangle_{\alpha\leq\delta}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\delta})$ is a *baby morass of length* $\delta + 1$ if there is a unique sequence of ordinals $\langle\sigma_\alpha\rangle_{\alpha<\delta}$ s.t.

(1) $\langle\theta_\alpha\rangle_{\alpha\leq\delta}$ is a sequence of strictly increasing ordinals with $\theta_0 > 0$.

(2) For each $\alpha < \delta$, $\sigma_\alpha < \theta_\alpha$ and $\theta_{\alpha+1} = \theta_\alpha + (\theta_\alpha - \sigma_\alpha)$.

(3) For each (β, α) with $\beta \leq \alpha \leq \delta$, $F_{\beta\alpha}$ is a set of order-preserving functions from θ_β to θ_α.

(4) For each $\alpha \leq \delta$, let id_α denote the identity function from θ_α to θ_α, then $F_{\alpha\alpha} = \{id_\alpha\}$.

(5) For each $\alpha < \delta$, let f_α denote the map from θ_α to $\theta_{\alpha+1}$ s.t. $f_\alpha \lceil \sigma_\alpha = id_\alpha \lceil \sigma_\alpha$ and $f_\alpha(i) = \theta_\alpha + (i - \sigma_\alpha)$ for all i with $\sigma_\alpha \leq i < \theta_\alpha$, then $F_{\alpha\alpha+1} = \{id_\alpha, f_\alpha\}$.

(6) For any $(\alpha, \beta_1, \beta_2, h_1, h_2)$ s.t. α is a limit ordinal, $\beta_1, \beta_2 < \alpha \leq \delta$, $h_1 \in F_{\beta_1\alpha}$ and $h_2 \in F_{\beta_2\alpha}$, there is (β, g_1, g_2, g) s.t. $\beta_1, \beta_2 < \beta < \alpha$, $g_1 \in F_{\beta_1\beta}$, $g_2 \in F_{\beta_2\beta}$, $g \in F_{\beta\alpha}$, $h_1 = g \circ g_1$ and $h_2 = g \circ g_2$.

(7) For any (β, α, γ) with $\beta \leq \alpha \leq \gamma \leq \delta$, $F_{\beta\gamma} = F_{\alpha\gamma} \circ F_{\beta\alpha}$.

(8) For any (β, α) with $\beta \leq \alpha \leq \delta$, $\theta_\alpha = \bigcup\{f''\theta_\beta : f \in F_{\beta\alpha}\}$.

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(1.5) Proposition. Let $(\langle\theta_\alpha\rangle_{\alpha\leq\delta}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\delta})$ be a baby morass. If $(\alpha, \beta, f, g, i, j)$ is s.t. $\beta \leq \alpha \leq \delta$, $f, g \in F_{\beta\alpha}$, $i, j \in \theta_\beta$ and $f(i) = g(j)$, then $i = j$ and $f \lceil i + 1 = g \lceil j + 1$.

Proof. By induction on $\alpha(\leq \delta)$ for all β. No use of (8) in (1.4) is made.

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(1.6) Definition. A pair of sequences $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$ is a *simplified* $(\kappa, 1)$-*morass* if

(1) $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$ is a baby morass of length $\kappa + 1$.

(2) For any $\alpha < \kappa$, $\theta_\alpha < \kappa$ and $\theta_\kappa = \kappa^+$.

(3) For any (β, α) with $\beta \leq \alpha < \kappa$, $|F_{\beta\alpha}| < \kappa$.

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We will consider simplified $(\kappa, 1)$-morasses as well as $(\kappa^+, 1)$-morasses. We sometimes use $\mathcal{A}$ to name a simplified morass.

(1.7) Proposition. Let $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$ be a simplified $(\kappa, 1)$-morass, then $\{f''\theta_\alpha : \alpha < \kappa,\ f \in F_{\alpha\kappa}\}$ is a cofinal subset of $[\kappa^+]^{<\kappa}$ with respect to $\subseteq$.

Proof. It takes some facts about simplified morasses other than (1.5). But this is well-known.

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(1.8) Definition. A simplified $(\kappa, 1)$-morass $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$ is *weakly nice* if For any nice Δ-system $\mathcal{X}$ in $[\kappa^+]^{<\kappa}$ (see (1.2) for definition), there is (α, z, f) s.t. $\alpha < \kappa$, $z \subseteq \theta_\alpha$, $z - \sigma_\alpha \neq \emptyset$, $f \in F_{\alpha+1\kappa}$ and $\{f \circ id''_\alpha z, f \circ f''_\alpha z\} \subset \mathcal{X}$.

For a simplified $(\kappa, 1)$-morass $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$, a sequence $\langle z_\alpha\rangle_{\alpha<\kappa}$ is *nice* if

(1) For each $\alpha < \kappa$, $z_\alpha \subseteq \theta_\alpha$.

(2) For any nice Δ-system $\mathcal{X}$ in $[\kappa^+]^{<\kappa}$, there is (α, f) s.t. $\alpha < \kappa$, $z_\alpha - \sigma_\alpha \neq \emptyset$, $f \in F_{\alpha+1\kappa}$ and $\{f \circ id''_\alpha z_\alpha, f \circ f''_\alpha z_\alpha\} \subset \mathcal{X}$.

A simplified $(\kappa, 1)$-morass is *nice* if there is a nice sequence for the morass.

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From (1.9) through (1.11), we shoot for an analogy with [1]. Note that for any simplified $(\kappa^+, 1)$-morass, the existence of a complete amalgamation system for the morass is equivalent to $2^\kappa = \kappa^+$.

(1.9) Proposition. If there is a nice simplified $(\kappa, 1)$-morass $\mathcal{A}$, then $2^{<\kappa} = \kappa$ holds.

Proof. Fix $\lambda < \kappa$. We show $2^\lambda \leq \kappa$. Let $\langle z_\alpha\rangle_{\alpha<\kappa}$ be a nice sequence for the morass $\mathcal{A} = (\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$. Given $X \subseteq \lambda$, let $\mathcal{X} = \{X \cup \{\lambda, \eta\} : \kappa < \eta < \kappa^+\}$. Then $\mathcal{X}$ is a nice Δ-system in $[\kappa^+]^{<\kappa}$. Since $\langle z_\alpha\rangle_{\alpha<\kappa}$ is a nice sequence for $\mathcal{A}$, there is (α, f) s.t. $\alpha < \kappa$, $z_\alpha - \sigma_\alpha \neq \emptyset$, $f \in F_{\alpha+1\kappa}$ and $\{f \circ id''_\alpha z_\alpha, f \circ f''_\alpha z_\alpha\} \subset \mathcal{X}$. So we have $X = (f''z_\alpha) \cap \lambda$ and $\lambda \in f''z_\alpha$. Hence $P(\lambda) = \{(f''z_\alpha) \cap \lambda : \alpha < \kappa,\ f \in F_{\alpha+1\kappa},\ \lambda \in f''z_\alpha\}$ holds. But for any $\alpha < \kappa$, if $f, g \in F_{\alpha+1\kappa}$ and $\lambda \in (f''z_\alpha) \cap (g''z_\alpha)$, then $(f''z_\alpha) \cap \lambda = (g''z_\alpha) \cap \lambda$ holds by (1.5). Therefore we conclude that $\mid P(\lambda) \mid \leq \kappa$.

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(1.10) Proposition. For a simplified $(\kappa^+, 1)$-morass $\mathcal{A}$, the following are equivalent.

(1) $\mathcal{A}$ is nice.

(2) $\mathcal{A}$ is weakly nice and $2^\kappa = \kappa^+$ holds.

Proof. (1)⇒(2): If $\mathcal{A}$ is nice, then it is trivial by definition that $\mathcal{A}$ is weakly nice and by (1.9), $2^\kappa = \kappa^+$ holds.

(2)⇒(1): This proof is a slight modification of [1]. For each $\alpha < \kappa^+$, since $\theta_\alpha < \kappa^+$ and $2^\kappa = \kappa^+$, we may fix an enumeration $\langle \vec{X}^\alpha_\delta\rangle_{\delta<\kappa^+}$ of ${}^\kappa P(\theta_\alpha)$(= the κ-sequences of subsets of θ_α). For each $\alpha < \kappa^+$, let $D_\alpha = \{f(\vec{X}^\beta_\delta) : \beta, \delta < \alpha, f \in F_{\beta\alpha}\}$, where $f(\vec{X}^\beta_\delta)$ denotes the sequence of length κ s.t. for each $\eta < \kappa$, the η-th value is given by $\{f(i) : i \in (\vec{X}^\beta_\delta)_\eta$ (= the

η-th value of $\vec{X}^{\beta}_{\delta}$)}. Since $\mid F_{\beta\alpha} \mid \leq \kappa$ for all (β, α) with $\beta \leq \alpha < \kappa^+$, we know $\mid D_\alpha \mid \leq \kappa$ holds for all $\alpha < \kappa^+$.

It is clear that we may choose $\langle \langle z^\eta_\alpha \rangle_{\alpha<\kappa^+} \rangle_{\eta<\kappa}$ s.t.

(1) $\forall \eta < \kappa \forall \alpha < \kappa^+ \; z^\eta_\alpha \subseteq \theta_\alpha$.

(2) $\forall \alpha < \kappa^+ \forall \vec{X} \in D_\alpha \exists \eta < \kappa^+ \; z^\eta_\alpha = (\vec{X})_\eta$ (= the η-th value of $\vec{X}$).

Claim. For some $\eta < \kappa$ $\langle z^\eta_\alpha \rangle_{\alpha<\kappa^+}$ is a nice sequence for $\mathcal{A}$.

Proof. For each $\alpha < \kappa^+$, let $D^P_\alpha = \{f(\vec{Y}) : \beta < \alpha, \vec{Y} \in D_\beta, f \in F_{\beta\alpha}\}$. Then $D^P_\alpha \subseteq D_\alpha$ and let $D^N_\alpha = D_\alpha - D^P_\alpha$. We define $\langle F_\alpha \rangle_{\alpha<\kappa^+}$ s.t. for each $\alpha < \kappa^+$

$(3)_\alpha$ $F_\alpha : D_\alpha \to \kappa^+$.

$(4)_\alpha$ $F_\alpha \lceil D^N_\alpha$ is one-to-one.

$(5)_\alpha$ $(F''_\alpha D^P_\alpha) \cap (F''_\alpha D^N_\alpha) = \emptyset$.

$(6)_\alpha$ $\forall \beta < \alpha \forall \vec{Y} \in D_\beta \forall f \in F_{\beta\alpha} \; F_\beta(\vec{Y}) = F_\alpha(f(\vec{Y}))$.

The construction is by a recursion on $\alpha < \kappa^+$. Suppose we have constructed $\langle F_\beta \rangle_{\beta<\alpha}$. We construct F_α. For $\vec{X} \in D^P_\alpha$, since we have $(6)_\beta$ for all $\beta < \alpha$, there is a unique element of κ^+ which fulfills $(6)_\alpha$. Let $F_\alpha(\vec{X})$ be this element of κ^+. Since $\mid D^P_\alpha \mid \leq \kappa$ and the codomain of F_α is κ^+, there is no problem in fulfilling $(4)_\alpha$ and $(5)_\alpha$. This completes the construction. We first observe that for any $\alpha < \kappa^+$

$(7)_\alpha$ If $\vec{X}, \vec{Y} \in D_\alpha$, $\vec{X} \neq \vec{Y}$ but $F_\alpha(\vec{X}) = F_\alpha(\vec{Y})$, then $\vec{X}, \vec{Y} \in D^P_\alpha$.

And so

$(8)_\alpha$ If $\vec{X}, \vec{Y} \in D_\alpha$, $\vec{X} \neq \vec{Y}$ and $F_\alpha(\vec{X}) = F_\alpha(\vec{Y})$, then there is $(\beta, \vec{Z}, f, g)$ s.t. $\beta < \alpha$, $\vec{Z} \in D_\beta$ $f, g \in F_{\beta\alpha}$, $f(\vec{Z}) = \vec{X}$ and $g(\vec{Z}) = \vec{Y}$.

Finally we define $F : {}^{\kappa}([\kappa^{++}]^{<\kappa^+}) \to \kappa^+$ by $F(\vec{X}) = F_\alpha(\vec{X}^{\beta}_{\delta})$ for any (β, α, δ) s.t. $\beta, \delta < \alpha$ and there is $f \in F_{\beta\kappa^+}$ with $\vec{X} = f(\vec{X}^{\beta}_{\delta})$. F is well-defined by $(6)_\alpha$ and (1.7). Now we are ready to prove our claim by contradiction.

Suppose for every $\eta < \kappa$, $\langle z^\eta_\alpha \rangle_{\alpha<\kappa^+}$ failed to be a nice sequence. So there is $\langle \mathcal{X}_\eta \rangle_{\eta<\kappa}$ s.t. for each $\eta < \kappa$

$(9)_\eta$ $\mathcal{X}_\eta$ is a nice Δ-system in $[\kappa^{++}]^{<\kappa^+}$, say, $\mathcal{X}_\eta = \{X_{\eta\xi}\}_{\xi<\kappa^{++}}$ s.t. for any (ξ, ζ) with $\xi < \zeta < \kappa^{++}$, $X_{\eta\xi} \cap X_{\eta\zeta} < X_{\eta\xi} - X_{\eta\zeta} < X_{\eta\zeta} - X_{\eta\xi}$.

$(10)_\eta$ $\neg \exists \alpha < \kappa^+ \exists f \in F_{\alpha+1\kappa^+}$ $(z^\eta_\alpha - \sigma_\alpha \neq \emptyset$ and $\{f \circ f''_\alpha z^\eta_\alpha, f \circ id''_\alpha z^\eta_\alpha\} \subset \mathcal{X}_\eta)$.

For each $\xi < \kappa^{++}$, let $\vec{X}_\xi = \langle X_{\eta\xi} \rangle_{\eta<\kappa}$. Then $\vec{X}_\xi$ is a κ-sequence of elements of $[\kappa^{++}]^{<\kappa^+}$. Since $2^\kappa = \kappa^+$, without loss of generality we may assume that for any (ξ, ζ) with $\xi < \zeta < \kappa^{++}$, $F(\vec{X}_\xi) = F(\vec{X}_\zeta)$ and the two structures $(\bigcup_{\eta<\kappa} X_{\eta\xi}, <, ..., X_{\eta\xi}, ...)$ and $(\bigcup_{\eta<\kappa} X_{\eta\zeta}, <, ..., X_{\eta\zeta}, ...)$ are isomorphic. Furthermore we may assume the collection $\mathcal{X} = \{\bigcup_{\eta<\kappa} X_{\eta\xi} : \xi < \kappa^{++}\}$ forms a nice Δ-system in $[\kappa^{++}]^{<\kappa^+}$.

Since the morass $\mathcal{A}$ is weakly nice, there is (α, z, f) s.t. $\alpha < \kappa^+$, $z \subseteq \theta_\alpha$, $z - \sigma_\alpha \neq \emptyset$, $f \in F_{\alpha+1\kappa^+}$ and $\{f \circ id''_\alpha z, f \circ f''_\alpha z\} \subset \mathcal{X}$. Let (ξ, ζ) be s.t. $\xi < \zeta < \kappa^{++}$, $f \circ id''_\alpha z = \bigcup_{\eta<\kappa} X_{\eta\xi}$, $f \circ f''_\alpha z = \bigcup_{\eta<\kappa} X_{\eta\zeta}$. So there is $\delta < \kappa^+$ s.t. $z = \bigcup ran(\vec{X}^{\alpha}_{\delta})$. The structures $(z, <, ..., (\vec{X}^{\alpha}_{\delta})_\eta, ...)$ and $(\bigcup_{\eta<\kappa} X_{\eta\xi}, <, ..., (\vec{X}_\xi)_\eta, ...)$ are isomorphic by $f \circ id_\alpha$

and the structures $(z, <, ..., (\vec{X}^{\alpha}_{\delta})_{\eta}, ...)$ and $(\bigcup_{\eta<\kappa} X_{\eta\zeta}, <, ..., (\vec{X}_{\zeta})_{\eta}, ...)$ are isomorphic by $f \circ f_{\alpha}$.

We claim $\vec{X}^{\alpha}_{\delta} \in D_{\alpha}$. To see this we fix (γ, g, h) s.t. $\alpha+1, \delta < \gamma < \kappa^{+}$, $g \in F_{\alpha+1\gamma}$, $h \in F_{\gamma\kappa+}$ and $h \circ g = f$. By the definition of D_{γ}, $g \circ id_{\alpha}(\vec{X}^{\alpha}_{\delta})$, $g \circ f_{\alpha}(\vec{X}^{\alpha}_{\delta}) \in D_{\gamma}$ and $F_{\gamma}(g \circ id_{\alpha}(\vec{X}^{\alpha}_{\delta})) = F(\vec{X}_{\xi}) = F(\vec{X}_{\zeta}) = F_{\gamma}(g \circ f_{\alpha}(\vec{X}^{\alpha}_{\delta}))$. By $(8)_{\gamma}$, we have $(\beta, \vec{z}, g_1, g_2)$ s.t. $\beta < \gamma$, $\vec{Z} \in D_{\beta}$, $g_1, g_2 \in F_{\beta\gamma}$, $g_1(\vec{Z}) = g \circ id_{\alpha}(\vec{X}^{\alpha}_{\delta})$ and $g_2(\vec{Z}) = g \circ f_{\alpha}(\vec{X}^{\alpha}_{\delta})$. So $\beta \leq \alpha$ and there is $g_3 \in F_{\beta\alpha}$ s.t. $g_3(\vec{Z}) = \vec{X}^{\alpha}_{\delta}$. And so $\vec{X}^{\alpha}_{\delta} \in D_{\alpha}$. Since $\vec{X}^{\alpha}_{\delta} \in D_{\alpha}$ and (2) hold, there is $\eta < \kappa$ s.t. $z^{\eta}_{\alpha} = (\vec{X}^{\alpha}_{\delta})_{\eta}$. Since $\{f \circ id''_{\alpha}(\vec{X}^{\alpha}_{\delta})_{\eta}, f \circ f''_{\alpha}(\vec{X}^{\alpha}_{\delta})_{\eta}\} \subset \mathcal{X}_{\eta}$. This contradicts $(10)_{\eta}$.

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(1.11) Proposition. $(2^{<\kappa} = \kappa)$ Let $\mathcal{A}$ be a weakly nice simplified $(\kappa, 1)$-morass, then there is a nice sequence for $\mathcal{A}$ in the generic extensions via the forcing notion $({}^{\kappa>}2, \supseteq)$.

Proof. This proof is a simple modification of [1]. We describe a natural forcing notion P for adding a nice sequence for $\mathcal{A}$. Then it is easy to see that the P is κ-closed, atomless (i.e. $\forall p \in P \exists p_1, p_2 \in P$ p_1 and p_2 are incompatible and $p_1, p_2 \leq p$) and $\mid P \mid = \kappa$. Since $({}^{\kappa>}\kappa, \supseteq)$ is densely embeddable into P. We will be done.

Now here is the p.o. set P defined by $P = \{\langle z_{\beta}\rangle_{\beta<\alpha} : \alpha < \kappa, \forall \beta < \alpha\ z_{\beta} \subseteq \theta_{\beta}\}$ and for $p, q \in P$, $q \leq p$ iff $q \supseteq p$.

Suppose G is an arbitrary P-generic filter over the ground model V. Let $\langle z_{\beta}\rangle_{\beta<\kappa} = \bigcup G$ (in $V[G]$). We show $\langle z_{\beta}\rangle_{\beta<\kappa}$ is a nice sequence for $\mathcal{A}$.

Suppose $p \Vdash$ "$\{\dot{X}_{\xi} : \xi < \kappa^{+}\}$ is a nice Δ-system s.t. for each (ξ, ζ) with $\xi < \zeta < \kappa^{+}$, $\dot{X}_{\xi} \cap \dot{X}_{\zeta} < \dot{X}_{\xi} - \dot{X}_{\zeta} < \dot{X}_{\zeta} - \dot{X}_{\xi}$". We want $(q, \alpha, f, \xi, \zeta)$ s.t. $q \leq p$, the length of q is $\alpha+1$, $f \in F_{\alpha+1\kappa}$, $\xi < \zeta < \kappa^{+}$, $z^{q}_{\alpha} - \sigma_{\alpha} \neq \emptyset$ and $q \Vdash$ "$\dot{z}^{q}_{\alpha} = \dot{z}_{\alpha}$, $f \circ id''_{\alpha}\dot{z}_{\alpha} = \dot{X}_{\xi}$ and $f \circ f''_{\alpha}\dot{z}_{\alpha} = \dot{X}_{\zeta}$". For each $\xi < \kappa^{+}$, since P is κ-closed, we know $p \Vdash$ "$\dot{X}_{\xi} \in V$". Since $\mathcal{A}$ is cofinal in $[\kappa^{+}]^{<\kappa}$ with respect to $\subseteq$, we may take $(p_{\xi}, \alpha_{\xi}, X_{\xi}, f_{\xi})$ s.t. $p_{\xi} \leq p$, $p_{\xi} \Vdash$ "$\dot{X}_{\xi} = \check{X}_{\xi}$", α_{ξ} is the length of p_{ξ}, $f_{\xi} \in F_{\alpha_{\xi}\kappa}$ and $X_{\xi} \subseteq f''_{\xi}\theta_{\alpha_{\xi}}$. By thinning, we may assume that there is (q, α, z) s.t. $\{p_{\xi} : \xi < \kappa^{+}\} = \{q\}$, α is the length of q, $z \subseteq \theta_{\alpha}$, for each $\xi < \kappa^{+}$ $f''_{\xi}z = X_{\xi}$, $\{X_{\xi} : \xi < \kappa^{+}\}$ forms a nice Δ-system s.t. for any (ξ, ζ) with $\xi < \zeta < \kappa^{+}$, $X_{\xi} \cap X_{\zeta} < X_{\xi} - X_{\zeta} < X_{\zeta} - X_{\xi}$.

Since $\mathcal{A}$ is weakly nice there is $(\alpha', z', f, \xi, \zeta)$ s.t. $\alpha' < \kappa$, $z' \subseteq \theta_{\alpha'}$, $z' - \sigma_{\alpha'} \neq \emptyset$, $f \in F_{\alpha'+1\kappa}$, $\xi < \zeta$, $f \circ id''_{\alpha'}z' = X_{\xi}$ and $f \circ f''_{\alpha'}z' = X_{\zeta}$. Since $z' - \sigma_{\alpha'} \neq \emptyset$, it must be the case that $\alpha \leq \alpha'$. By extending q, if necessary, we may assume that the length of q is $\alpha'+1$ and $z^{q}_{\alpha'} = z'$. Since $f \circ id''_{\alpha'}z^{q}_{\alpha'} = X_{\xi}$ and $f \circ f''_{\alpha'}z^{q}_{\alpha'} = X_{\zeta}$ hold, we have $q \Vdash$ "$f \circ id''_{\alpha'}\dot{z}_{\alpha'} = \dot{X}_{\xi}$ and $f \circ f''_{\alpha'}\dot{z}_{\alpha'} = \dot{X}_{\zeta}$".

⊣

§2 Forcing Nice Simplified Morasses.

In this section we force a weakly nice simplified morass. We have a forcing notion to add a simplified morass (see for example [2]). It turns out that the generic simplified morasses are weakly nice ones. For reader's convenience we list relevant definitions and facts.

(2.1) Definition. For baby morasses $B_1 = (\langle\theta^1_\alpha\rangle_{\alpha\le\delta^1}, \langle F^1_{\beta\alpha}\rangle_{\beta\le\alpha\le\delta^1})$ and $B_2 = (\langle\theta^2_\alpha\rangle_{\alpha\le\delta^2}, \langle F^2_{\beta\alpha}\rangle_{\beta\le\alpha\le\delta^2})$ (for definition see (1.4)), we introduce a partial order by $B_2 \le B_1$ iff

(1) $\delta^1 \le \delta^2$.

(2) For any α with $\alpha \le \delta^1$, $\theta^2_\alpha = \theta^1_\alpha$.

(3) For any (β,α) with $\beta \le \alpha \le \delta^1$, $F^2_{\beta\alpha} = F^1_{\beta\alpha}$.

⊣

(2.2) Proposition. For any baby morass $B = (\langle\theta_\alpha\rangle_{\alpha\le\delta}, \langle F_{\beta\alpha}\rangle_{\beta\le\alpha\le\delta})$ and any ordinal $\sigma < \theta_\delta$, we have a baby morass $B' = (\langle\theta'_\alpha\rangle_{\alpha\le\delta'}, \langle F'_{\beta\alpha}\rangle_{\beta\le\alpha\le\delta'})$ s.t.

(1) $\delta' = \delta + 1$.

(2) $B' \le B$.

(3) $\theta'_{\delta+1} = \theta_\delta + (\theta_\delta - \sigma)$.

Proof. This is an easy exercise.

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(2.3) Definition. A subset X of κ^+ is said *good* if for any $\alpha < \kappa^+$, $X \cap [\kappa\alpha, \kappa(\alpha+1))$ is down-ward closed in it. (i.e. if $\kappa\alpha \le j \le i < \kappa(\alpha+1)$ and $i \in X$, then $j \in X$.)

⊣

(2.4) Definition. We describe a forcing notion P for adding a $(\kappa,1)$-morass. Let $p \in P$ iff $p = (B,h)$ s.t.

(1) $B = (\langle\theta_\alpha\rangle_{\alpha\le\delta}, \langle F_{\beta\alpha}\rangle_{\beta\le\alpha\le\delta})$ is a baby morass of length $\delta+1$.

(2) $\delta < \kappa$ and $\theta_\delta < \kappa$.

(3) For any (β,α) with $\beta \le \alpha \le \delta$, $|\, F_{\beta\alpha} \,| < \kappa$.

(4) h is an order-preserving function from θ_δ to κ^+ s.t. $h''\theta_\delta$ is a good subset of κ^+.

It is our convention to use script p to denote the relevant parts of the forcing condition p such as B_p, h_p, θ^p_α, $F^p_{\beta\alpha}$ and δ_p.

For $p,q \in P$ we define $q \le p$ iff

$B_q \le B_p$ as baby morasses and there is a unique $h \in F^q_{\delta_p\delta_q}$ s.t. $h_q \circ h = h_p$.

⊣

(2.5) Lemma.

(1) $(P,\le)$ is a p.o. set.

(2) For any pair of conditions of P with the same baby morass parts, say, $p = (B,h_p)$, $q = (B,h_q)$, suppose there is $\sigma < \theta_\delta$ s.t. $h_p{\restriction}\sigma = h_q{\restriction}\sigma$ and $h_q(\sigma) > h''_p\theta_\delta$. Then there is $r = (B_r,h_r) \in P$ s.t.

  (a) $\delta_r = \delta+1$ and $\theta^r_{\delta+1} = \theta_\delta + (\theta_\delta - \sigma)$.

  (b) $r \le p,q$.

  (c) $h_r \circ f_\delta = h_q$ and $h_r \circ id_\delta = h_p$.

(3) Suppose we have gotten a sequence $\langle p_\xi\rangle_{\xi<\rho}$ of descending conditions of P (i.e. $p_\xi \ge p_\zeta$ for all (ξ,ζ) with $\xi < \zeta < \rho$) s.t. $\rho < \kappa$ and ρ is a limit ordinal. For each $\xi < \rho$,

denote $p_\xi = (B_\xi, h_\xi)$ and B_ξ has the length $\delta_\xi + 1 < \kappa$. Let $\delta = sup\{\delta_\xi : \xi < \rho\}$, then there is $p = (B_p, h_p) \in P$ s.t. the baby morass B_p has the length $\delta + 1$, for each $\xi < \rho$ $p \leq p_\xi$ with $h_p^{-1} \circ h_\xi \in F^p_{\delta_\xi \delta}$ and θ^p_δ is the order-type of $\bigcup_{\xi<\rho} h''_\xi \theta^{p_\xi}_{\delta_\xi}$.

(4) For any condition p of P and any ordinal $\tau < \kappa$, there is $q \in P$ s.t. $q \leq p$, $\delta_q > \tau$ and $\theta^q_{\delta_q} > \tau$.

(5) For any condition p of P and any $i \in \kappa^+$, there is $q \in P$ s.t. $q \leq p$ and $i \in h''_q \theta^q_{\delta_q}$.

(6) $(P, \leq)$ is κ-closed.

(7) $(2^{<\kappa} = \kappa)$ $(P, \leq)$ has the κ^+-c.c..

Proof. We just comment that we required that the image of h_p, where $p = (B_p, h_p)$, is a good subset of κ^+ in order to get (5). The rest is well-known. (see for example [2].)

⊣

(2.6) Theorem. $(2^{<\kappa} = \kappa)$ There is a weakly nice simplified $(\kappa, 1)$-morass in the generic extensions via the P of (2.4).

Proof. Let G be an arbitrary P-generic filter over the ground model V. For each $\alpha < \kappa$, let $\theta_\alpha = \theta^p_\alpha$ for any $p \in G$ with $\alpha < \delta_p$ and let $\theta_\kappa = \kappa^+$ (in $V[G]$). For (β, α) with $\beta \leq \alpha < \kappa$, let $F_{\beta\alpha} = F^p_{\beta\alpha}$ for any $p \in G$ with $\alpha < \delta_p$. For each $\alpha < \kappa$, let $F_{\alpha\kappa} = \{h_p \circ f : p \in G, \alpha \leq \delta_p, f \in F^p_{\alpha\delta_p}\}$.

Then it is known that $(\langle\theta_\alpha\rangle_{\alpha\leq\kappa}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\kappa})$ is a simplified $(\kappa, 1)$-morass in $V[G]$. We show this is a weakly nice one.

Suppose $p \Vdash$"$\{\dot{X}_\xi : \xi < \kappa^+\}$ is a nice Δ-system s.t. for any (ξ, ζ) with $\xi < \zeta < \kappa^+$, $\dot{X}_\xi \cap \dot{X}_\zeta < \dot{X}_\xi - \dot{X}_\zeta < \dot{X}_\zeta - \dot{X}_\xi$". We want $(q, \alpha, z, \xi, \zeta)$ s.t. $q \leq p$, $\alpha < \delta_q = \alpha + 1$, $z \subseteq \theta^q_\alpha$, $z - \sigma^q_\alpha \neq \emptyset$, $\xi < \zeta < \kappa^+$ and $q \Vdash$"$h_q \circ id''_\alpha z = \dot{X}_\xi$ and $h_q \circ \dot{f}''_\alpha z = \dot{X}_\zeta$". (Note that $q \Vdash$"$h_q \in \dot{F}_{\alpha+1\kappa}$, $id^q_\alpha = id_\alpha$ and $f^q_\alpha = \dot{f}_\alpha$".)

Since P is κ-closed, we know for each $\xi < \kappa^+$, $p \Vdash$"$\dot{X}_\xi \in V$". So we have (p_ξ, X_ξ) s.t. $p_\xi \Vdash$"$\dot{X}_\xi = \check{X}_\xi$" and $X_\xi \subseteq h''_{p_\xi} \theta^{p_\xi}_{\delta_{p_\xi}}$. By thinning, using $2^{<\kappa} = \kappa$, we may assume that $\{X_\xi : \xi < \kappa^+\}$ forms a nice Δ-system and that there are a baby morass $B = (\langle\theta_\alpha\rangle_{\alpha\leq\delta}, \langle F_{\beta\alpha}\rangle_{\beta\leq\alpha\leq\delta})$ and $z \subseteq \theta_\delta$ s.t. for each $\xi < \kappa^+$, $p_\xi = (B, h_{p_\xi})$ and $h''_{p_\xi} z = X_\xi$ hold. Since any pair of conditions from $\{p_\xi : \xi < \kappa^+\}$ satisfy (2) of (2.5), we are done.

⊣

(2.7) Corollary. It is consistent that there exists a nice simplified $(\kappa, 1)$-morass relative to the consistency of ZFC.

Proof. By (2.6) and (1.11).

⊣

§3 Destroying Weakly Nice Simplified Morasses.

In this section we destroy the weakly niceness of a weakly nice simplified $(\kappa, 1)$-morass. Note that if a κ-closed forcing notion P satisfies the following stronger form of the κ^+-c.c., then the weakly nice simplified morasses remain weakly nice: For any $\{p_\xi\}_{\xi<\kappa^+} \subseteq P$, there

is I s.t. $I \subseteq \kappa^+$, $| I | = \kappa^+$ and for any $\xi, \zeta \in I$ with $\xi \neq \zeta$, p_ξ and p_ζ are compatible in P. So our p.o. set does not satisfy this property.

(3.1) Definition. Let $\mathcal{A}$ be an arbitrary weakly nice simplified $(\kappa, 1)$-morass. We define a forcing notion P designed to kill the weakly niceness of $\mathcal{A}$ by forcing a subset of κ^+. Let $p \in P$ iff

(1) $p \subset \kappa^+$ and $| p | < \kappa$.

(2) For any $i, j \in p$ with $i < j$, $\neg\{\exists\alpha\exists k \in \theta_\alpha - \sigma_\alpha \exists f \in F_{\alpha+1\kappa}$ $(f \circ id_\alpha(k) = i$ and $f \circ f_\alpha(k) = j)\}$ holds.

For $p, q \in P$, $q \leq p$ iff $q \supseteq p$.

⊣

(3.2) Lemma. Let $(p_0, p_1, p_2, p_3, \alpha, z, g_1, g_2)$ be s.t. $p_0, ..., p_3 \in P$, $\alpha < \kappa$, $z \subseteq \theta_\alpha$, $z - \sigma_\alpha \neq \emptyset$, $g_1, g_2 \in F_{\alpha+1\kappa}$, $g_1 \lceil id_\alpha(\sigma_\alpha) = g_2 \lceil id_\alpha(\sigma_\alpha)$, $g_2(id_\alpha(\sigma_\alpha)) > g_1'' \theta_{\alpha+1}$, $p_0 = g_1 \circ id_\alpha'' z$, $p_1 = g_1 \circ f_\alpha'' z$, $p_2 = g_2 \circ id_\alpha'' z$ and $p_3 = g_2 \circ f_\alpha'' z$. Then $p_1 \cup p_2$, $p_0 \cup p_3 \in P$. (But $p_0 \cup p_1 \notin P$, $p_2 \cup p_3 \notin P$.)

Proof. Since $f_\alpha''(z - \sigma_\alpha) > id_\alpha''(z - \sigma_\alpha)$ but $g_2 \circ id_\alpha''(z - \sigma_\alpha) > g_1 \circ f_\alpha''(z - \sigma_\alpha)$ holds, it must be that $p_1 \cup p_2 \in P$.

Since for any $g \in F_{\alpha+1\kappa}$ if $g \circ f_\alpha''(z - \sigma_\alpha) = p_3 - p_0$, then $g \circ id_\alpha''(z - \sigma_\alpha) = p_2 - p_0$, so $p_0 \cup p_3 \in P$.

⊣

(3.3) Lemma.

(1) $(P, \leq)$ is κ-closed.

(2) $(2^{<\kappa} = \kappa)$ $(P, \leq)$ has the κ^+-c.c..

Proof. Since (1) is trivial, we show (2). Given $\{p_\xi : \xi < \kappa^+\} \subseteq P$, by $2^{<\kappa} = \kappa$, we may asume $\{p_\xi : \xi < \kappa^+\}$ forms a nice Δ-system in $[\kappa^+]^{<\kappa}$ s.t. for any (ξ, ζ) with $\xi < \zeta < \kappa^+$, $p_\xi \cap p_\zeta < p_\xi - p_\zeta < p_\zeta - p_\xi$ holds. Since $\mathcal{A}$ is weakly nice, we may further assume that for each $\eta < \kappa^+$, there is $(\alpha_\eta, z_\eta, g_\eta)$ s.t. $\alpha_\eta < \kappa$, $z_\eta \subseteq \theta_{\alpha_\eta}$, $z_\eta - \sigma_{\alpha_\eta} \neq \emptyset$, $g_\eta \in F_{\alpha_\eta+1\kappa}$, $p_{2\eta} = g_\eta \circ id_{\alpha_\eta}'' z_\eta$ and $p_{2\eta+1} = g_\eta \circ f_{\alpha_\eta}'' z_\eta$.

Since $| \{(\alpha, z) : \alpha < \kappa, z \subseteq \theta_\alpha\} | = \kappa$ by $2^{<\kappa} = \kappa$, we may assume that there is (α, z) s.t $\{\alpha_\eta : \eta < \kappa^+\} = \{\alpha\}$, $\{z_\eta : \eta < \kappa^+\} = \{z\}$ and $z \subseteq \theta_\alpha$. So there are many combinations of conditions as in (3.2) among $\{p_\xi : \xi < \kappa^+\}$. In particular, $\{p_\xi : \xi < \kappa^+\}$ is not an antichain.

⊣

(3.4) Theorem. $(2^{<\kappa} = \kappa)$ For any weakly nice simplified $(\kappa, 1)$-morass $\mathcal{A}$, there is a notion of forcing Q which is κ-closed and has the κ^+-c.c. s.t. $\mathcal{A}$ is no more weakly nice in the generic extensions via Q.

Proof. Consider P in (3.1). Since P has the κ^+-c.c., there is $p_0 \in P$ s.t. $p_0 \Vdash_P$"$\bigcup \dot{G}$ is cofinal in κ^+". Let $Q = \{p \in P : p \leq p_0\}$ with the induced order. It is easy to see that for any $q \in Q$, $q \Vdash_Q$"Because of $\bigcup \dot{G} \subset \kappa^+$, $\mathcal{A}$ is no more weakly nice in $V[\dot{G}]$".

⊣

(3.5) Corollary. MA_{ω_1} implies that there is no weakly nice simplified $(\omega_0, 1)$-morass. (there are simplified $(\omega_0, 1)$-morasses though.)

Proof. By contradiction. Suppose there were a weakly nice simplified $(\omega_0, 1)$-morass $\mathcal{A}$. Then we have a c.c.c. p.o. set Q s.t. $\Vdash_Q$"$\bigcup \dot{G}$ is a cofinal subset of ω_1 and $\bigcup \dot{G}$ witnesses that $\mathcal{A}$ is not a weakly nice one". By MA_{ω_1}, we would have such a subset of ω_1 in V. This is a contradiction.

⊣

§4 Open Questions.

We list a number of typical open questions.

1. Are simplified $(\kappa, 1)$-morasses constructed in L nice (weakly nice) in L ?
2. Is every weakly nice simplified $(\kappa, 1)$-morass nice under $2^{<\kappa} = \kappa$?
3. Is it possible to iterate the Q in (3.4) to obtain that no simplified $(\kappa, 1)$-morass is weakly nice for κ $(\geq \omega_1)$?
4. Is it possible that a complete amalgamation system exists for some $(\omega_0, 1)$-morass while there exist no weakly nice $(\omega_0, 1)$-morasses ?

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