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Hypergeometric functions and modular embeddings

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I. Discontinuous groups acting on irreducible complex symmetric domains of dim > 1 with finite covolume are arithmetically defined with the possible exception of groups on the complex ball B_N

$$|z_1|^2 + \dots + |z_N|^2 < |z_0|^2$$

(Conjecture of Selberg, proven by Margoulis and Selberg)

Mostow: Examples of non-arithmetic groups acting on B_2 and B_3

History: Picard 1885
Terada 1973/83

Deligne - Mostow and Mostow 1986
Hirzebruch - Höfer - Yoshida 1983 - 87
Sauter 1990

„Picard - Terada - Mostow - Deligne" groups
PTMD - groups Δ

Construction of Δ as monodromy groups of the Appell - Lauricella - functions

From now on $N = 2$

$\mu_0, \mu_1, \ldots, \mu_4 \in \mathbb{Q} \cap]0,1[$, $\mu_0 + \ldots + \mu_4 = 2$

$$F_1(x,y) := \ldots \int_1^\infty \underbrace{u^{-\mu_0}(u-1)^{-\mu_1}(u-x)^{-\mu_2}(u-y)^{-\mu_3}du}_{\omega :=}$$

solution of a system of linear PDE's holomorphic outside the „characteristic surfaces"

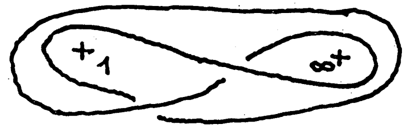
$x = y$ and $x, y = 0, 1, \infty$.

$Q := \mathbb{C}^2 - \{\text{characteristic surfaces}\}$

fundamental solutions e.g. $\int_1^\infty \omega$, $\int_0^x \omega$, $\int_0^y \omega$

Integration paths avoiding other singularities of ω, can be chosen as cycles on the Riemann surface of ω

„Pochhammer cycles"



Δ can be calculated moving integration paths [Felix Klein .... Yoshida] $\Rightarrow$ Δ is induced by some automorphism group of H_1 of the Riemann surface of ω.

Thm (P-T-M-D):

$Q \longrightarrow \mathbb{P}^2(\mathbb{C}) : (x,y) \mapsto \left(\int_1^\infty \omega, \int_0^x \omega, \int_0^y \omega\right)$
defines a $PGL_3(\mathbb{C})$ - multivalent, locally biholo map ψ onto a dense subset of a complex ball $B \cong B_2$. The non-uniqueness of ψ is described by the action of Δ on B. This action is discontinuous if e.g.

$$(1-\mu_i-\mu_j)^{-1} \in \begin{cases} \frac{1}{2}\mathbb{Z} \cup \{\infty\} & \text{if } \mu_i = \mu_j \\ \mathbb{Z} \cup \{\infty\} & \text{otherwise} \end{cases} \quad \text{for all } i \neq j \in \{0,\dots,4\}$$

In the second case ψ is the inverse of the canonical projection $B \to \Delta\backslash B$.

II. Main result (P. Cohen, J.W.) For any PTMD group Δ there is an arithmetic group Γ acting on a power B^m of the ball and a „modular embedding" consisting of two compatible injections

$h : \Delta \hookrightarrow \Gamma$ (group homomorphism)

$F : B \hookrightarrow B^m$ (analytic)

with $F(\gamma\tau) = h(\gamma)\,F(\tau)$ for all $\tau \in B$ and $\gamma \in \Delta$.

F induces a morphism of algebraic varieties

$\overline{F} : \overline{\Delta\backslash B} \longrightarrow \overline{\Gamma\backslash B^m}$ (compactified if necessary)

defined over $\overline{\mathbb{Q}}$.

III. Elements of the proof.

Easy part: construction of h

$d := \operatorname{lcd}(\mu_0, \dots, \mu_4) \Rightarrow \Delta \subset PSU(2,1;\mathbb{Z}[\zeta_d])$

by direct calculation of generators, $\zeta_d := \exp \frac{2\pi i}{d}$.

Often $\Gamma = PSU(2,1;\mathbb{Z}[\zeta_d])$, $h = id$.

e.g. in the example

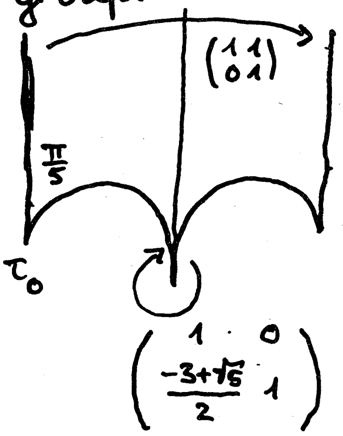
$\mu_0 = \frac{7}{12}$, $\mu_1 = \frac{5}{12}$, $\mu_2 = \frac{6}{12}$, $\mu_3 = \mu_4 = \frac{3}{12}$

Γ acts on $B^2 = B \times B$ discontinuously by

$(\tau_1, \tau_2) \overset{\gamma}{\mapsto} (\gamma\tau_1, \gamma^{\sigma_5}\tau_2)$ where $\sigma_5 : \zeta_{12} \mapsto \zeta_{12}^5$

How to construct F ? ?

Digression to the easier case $N = 1$ of triangle groups Δ. Example: Signature $[5, \infty, \infty]$



$\leftarrow$ generators of Δ in $\mathfrak{H}$

$\Rightarrow \Delta \hookrightarrow PSL_2\, \mathcal{O}_{\sqrt{5}}$

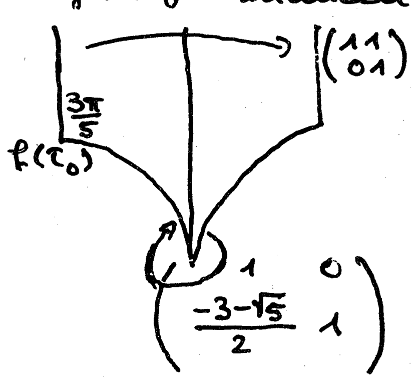
Wanted: A modular embedding $F : \mathfrak{H} \to \mathfrak{H} \times \mathfrak{H}$

$F(\tau) = (\tau, f(\tau))$ with

$f : \mathfrak{H} \to \mathfrak{H}$ holo and

$$f(\gamma\tau) = \gamma^\sigma f(\tau)$$

$\gamma \mapsto \gamma^\sigma$ induced by $\sqrt{5} \mapsto -\sqrt{5}$



Construction of f by Riemann theorem and Schwarz' reflection principle <u>or</u>

using triangle functions $\quad f = D_3 \circ D_1^{-1}$

<u>or</u> (projectively, neglecting constants and PGL_2 - transformations)

$$\Big(\underbrace{\int_1^\infty \omega, \int_0^x \omega}_{\tau}\Big) \mapsto \Big(\underbrace{\int_1^\infty \omega, \int_0^x \omega}_{\tau=\tau_1} ; \underbrace{\int_1^\infty \omega_3, \int_0^x \omega_3}_{f(\tau)=\tau_2}\Big)$$

where

$$\omega = u^{-3/5}(u-1)^{-3/5}(u-x)^{-2/5}\,du \quad = \frac{du}{w}$$

on the curve $\quad w^5 = u^3(u-1)^3(u-x)^2$

$$\omega_3 = \ldots = \frac{u(u-1)(u-x)}{w^3}\,du \quad \text{on the same curve}$$

Digression: Number - theoretic motivation.

There are generating Δ - automorphic functions with Taylor expansions

$$j(\tau) = \sum_{n \geq 0} c_n \,r^n \left(\frac{\tau-\tau_0}{\tau-\bar{\tau}_0}\right)^n, \quad \text{all } c_n \in \overline{\mathbb{Q}}$$

and $\quad r = \dfrac{B(\frac{1}{5}, \frac{2}{5})}{B(\frac{4}{5}, \frac{3}{5})}$.

The same constants play the same role for Hilbert modular functions at the corresponding fixed point of $PSL_2\, \mathcal{O}_{\sqrt{5}}$. Why Beta - values?

End of the digressions, back to the $N=2$ - example: ~~For~~ $F : B \to B \times B$ take

$$\Big(\int_1^\infty \omega, \int_0^x \omega, \int_0^y \omega\Big) \mapsto \Big(\int_1^\infty \omega, \int_0^x \omega, \int_0^y \omega ; \int_1^\infty \omega_5, \int_0^x \omega_5, \int_0^y \omega_5\Big)$$

with differentials

$$\omega = u^{-7/12}(u-1)^{-5/12}(u-x)^{-6/12}(u-y)^{-3/12}\,du$$

$= \frac{du}{w}$ on the curve $X_s(x,y)$ given by

$$w^{12} = u^7(u-1)^5(u-x)^6(u-y)^3$$

and

$$\omega_5 = u^{-11/12}(u-1)^{-1/12}(u-x)^{-6/12}(u-y)^{-3/12}$$

$= \frac{u^2(u-1)^2(u-x)^2(u-y)}{w^5}\,du$ on the same curve.

IV. Principles behind this construction.

Let $X(x,y)$ a non-singular projective model of $X_s(x,y)$, $\mathrm{Jac}\,X(x,y)$ its Jacobian, m_4 and m_3 morphisms of $\mathrm{Jac}\,X(x,y)$ on other Jac's induced by

$$X_s(x,y) \begin{array}{l} \nearrow\; w^4 = u^7(u-1)^5\cdots \\ \searrow\; w^6 = u^7(u-1)^5\cdots \end{array}$$

and $T(x,y) :=$ connected component of 0 of $\mathrm{Ker}\,m_4 \cap \mathrm{Ker}\,m_3$

$T(x,y)$ is a p.p. abelian variety of dimension 6 $(= \frac{3}{2}\varphi(d),\ d = 12)$:

$\chi : X_s(x,y) \to X_s(x,y) : (u,w) \mapsto (u, \zeta_{12}^{-1} w)$

induces $\mathbb{Z}[\zeta_{12}] \subset \mathrm{End}\,T(x,y)$.

$H^0(T(x,y),\Omega)$ splits into χ-Eigenspaces

$V_n := \{\omega \text{ (first kind)} \mid \omega\circ\chi = \zeta_{12}^n\cdot\omega\}$

with $n \in (\mathbb{Z}/12\mathbb{Z})^*$. The dimensions $r_n = \dim V_n$

can be calculated by an old theorem of Chevalley and Weil :

$$\tau_n = -1 + \sum_{j=0}^{4} \langle n\mu_j \rangle$$

where $\langle \alpha \rangle$ denotes the fractional part $\alpha - [\alpha]$ of $\alpha \in \mathbb{R}$. In our example

$$\tau_1 = \tau_5 = 1 \qquad \tau_{-1} = \tau_{-5} = 2$$

(always $\tau_n + \tau_{-n} = 3$, so $\dim T(x,y) = \frac{3}{2}\varphi(d)$)

ω and ω_5 generate V_1 and V_5

(if $\dim V_n = 1$, it has a generator on $X_3(x,y)$

$$u^{-\langle n\mu_0 \rangle}(u-1)^{-\langle n\mu_1 \rangle}(u-x)^{-\langle n\mu_2 \rangle}(u-y)^{-\langle n\mu_3 \rangle}du)$$

$T(x,y)$ belongs to a family of p.p. abelian varieties with „generalized complex multiplication" by $\mathbb{Q}(\zeta_{12})$ and „type"

$$\sum \tau_n \sigma_n = 1\cdot\sigma_1 + 1\cdot\sigma_5 + 2\cdot\sigma_{-5} + 2\cdot\sigma_{-1} .$$

[Siegel / Shimura] : This family is parametrized by B^m, $m = \#\, V_n$ of dimension 1, i.e. $m=2$ in our case, its coordinates are given by

$$\underbrace{\int_{\gamma_0}\omega_1 , \int_{\gamma_1}\omega_1 , \int_{\gamma_2}\omega_1}_{\psi(x,y)\,\in\, B} \quad ; \quad \underbrace{\int_{\gamma_0}\omega_5 , \int_{\gamma_1}\omega_5 , \int_{\gamma_2}\omega_5}_{\psi_5(x,y)\,\in\, B}$$

(neglecting linear transformations) where $\omega_1 = \omega$ and ω_5 generate the dim - 1 - eigen= spaces of $H^0(\quad , \Omega)$

and $\gamma_0, \gamma_1, \gamma_2$ generate the cycles of the abelian variety as $\mathbb{Z}[\zeta_d]$ - module. So

$$F : \psi(x,y) \longmapsto (\psi(x,y),\ \psi_5 \Psi^{-1} \psi(x,y)),$$

at least in $\psi Q \subset B$.

F is clearly injective and holomorphic.

Since Δ only changes the $\mathbb{Z}[\zeta_d]$ - basis of $H_1(\ ,\mathbb{Z})$, $T(x,y)$ remains the same, only its coordinates in B^2 change $\Rightarrow$ Δ is in a natural way a subgroup of the modular group for the family of abelian varieties considered. This modular group Γ is always arithmetic.

V. Singularities.

$B - \psi Q$ = images of „stable singular points" under (a continuous extension of) ψ

e.g. of $y = 0$ $\quad (\mu_0 + \mu_3 = \frac{10}{12} < 1)$

= locally finite union of analytic subsets of B of codimension ≥ 1

components of F holomorphic and bounded outside $\Rightarrow$ singularities removable (Riemann).

Behaviour of the $T(x,y)$ in the characteristic surfaces:

In $y = 0$ $\qquad \omega = u^{-10/12} (u-1)^{-5/12} (u-x)^{-6/12}\, du$

same procedure as before leads to a family $T(x)$ of abelian varieties with CM by $\mathbb{Q}(\zeta_{12})$ and

of type $1 \cdot \sigma_1 + 2\sigma_{-5} + 1 \cdot \sigma_{-1}$ and dim = 4
($\varphi(d)$ in general) belonging to Gauß hyper=
geometric functions with arithmetic (!) monodromy
group $\Delta_{y=0}$ of signature $[3,4,12]$)
$\Rightarrow$ On $y = 0$, $T(x,y) = T(x) \oplus A_{y=0}$
with a constant p.p. abelian variety with
CM by $\mathbb{Q}(\zeta_{12})$ and type $1 \cdot \sigma_5 + 1 \cdot \sigma_{-1}$
(in the narrow sense of [Shimura - Taniyama])
and periods of first kind

$B(\mu_0, \mu_3) = B(\frac{7}{12}, \frac{3}{12})$ and $B(\langle -5\mu_0 \rangle, \langle -5\mu_3 \rangle) = B(\frac{1}{12}, \frac{9}{12})$

In $(x,y) = (1,0)$ $T(x,y)$ splits into three abelian
varieties of CM - type.
Shimura $\Rightarrow$ Their periods occur in the Taylor
expansions of suitably normalized Γ - automorphic
functions
$\Rightarrow \ldots$ $\overline{F}$ defined over $\overline{\mathbb{Q}}$.

$x = 0$ is non - stable $(\mu_0 + \mu_2 = \frac{13}{12} > 1)$
ψ blows down $x = 0$ to a Δ - orbit of points in B
$\Rightarrow T(x,y) \cong A \oplus A' \oplus A'$, all of CM type
$\Rightarrow F_1(0,y)$ is an algebraic hypergeometric function;
its monodromy group $\Delta_{x=0}$ (tetrahedral) is the
fixgroup in Δ for $\psi(0,y)$.

Literature

Paula Cohen, Jürgen Wolfart:
Modular embeddings for some non-arithmetic Fuchsian groups,
Acta Arithmetica 56 (1990), 93 - 110
(N=1 case)

— " — : Fonctions hypergéométriques en plusieurs variables et espaces de modules de variétés abéliennes, preprint.
(32 p., not in final form, but copies can be made)

— " — : Monodromie des fonctions d'Appell, variétés abéliennes et plongement modulaire, preprint MPI Bonn 1989 - 80, to appear in the proceedings of the Journées Arithmétiques Luminy 1989 in Astérisque.
(6 p., very short form)

— " — : Algebraic Appell - Lauricella Functions, to appear in the proceedings of the Katata workshop.