数理解析研究所講究録
第 780 巻 1992 年 141-156

ON THE PARTIAL REGULARITY OF BOUNDED WEAK SOLUTIONS TO NONLINEAR DEGENERATE PARABOLIC SYSTEMS OF P-HARMONIC TYPE

MASASHI MISAWA (三沢正史)

Department of Mathematics,
Faculty of Science and Technology,
Keio University

ABSTRACT. We establish partial regularity for bounded weak solutions of nonlinear parabolic systems of p-harmonic type. It's necessary to consider L^q- estimate for the spatial gradient of solutions by carefully using so-called Gehring inequality.

1.Introduction.

In this paper we establish Hölder estimates for bounded weak solutions to nonlinear degenerate parabolic systems of the form

$$\frac{\partial u^i}{\partial t} - \mathrm{div}(|Du|^{p-2}Du^i) = f^i(t,x,u,Du), \quad 1 \le i \le n \tag{1.1}$$

in an open set $Q = (0,T) \times \Omega \subset R^{m+1}$, $m \ge 2$. Here Ω is an open set in R^m, $x \in \Omega \subset R^m$, $t > 0$, T is a given positive number, $u = (u^1, u^2, \cdots, u^n)$ is a mapping: $Q \to R^n$ and $Du = (D_1u, D_2u, \cdots, D_mu)$, $D_\alpha u = \partial u/\partial x^\alpha$ $(1 \le \alpha \le m)$ is the spatial gradient of u, p is any positive number satisfying

$$2 < p < \infty$$

and $f(t,x,u,p)$ is a Caratheodory function : $(0,T) \times \Omega \times R^n \times R^{nm} \to R^n$, satisfying the growth condition with some positive constant a

$$|f(t,x,u,p)| \le a|p|^p \tag{1.2}$$

Let us introduce the parabolic metric with some positive constant θ

$$\delta_\theta(z_1,z_2) = \max(|x_1 - x_2|,\ |t_1-t_2|^{1/\theta}), \quad z_i = (t_i,x_i),\ i = 1,2. \tag{1.3}$$

and denote by $H^k(\cdot,\delta_\theta)$ the k-dimensional Hausdorff measure with respect to δ_θ. Here we recall some function spaces: Hölder space $C^{0,\mu}(Q,\delta_\theta)$, denoted the spaces of Hölder continuous functions in Q (with respect to the metric δ_θ) with an exponent μ, the usual Lebegue space $L^p(\Omega) = L^p(\Omega, R^n)$ and Sobolev spaces: $W_p^k(\Omega) = W_p^k(\Omega, R^n)$, $\overset{\circ}{W}{}_p^k(\Omega) = \overset{\circ}{W}{}_p^k(\Omega)(\Omega, R^n)$, $V_{2,p}(Q) = L^\infty((0,T); L^2(\Omega)) \cap L^p((0,T); W_p^1(\Omega))$, $\overset{\circ}{W}{}_{2,p}^{1,1}(Q) = W_2^1((0,T); L^2(\Omega)) \cap L^p((0,T); \overset{\circ}{W}{}_p^1(\Omega))$. By a weak solution u of (1.1) in Q we mean a vector-valued function $u = (u^1, u^2, \cdots, u^n) \in V_{2,p}(Q) \cap L^\infty(Q)$ satisfying (1.1) in the weak sense:

$$\iint_Q [-u^i\partial_t\varphi^i + |Du|^{p-2}Du^iD\varphi^i]dtdx = \iint_Q f^i\varphi^i dtdx \quad \text{for any } \varphi \in \overset{\circ}{W}{}_{2,p}^{1,1}(Q) \cup L^\infty(Q). \tag{1.4}$$

In (1.4) and in what follows, the summation notation over repeated indices is adopted.

Then our main theorem is the following:

Theorem. *Let u be a bounded weak solution of (1.1), set $M = \sup_Q |u|$ and assume that*

$$1 > 2aM. \tag{1.5}$$

Then there exist positive constants ε, $\beta, 0 < \beta < 1$, and an open set $Q_0 \subset Q$ such that $u \in C^{0,\beta}_{\rm loc}(Q_0, \delta_2)$ and $H^{m-\varepsilon}(Q - Q_0, \delta_2) = 0$.

The proof of Theorem relies on a perturbation argument (see [8],[9],[13]) and an L^q–estimate for $|Du|$ which is of some interest in itself(refer to [9]). We prove such L^q–estimate by exploiting so-called Gehring-inequality in Sect.3 (see [8],[9]).

Remark. In a scalar case everywhere regularity for bounded weak solutions is established without assuming (1.5) (see [4],[14]). In a case of $p = 2$ the analogue result is obtained in [9], [10].

Some standard notations: For $z_0 = (t_0, x_0) \in Q$ and $r, \tau > 0$

$$B_r(x_0) = \{x \in R^n : |x - x_0| < r\}, \;\; Q_{r,\tau}(z_0) = (t_0 - \tau, t_0) \times B_r(x_0).$$

For $\theta > 0$ and $z_0 \in Q$, $r > 0$ put the cylinders

$$Q_r^\theta(z_0) = (t_0 - r^\theta, t_0) \times B_r(x_0). \tag{1.6}$$

When $\theta = p$ we let $Q_r(z_0) = Q_r^p(z_0)$. In the above notations the center points x_0, z_0 are omitted when no confusion may arise. For an integrable function $f : Q \to R^n$ and a measurable set $A \subset Q$

$$\bar{f}_A = \frac{1}{|A|} \int_A f dz$$

where $|A|$ denote Lebegue measure of A. For any positive number l we mean by $[l]$ the greatest positive integer not greater than l.

2.Some preminalies.

In this section we collect a few results which we shall use in the following.

We now introduce another function space. Assume that Ω is 'of type A' (see [9],[11]), namely there exists a constant $A > 0$ such that for any $R > 0$ and all $x_0 \in \Omega$

$$|\Omega \cap B_R(x_0)| \geq AR^n$$

and denote by $L^{p,\mu}(Q)$, $p \geq 1, \mu > 0$, the space of all functions u in $L^p(Q)$ satisfying

$$([u]_{p,\mu.Q})^p = \sup_{z_0 \in Q, R>0} R^{-\mu} \iint_{Q_R^\theta(z_0)} |u - \bar{u}_{Q_R^\theta(z_0)}|^p dz < \infty. \tag{2.1}$$

$L^{p,\mu}(Q)$ is a Banach space with the norm

$$\{|u|^p_{L^p(Q)} + ([u]_{p,\mu.Q})^p\}^{1/p}.$$

These spaces have been introduced in [8] for the Euclidean metric and in [3] for a general class of metrics including the parabolic one δ_θ. We have the following result([3], Theorem 3.1).

Proposition 2.1. *The spaces $L^{p,m+\theta+\theta\mu}(Q)$ and $C^{0,\mu}(Q,\delta_\theta)$, $0 < \mu < 1$ are topological and algebraically isomorphic.*

We actually exploit Proposition 2.1 on a local cylinder.

Let us now recall the estimate for solutions of nonlinear degenerate parabolic systems (refer to [5],[13]).

Proposition 2.2. *Let v be a weak solution of (1.1) with $f \equiv 0$ in some cylinder $Q_R^\theta \subset Q$ where $\theta = 2 + \alpha(p-2)$ with $\alpha > 0$. Then, for $0 < \alpha < 1$, there exist positive constants γ, $q > 1$ depending only on m, p and α such that*

$$\iint_{Q_r^\theta} |Dv|^p dtdx \leq \gamma \Big(\frac{r}{R}\Big)^{m+\theta-\alpha p} \Big\{ \iint_{Q_R^\theta} |Dv|^p dtdx + 1 \Big\}. \tag{2.2}$$

holds for all $0 < r < R/2$.

Finally we need the following result that can be proved similarly as [8], Prop. 5.1 (also refer to [9]) only by changing Euclidean cubes with parabolic ones:

Proposition 2.3. *Let g be a nonnegative L^q–integrable function defined in some cylinder Q_R with some $q > 1$. Let us suppose that g satisfys with some positive constants:$b > 1, \delta > 0$*

$$\begin{aligned} \frac{1}{|Q_r^\theta|} \iint_{Q_r^\theta(z_0)} g^q dtdx \leq b \Big(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} g dtdx \Big)^q + \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} f^q dtdx \\ + \delta \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} g^q dtdx \end{aligned} \tag{2.3}$$

for all $z_0 \in Q_R$ and any $0 < r < (1/4)dist(z_0, \partial Q_R)$. Then there exist positive constants γ, ε, depending on b, q, δ and m, and δ_0, depending only on q and m, such that, if $\delta < \delta_0$, $g \in L^{\tilde q}(Q_{R/4})$ for $\tilde q \in [q, q+\varepsilon)$ and

$$\Big(\frac{1}{|Q_{R/4}|} \iint_{Q_{R/4}} g^{\tilde q} dtdx \Big)^{1/\tilde q} \leq \gamma \Big(\frac{1}{|Q_R|} \iint_{Q_R} g^q dtdx \Big)^{\frac{1}{q}} + \Big(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} f^{\tilde q} dtdx \Big)^{\frac{1}{\tilde q}} \tag{2.4}$$

Now we state a fundamental inequality for solutions to (1.1). In the following Q_R is a arbitrarily fixed cylinder such that $Q_R \subset Q$, $0 < R \leq 1$. We also take a positive number θ as $0 < \theta \leq p$ and $\chi = \chi(x)$ as a function in $C_0^\infty(B_2)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on B_1 and $|D\chi| \leq 2$. We denote by $\chi_{x_0,2r}$ the function $\chi_{x_0,2r}(x) = \chi((x - x_0)/r)$ for any $x_0 \in Q$ and replace the notation $\chi_{x_0,2r}$ by χ when no confusion may arise. We also use the weighted means of u in $B_{2r}(x_0)$ as

$$\begin{aligned} \bar u^\chi_{B_{2r}(x_0)}(t) &= \int_{B_{2r}(x_0)} u(t,x)\chi^p_{x_0,2r}(x)dx \Big/ \int_{B_{2r}(x_0)} \chi^p_{x_0,2r}(x)dx, \quad x_0 \in \Omega, \\ \bar u^\chi_{Q_{2r}^\theta(z_0)} &= \iint_{Q_{2r}^\theta(z_0)} u(t,x)\chi^p_{x_0,2r}(x)dtdx \Big/ \iint_{Q_{2r}^\theta(z_0)} \chi^p_{x_0,2r}(x)dtdx, \quad z_0 \in Q. \end{aligned} \tag{2.5}$$

Lemma 2.4. *(Caccioppoli type estimate) There exists a positive constant γ depending only on m, M and θ such that*

$$\sup_{t_0-r^\theta<t<t_0}\int_{B_r(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_{2r}(x_0)}|^2dx+\iint_{Q^\theta_r(t_0,x_0)}|Du|^pdtdx$$
$$\leq\gamma\Big(r^{-\theta}\iint_{Q^\theta_{2r}(t_0,x_0)}|u-\bar{u}^\chi_{B_{2r}(x_0)}(t)|^2dtdx+r^{-p}\iint_{Q^\theta_{2r}(t_0,x_0)}|u-\bar{u}^\chi_{B_{2r}(x_0)}(t)|^pdtdx\Big) \tag{2.6}$$

holds for any $Q^\theta_{2r}(t_0,x_0)\subset Q_R$.

Proof.Let $\tau\in C^\infty(R,R)$ depend only on a time-variable t satisfying $0\leq\tau\leq 1$, $\tau=1$ on $[t_0-r^\theta,t_0]$, $\tau=0$ on $t<t_0-(2r)^\theta$ and $|\partial_t\tau|\leq 2/(2^\theta-1)r^{-\theta}$. Testing (1.1) by a function $\varphi=(u-\bar{u}^\chi_{B_{2r}(x_0)}(t))\chi^p\tau^p\mathbf{1}_{-\infty,t_0}$, we have

$$\int_{B_{2r}\times\{t_0\}}|u-\bar{u}^\chi_{B_{2r}}|^2\chi^p\tau^pdx+\iint_{Q^\theta_{2r}}[|Du|^p-f(t,x,u,Du)(u-\bar{u}^\chi_{B_{2r}}(t))]\chi^p\tau^pdtdx$$
$$\leq\gamma\iint_{Q^\theta_{2r}}|u-\bar{u}^\chi_{B_{2r}}|^2\chi^p\tau^{p-1}\partial_t\tau dtdx \tag{2.7}$$
$$+\gamma\iint_{Q^\theta_{2r}}|Du|^{p-2}DuD\chi(u-\bar{u}^\chi_{B_{2r}})\chi^{p-1}\tau^pdtdx.$$

Since by our choice of a test function the remaining term

$$\int_{t_0-2r^\theta}^{t_0}[\int_{B_{2r}}(u-\bar{u}^\chi_{B_{2r}}(t))\chi^pdx]\partial_t\bar{u}^\chi_{B_{2r}}(t)\tau^pdt$$

is equal to zero, we obtain the lemma from applying Young's inequality and (1.5) to (2.7). Note that the time derivative $\partial_t\bar{u}^\chi_{B_{2r}}(t)$is integrable. In fact, testing the identity by $\varphi=\chi^p\mathbf{1}_{(t,t_0)}$one immediately sees that $\bar{u}^\chi_{B_{2r}}(t)$ is absolutely continuous.

Remark. $(u-\bar{u}^\chi_{B_{2r}(x_0)}(t))\chi^p\tau^p\mathbf{1}_{-\infty,t_0}$ is not admissible as a test function in (1.1). But, by substituting $[(u-\bar{u}^\chi_{B_{2r}(x_0)}(t))_h\chi^p\tau^p\mathbf{1}^\varepsilon_{-\infty,t_0}]_{\bar{h}}$ where $\eta_h(t)=(1/h)\int_t^{t+h}\eta(s)ds$, $\eta_{\bar{h}}(t)=(1/h)\int_{t-h}^t\eta(s)ds$ and $\mathbf{1}^\varepsilon_{-\infty,t_0}\in C^\infty(R)$, $\mathbf{1}^\varepsilon_{-\infty,t_0}=1$ on $t<t_0-\varepsilon$, $\mathbf{1}^\varepsilon_{-\infty,t_0}=0$ on $t>t_0$, (which is admissible as a test function in (1.1)) into (1.1), and calculating similarly as (2.7) and letting $h,\varepsilon\downarrow 0$ in the resulting inequality, we have (2.7).

Lemma2.5. *There exists a positive constant γ depending only on m, M and θ such that*

$$\begin{aligned}&\sup_{t_0-r^\theta<t<t_0}\int_{B_r(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_r(x_0)}(t)|^2dx\\&\leq\gamma\Big(r^{2-\theta}\iint_{Q^\theta_{2r}(t_0,x_0)}|Du|^2dtdx+\iint_{Q^\theta_{2r}(t_0,x_0)}|Du|^pdtdx\Big)\end{aligned}\tag{2.8}$$

holds for any $Q^\theta_{2r}(t_0,x_0)\subset Q_R$.

Proof. As in the proof of Lemma 2.4, testing (1.1) with

$$(u-\bar{u}^\chi_{B_{2r}(x_0)}(t))\chi^p\tau^p\mathbf{1}_{-\infty,t_0},$$

we obtain, from applying a simple variation of Poincaré inequality for the resulting inequality,

$$\begin{aligned}&\sup_{t_0-2r^\theta<t<t_0}\int_{B_r(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_{2r}(x_0)}(t)|^2dx\\&\leq\gamma\Big(r^{2-\theta}\iint_{Q^\theta_{2r}(t_0,x_0)}|Du|^2dtdx+\iint_{Q^\theta_{2r}(t_0,x_0)}|Du|^pdtdx\Big).\end{aligned}$$

Since, for any $t\in(t_0-r^\theta,t_0)$

$$\begin{aligned}&\int_{B_r(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_r(x_0)}(t)|^2dx\\&\leq\int_{B_{2r}(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_{2r}(x_0)}(t)|^2dx+2|B_r||\bar{u}^\chi_{B_{2r}}(t)-\bar{u}^\chi_{B_r}(t)|^2\\&\leq\gamma\int_{B_r(x_0)\times\{t\}}|u-\bar{u}^\chi_{B_{2r}(x_0)}(t)|^2dx.\end{aligned}\tag{2.9}$$

the result follows.

Lemma 2.6. *There exists a positive constant γ depending only on m and M such that*

$$\sup_{t_0-r^\theta<t<t_0}\int_{B_r\times\{t\}}|u(t,x)-\bar{u}^\chi_{B_r}(t)|^pdx\leq\gamma r^{p(\theta-p)/(p-1)}\iint_{Q^\theta_{2r}}|Du|^pdtdx\tag{2.10}$$

holds for any $Q^\theta_{2r}\subset Q_R$.

Proof. Let τ be the same function as in Lemma 2.4. Testing (1.1) with

$$\varphi=(u-\bar{u}^\chi_{Q^\theta_{2r}})|u-\bar{u}^\chi_{Q^\theta_{2r}}|^{p-2}\chi^p\tau^p$$

(note Remark after Lemma 2.4) and using Young's inequality, we have

$$\begin{aligned}&(1/p)\int_{B_{2r}}|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p\chi^p\tau^{p-1}dx-(1/p)\iint_{Q^{\theta}_{2r}}|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p\chi^p\partial_t\tau\tau^{p-1}dtdx\\&+(1-p\varepsilon)\iint_{Q^{\theta}_{2r}}|Du|^p|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^{p-2}\chi^p\tau^p dtdx\\&+(p-2)/4\iint_{Q^{\theta}_{2r}}|Du|^{p-2}|D|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^2|^2|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^{p-4}\chi^p\tau^p dtdx\\&-p\gamma(p,\varepsilon)\iint_{Q^{\theta}_{2r}}|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^{2(p-1)}|D\chi|^p\tau^p dtdx\le a\iint_{Q^{\theta}_{2r}}|Du|^p|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^{p-1}\chi^p\tau^p dtdx.\end{aligned}$$

Putting ε so small in the above and noticing $p>2$, we obtain from the boundedness of u

$$\begin{aligned}&\sup_{t_0-r^{\theta}<t<t_0}\int_{B_r}|u(t,x)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p dx\le\gamma\iint_{Q^{\theta}_{2r}}|u(t,x)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p\partial_t\tau dtdx\\&+\gamma\iint_{Q^{\theta}_{2r}}|u(t,x)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^{2(p-1)}|D\chi|^p dtdx.+a(2M)^{p-1}\iint_{Q^{\theta}_{2r}}|Du|^p dtdx.\end{aligned}\tag{2.11}$$

Note the following estimate: For $t_0-(2r)^{\theta}<s<t<t_0$

$$\begin{aligned}&\int_{B_r\times\{t\}}|u-\bar{u}^{\chi}_{B_{2r}}(t)|^p dx\\&\le 2^{p-1}\int_{B_r\times\{t\}}|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p dx+2^{p-1}|B_r||\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p,\\&\iint_{Q^{\theta}_{2r}}|u-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p dtdx\\&\le 2^{p-1}\iint_{Q^{\theta}_{2r}}|u-\bar{u}^{\chi}_{B_{2r}}(t)|^p dtdx+2^{p-1}|B_{2r}|\int_{t_0-(2r)^{\theta}}^{t_0}|\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p dt.\end{aligned}\tag{2.12}$$

Now we estimate $|\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p$ for $t_0-(2r)^{\theta}<t<t_0$. Testing the identity (1.1) by

$$\chi^p\mathbf{1}_{s,t}(\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{B_{2r}}(s))|\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{B_{2r}}(s)|^{p-2},\quad t,s\in(t_0-2r^{\theta},t_0)$$

and noting the boundedness of u, we have, for any $t_0-(2r)^{\theta}<s<t<t_0$

$$|B_{2r}||\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{B_{2r}}(s)|^p\le\gamma(M)(r^{(\theta-p)/(p-1)}+1)\iint_{Q^{\theta}_{2r}}|Du|^p dtdx.\tag{2.13}$$

Noticing that $\bar{u}^{\chi}_{Q^{\theta}_{2r}}=\int_{t_0-(2r)^{\theta}}^{t_0}\bar{u}^{\chi}_{B_{2r}}(s)ds/(2r)^{\theta}$, we find that, for any $t_0-(2r)^{\theta}<t<t_0$

$$|\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{Q^{\theta}_{2r}}|^p\le\sup_{t_0-(2r)^{\theta}<s<t<t_0}|\bar{u}^{\chi}_{B_{2r}}(t)-\bar{u}^{\chi}_{B_{2r}}(s)|^p,\tag{2.14}$$

so that, substituting (2.13) into (2.14) gives that

$$\sup_{t_0-(2r)^\theta<s<t<t_0} |\bar{u}^\chi_{B_{2r}}(t) - \bar{u}^\chi_{B_{2r}}(s)|^p \leq \gamma |B_{2r}|^{-1} r^{(\theta-p)/(p-1)} \iint_{Q^\theta_{2r}} |Du|^p dtdx. \tag{2.15}$$

Combining (2.12) and (2.15) with (2.11), we obtain from the boundedness of u and a simple variation of Poincaré inequality

$$\sup_{t_0-(2r)^\theta<s<t<0} \int_{B_r\times\{t\}} |u(t,x) - \bar{u}^\chi_{B_{2r}}(t)|^p \leq \gamma(M) r^{\theta-p+(\theta-p)/(p-1)} \iint_{Q^\theta_{2r}} |Du|^p dtdx,$$

where we note $0 < \theta \leq p$ and $0 < r < 1$. Noting (2.9) in the proof of Lemma 2.5, the result immediately follows.

3. L^q– estimates.

Take a cylinder $Q_R \subset Q$, $0 < R \leq 1$, arbitrarily and fix it. Now we prove

Lemma 3.1. *(Reverse Hölder inequality) There exist positive constants γ and ε such that $|Du| \in L^{p+\varepsilon}_{\rm loc}(Q_{R/4})$. Moreover there exist exponents $0 < \tilde{p} < p$ and $1 < \bar{p}$ such that*

$$\left(\frac{1}{|Q_{R/4}|}\iint_{Q_{R/4}} |Du|^{p+\varepsilon} dtdx\right)^{1/(p+\varepsilon)} \leq \gamma \left\{ \left(\frac{1}{|Q_R|}\iint_{Q_R} |Du|^p dtdx\right)^{1/p} + \left(\iint_{Q_R} |Du|^{\tilde{p}} dtdx\right)^{\bar{p}} \right\}. \tag{3.1}$$

Proof. In the following θ is a positive constant satisfying $\theta \leq p$, which is chosen exactly later. Taking a exponent γ_1, α_2 as follows

$$\begin{aligned} &\gamma_1 = \frac{p}{m}\left(2 + \frac{1}{m+2}\right), \\ &\max\left\{\frac{2}{p+2}, \frac{2}{m+2}, \frac{2\gamma_1}{m+2}\Big/\left(\frac{2\gamma_1}{m+2} + \frac{m}{m+2}\right)\right\} < \alpha_2 < 1. \end{aligned} \tag{3.2}$$

Moreover we set

$$\alpha_1 = \frac{\gamma_1}{\alpha_2 + \gamma_1}, \quad \beta_1 = \frac{m}{m - (1-\alpha_2)(m+2)}, \quad \beta_2 = \frac{m}{(1-\alpha_2)(m+2)}. \tag{3.3}$$

Noting that

$$0 < \alpha_1,\ \ \alpha_2 < 1,$$

$$\beta_1,\ \ \beta_2 > 1 \ \text{and}\ 1/\beta_1 + 1/\beta_2 = 1$$

and using Hölder inequality, Lemma 2.6 and a simple variation of Sovolev inequality, we have, for any $Q^\theta_{4r} \subset Q_R$

$$\begin{aligned}
\iint_{Q^\theta_{2r}} |u - \bar{u}^\chi_{B_{2r}}(t)|^p dtdx \le & \sup_{t_0 - 2r^\theta < t < t_0} \left(\int_{B_{2r}(x_0)\times\{t\}} |u - \bar{u}^\chi_{B_{2r}}|^p \chi^p \tau^p dx \right)^{1-\alpha_1} \\
& \times \int_{t_0-(2r)^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}^\chi_{B_{2r}}|^p dx \right)^{\alpha_1} dt \\
\le & \left(r^{p(\theta-p)/(p-1)} \iint_{Q^\theta_{4r}} |Du|^p dtdx \right)^{1-\alpha_1} \int_{t_0-(2r)^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}^\chi_{B_{2r}}|^{\alpha_2 \beta_1 p} dx \right)^{\alpha_1/\beta_1} \\
& \times \left(\int_{B_{2r}} |u - \bar{u}^\chi_{B_r(x_0)}|^{p(1-\alpha_2)\beta_2} dx \right)^{\alpha_1/\beta_2} dt \\
\le & \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} \left(\iint_{Q^\theta_{4r}} |Du|^p dtdx \right)^{1-\alpha_1} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\alpha_1/\beta_2} dt \\
& \times \int_{t_0-(2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^{\alpha_2\beta_1 mp/(m+\alpha_2\beta_1 p)} dx \right)^{\alpha_1(m+\alpha_2\beta_1 p)/\beta_1 m} \\
\le & \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} |B_{2r}|^{\alpha_1(m+\alpha_2\beta_1 p)/\beta_1 m - \alpha_1\alpha_2} \\
& \times \left(\iint_{Q^\theta_{4r}} |Du|^p dtdx \right)^{1-\alpha_1} \int_{t_0-(2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^p dx \right)^{\alpha_1\alpha_2} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\frac{\alpha_1}{\beta_2}} dt \\
\le & \gamma r^{p(1-\alpha_2)\alpha_1} r^{\frac{p(\theta-p)(1-\alpha_1)}{p-1}} |B_{2r}|^{\frac{\alpha_1(m+\alpha_2\beta_1 p)}{\beta_1 m} - \alpha_1\alpha_2} \left(\iint_{Q^\theta_{4r}} |Du|^p dtdx \right)^{1-\alpha_1+\alpha_1\alpha_2}, \\
& \times \left[\int_{t_0-(2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\frac{\alpha_1}{\beta_2(1-\alpha_1\alpha_2)}} dt \right]^{1-\alpha_1\alpha_2} \\
\le & \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} |B_{2r}|^{\alpha_1(m+\alpha_2\beta_1 p)/\beta_1 m - \alpha_1\alpha_2} r^{\theta(1-\alpha_1\alpha_2-\alpha_1/\beta_2)(1-\alpha_1\alpha_2)} \\
& \times \left(\iint_{Q^\theta_{4r}} |Du|^p dtdx \right)^{1-\alpha_1+\alpha_1\alpha_2} \left(\iint_{Q^\theta_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dtdx \right)^{\alpha_1/\beta_2}
\end{aligned} \tag{3.4}$$

By applying Young's inequality for (3.4), the latter is

$$\le \delta r^p \iint_{Q^\theta_{4r}} |Du|^p dtdx + \gamma(\delta) r^p |Q^\theta_r| \left(\frac{1}{|Q^\theta_{2r}|} \iint_{Q^\theta_{2r}} |Du|^{mp/(m+2)} dtdx \right)^{(m+2)/m} \tag{3.5}$$

We estimate $\iint_{Q^\theta_{2r}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dtdx$ for any $Q^\theta_{4r} \subset Q_R$. By Hölder inequality and Lemma 2.5, we have

$$\begin{aligned}
&\iint_{Q^\theta_{2r}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dtdx \\
\le& \Big(\sup_{t_0-2r^\theta<t<t_0} \int_{B_{2r}\times\{t\}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dx\Big)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \Big(\int_{B_{2r}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dx\Big)^{\alpha_1} dt \\
\le& \gamma\Big(r^{-\theta} \iint_{Q^\theta_{4r}} |u-\bar{u}^\chi_{B_{4r}}(t)|^2 dtdx\Big)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \Big(\int_{B_{2r}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dx\Big)^{\alpha_1} dt \\
&+\gamma\Big(r^{-p} \iint_{Q^\theta_{4r}} |u-\bar{u}^\chi_{B_{4r}}(t)|^p dtdx\Big)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \Big(\int_{B_{2r}} |u-\bar{u}^\chi_{B_{2r}}(t)|^2 dx\Big)^{\alpha_1} dt \\
=& I_1 + I_2.
\end{aligned} \tag{3.6}$$

First we consider I_1. Set α_1, α_2, β_1 and β_2 as follows:

$$\begin{aligned}
&0<\alpha_1<\min\{1/2,\ 2/m\}, \quad 0<\alpha_2<1 \\
&\frac{p}{p-2+2\alpha_2} \le \beta_1 < \frac{m}{\alpha_2(m-2)} \quad \beta_2 = \frac{\beta_1}{\beta_1-1}.
\end{aligned} \tag{3.7}$$

We also set θ as

$$\theta = \Big(2-\frac{m\alpha_1}{\beta_2}\Big)/\Big(1+\frac{\alpha_1}{\beta_2}\Big). \tag{3.8}$$

Note that

$$2(1-\alpha_2)\beta_2 \le p, \quad \beta_1,\ \beta_2 > 1,$$

so that, calculating similarly as in (3.4) gives that

$$I_1 \le \gamma r^\theta |Q^\theta_r| \Big(\frac{1}{|Q^\theta_{4r}|} \iint_{Q^\theta_{4r}} |Du|^2 dtdx\Big)^{1-\alpha_1+\alpha_1\alpha_2} \Big(\iint_{Q^\theta_{2r}} |Du|^{2(1-\alpha_2)\beta_2} dtdx\Big)^{\alpha_1/\beta_2}$$

Noting that

$$\frac{p}{2(1-\alpha_1+\alpha_1\alpha_2)} > 1$$

and using Young's and Hölder inequalities, we obtain

$$\begin{aligned}
I_1 \le& \delta r^\theta |Q^\theta_r| \frac{1}{|Q^\theta_{4r}|} \iint_{Q^\theta_{4r}} |Du|^p dtdx \\
&+\gamma(\delta) r^\theta |Q^\theta_r| \Big(\iint_{Q^\theta_{2r}} |Du|^{2(1-\alpha_2)\beta_1} dtdx\Big)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))}.
\end{aligned} \tag{3.9}$$

Next, to estimate I_2 we put the exponents as follows:

θ and α_1 are the same as in (3.7) and (3.8),

$$1-\frac{p(2-\theta)}{2(m+\theta)}<\tilde{\alpha}_2<1, \tag{3.10}$$

$$\frac{p}{p-2+2\tilde{\alpha}_2}<\tilde{\beta}_1<\min\{\frac{m}{\tilde{\alpha}_2(m-2)},\ \frac{m+\theta}{2-\theta}/\Big(\frac{m+\theta}{2-\theta}-1\Big)\},\quad \tilde{\beta}_2=\frac{\tilde{\beta}_1}{\tilde{\beta}_1-1}.$$

Noting that

$$\tilde{\beta}_1,\ \tilde{\beta}_2>1,\quad 2(1-\tilde{\alpha}_2)\tilde{\beta}_2\leq p$$

and estimating similarly as (3.4), we have

$$\begin{aligned}I_2\leq{}&\gamma r^{(2-\theta)\alpha_1-(m+\theta)\alpha_1/\tilde{\beta}_1}r^\theta|Q_r^\theta|\\&\times\Big(\frac{1}{|Q_{4r}^\theta|}\iint_{Q_{4r}^\theta}|Du|^p dtdx\Big)^{1-\alpha_1}\Big(\frac{1}{|Q_{2r}^\theta|}\iint_{Q_{2r}^\theta}|Du|^2 dtdx\Big)^{\alpha_1\tilde{\alpha}_2}\\&\times\Big(\iint_{Q_{2r}^\theta}|Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2}dtdx\Big)^{\alpha_1/\tilde{\beta}_2}\end{aligned}$$

Note that

$$(2-\theta)\alpha_1-(m+\theta)\alpha_1/\tilde{\beta}_1\geq 0.$$

Since

$$\frac{1}{1-\alpha_1+2\alpha_1\tilde{\alpha}_2/p}>1,$$

from Young's and Hölder inequality it follows that

$$I_2\leq\delta r^\theta|Q_r^\theta|\frac{1}{|Q_{4r}^\theta|}\iint_{Q_{4r}^\theta}|Du|^p dtdx+\gamma(p,\delta)r^\theta|Q_r^\theta|\Big(\iint_{Q_{2r}^\theta}|Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2}dtdx\Big)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)} \tag{3.11}$$

Combining (3.9) and (3.11) with (3.6), we have

$$\begin{aligned}&\iint_{Q_{2r}^\theta}|u-\bar{u}_{B_{2r}}^\chi(t)|^2 dtdx\\&\leq\delta r^\theta|Q_r^\theta|\frac{1}{|Q_{4r}^\theta|}\iint_{Q_{4r}^\theta}|Du|^p dtdx+\gamma(p,\delta)r^\theta|Q_r^\theta|\Big(\iint_{Q_{2r}^\theta}|Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2}dtdx\Big)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)}\\&+\gamma(p,\delta)r^\theta|Q_r^\theta|\Big(\iint_{Q_{2r}^\theta}|Du|^{2(1-\alpha_2)\beta_2}dtdx\Big)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))}\end{aligned} \tag{3.12}$$

Thus, substituting (3.5) and (3.12) into (2.6) in Lemma 2.4 we obtain, for any $Q^{\theta}_{4r} \subset Q_R$

$$\begin{aligned}&\frac{1}{|Q^{\theta}_{r}|}\iint_{Q^{\theta}_{r}} |Du|^{p} dtdx \\ \leq &\delta\frac{1}{|Q^{\theta}_{4r}|}\iint_{Q^{\theta}_{4r}} |Du|^{p} dtdx + \gamma(p,\delta)\Big(\iint_{Q^{\theta}_{2r}} |Du|^{2(1-\alpha_2)\beta_2} dtdx\Big)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \\ &+\gamma(p,\delta)\Big(\frac{1}{|Q^{\theta}_{2r}|}\iint_{Q^{\theta}_{2r}} |Du|^{\frac{mp}{m+2}} dtdx\Big)^{\frac{m+2}{m}} + \gamma(p,\delta)\Big(\iint_{Q^{\theta}_{2r}} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dtdx\Big)^{\frac{p}{\tilde{\beta}_2(p-2\tilde{\alpha}_2)}}\end{aligned} \tag{3.13}$$

The desired estimate follows from Prop.2.3 with setting $g = |Du|^{mp/(m+2)}$, $q = (m+2)/m$ and

$$\begin{aligned} f =&\gamma\Big\{\Big(\iint_{Q_R} |Du|^{2(1-\alpha_2)\beta_2} dtdx\Big)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \\ &+\Big(\iint_{Q_R} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dtdx\Big)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)}\Big\}^{1/q}\end{aligned}$$

4.Proof of Theorem.

In the following we take $Q^2_{R_0}(\bar{t},\bar{x}) \subset Q$, $0 < R_0 \leq 1$, and fix it.

Lemma 4.1. *Suppose that there exists a sufficiently small $\delta > 0$ such that*

$$\overline{\lim_{r\downarrow 0}}\Big(\frac{1}{|B_r|}\iint_{Q^2_r(\bar{t},\bar{x})} |Du|^{p} dtdx\Big) < \delta \tag{4.1}$$

Then, taking $R_0 > 0$ sufficiently small, for $0 < \alpha < 1$, there exists a positive constant γ depending only on m, p, α, δ and $\iint_Q |Du|^p dtdx$ such that

$$\frac{1}{|Q^2_r|}\iint_{Q^2_r(t_0,x_0)} |Du|^{p} dtdx \leq \gamma r^{-\alpha p} \tag{4.2}$$

holds for any $(t_0, x_0) \in Q^2_{R_0/4}$ and all $0 < r < R_0/4$.

Proof.Let $Q^2_{4R}(t_0,x_0) \subset Q^2_{R_0}$ be fixed arbitrarily. Consider the Dirichlet problem:

$$\partial_t v^i - \mathrm{div}(|Dv|^{p-2}Dv^i) = 0 \quad \text{in } Q^{\theta}_R,\ i = 1,\cdots,n, \tag{4.3}$$

$$v = u \quad \text{on the parabolic boundary of } Q^{\theta}_R \tag{4.4}$$

where $\theta = 2 + \alpha(p-2)$.

Existence of weak solutions to (4.3) in the sense of (1.4) and to (4.4) in the sense of traces of $W_p^1(Q_R^\theta)$ functions can be established by a straightforward adoptation of Galerkin method as presented for example in [12].

Substracting (1.1) by (4.3) and testing the resulting inequality by $v - u$ on Q_R^θ(note Remark after Lemma 2.4), we have

$$\frac{1}{2}\int_{B_R\times\{t_0\}} |v-u|^2dx + \iint_{Q_R^\theta} |Dv - Du|^p dtdx \le a \iint_{Q_R^\theta} |Du|^p|v-u|dtdx. \tag{4.5}$$

Noticing the maximum estimate of the solution to (4.3) and (4.4) (see [13]), from (4.5) we deduce two inequalities for $0 < r < R$:

$$\iint_{Q_R^\theta} |Dv|^p dtdx \le \gamma \iint_{Q_R^\theta} |Du|^p dtdx, \tag{4.6}$$

$$\iint_{Q_R^\theta} |Du|^p dtdx \le 2^{p-1} \iint_{Q_R^\theta} |Dv|^p dtdx + 2^{p-1} \iint_{Q_R^\theta} |Dv - Du|^p dtdx. \tag{4.7}$$

From (2.2) in Prop.2.2 and (4.6) we obtain for $0 < r < R$

$$\iint_{Q_r^\theta} |Dv|^p dtdx \le \gamma \left(\frac{r}{R}\right)^{m+\theta-\alpha p} \left\{\iint_{Q_R^\theta} |Du|^p dtdx + 1\right\} \tag{4.8}$$

Combining (4.8) with (4.7) gives that

$$\iint_{Q_r^\theta} |Du|^p dtdx \le \gamma \left(\frac{r}{R}\right)^{m+\theta-\alpha p} \left(\iint_{Q_R^\theta} |Du|^p dtdx + 1\right) + \gamma \iint_{Q_r^\theta} |Du - Dv|^p dtdx. \tag{4.9}$$

Now we estimate $\iint_{Q_r^\theta} |Du - Dv|^p dtdx$. In the following ε is determined in Lemma 3.1. By Hölder inequality we have

$$\iint_{Q_R^\theta} |Du|^p|v-u|dtdx \le \left(\iint_{Q_R^\theta} |Du|^{p+\varepsilon} dtdx\right)^{p/(p+\varepsilon)} \left(\iint_{Q_R^\theta} |v-u|^{(p+\varepsilon)/\varepsilon} dtdx\right)^{\varepsilon/(p+\varepsilon)} \tag{4.10}$$

Noting the boundedness of v, we obtain from Poincarè inequality and (4.5)

$$\iint_{Q_R^\theta} |v-u|^{(p+\varepsilon)/\varepsilon} dtdx \le \gamma(M) \iint_{Q_R^\theta} |v-u|^p dtdx \le \gamma R^p \iint_{Q_R^\theta} |Du|^p dtdx. \tag{4.11}$$

To estimate $\frac{1}{|Q_R^\theta|}\iint_{Q_R^\theta}|Du|^{p+\varepsilon}dtdx$ we use a partition argument (refer to [13]). Set, for a subset $\tilde{Q}\subset Q$

$$\begin{aligned}
&f(\tilde{Q})\\
&=\gamma\Big\{\Big(\iint_{\tilde{Q}}|Du|^{2(1-\alpha_2)\beta_2}dtdx\Big)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))}\\
&+\Big(\iint_{\tilde{Q}}|Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2}dtdx\Big)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)}\Big\}^{m/(m+2)}
\end{aligned}$$

where the parameters are determined in Lemma 3.1. We assume that r^θ/r^p is an integer where note $\theta\le p$, and subdivide Q_r^θ into $s=r^{\theta-p}$ boxes with vertices (t_0,x_0), $\cdots$, (t_{s-1},x_0). Then, from (3.1) in Lemma 3.1 we obtain

$$\begin{aligned}
&\frac{1}{|Q_R^\theta|}\iint_{Q_R^\theta}|Du|^{p+\varepsilon}dtdx\le\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}\frac{1}{|Q_R^p|}\iint_{Q_R^p(t_i,x_0)}|Du|^{p+\varepsilon}dtdx\\
&\le\gamma\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}\Big\{\Big(\frac{1}{|Q_{4R}|}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\Big)^{\frac{p+\varepsilon}{p}}+(f(Q_{4r}(t_i,x_0)))^{p+\varepsilon}\Big\}\\
&\le\gamma\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}\Big(\frac{1}{|Q_{4R}|}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\Big)\Big(\frac{1}{|Q_{4R}|}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\Big)^{\frac{\varepsilon}{p}}\\
&+\gamma\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}(f(Q_{4r}(t_i,x_0)))^{p+\varepsilon}.
\end{aligned}\tag{4.12}$$

Taking $R_0>0$ sufficiently small we obtain from (4.1) and Lebegue absolute continuous theorem

$$\frac{1}{|B_{4R}|}\iint_{Q_{4R}^\theta(t_i,x_0)}|Du|^pdtdx<\delta\quad\text{for } i=0,1,\cdots,s-1.\tag{4.13}$$

Note that at most $([4^p]+1)$ cylinders $Q_{4R}(t_i,x_0)$ $(i=0,1,\cdots,s-1)$ are overlapped with each $Q_{4R}(t_i,x_0)$ $(i=0,1,\cdots,s-1)$, so that we have

$$\sum_{i=0}^{s-1}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\le([4^p]+1)\iint_{Q_{4R,R^\theta+(4^p-1)R^p}(t_0,x_0)}|Du|^pdtdx.\tag{4.14}$$

From (4.13) and (4.14) we obtain

$$\begin{aligned}
&\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}\Big(\frac{1}{|Q_{4R}|}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\Big)\Big(\frac{1}{|Q_{4R}|}\iint_{Q_{4R}(t_i,x_0)}|Du|^pdtdx\Big)^{\frac{\varepsilon}{p}}\\
&\le\frac{4^\theta([4^p]+1)}{4^p}(4R)^{-\varepsilon}\delta^{\varepsilon/p}\frac{1}{|Q_{4R}^\theta|}\iint_{Q_{4R,R^\theta+(4^p-1)R^p}(t_0,x_0)}|Du|^pdtdx.
\end{aligned}\tag{4.15}$$

We also find that

$$\frac{R^p}{R^\theta}\sum_{i=0}^{s-1}(f(Q_{4R}(t_i,x_0)))^{p+\varepsilon}\le\frac{R^p}{R^\theta}s(f(Q_{4R,R^\theta+(4^p-1)R^p}(t_0,x_0)))^{p+\varepsilon}$$
$$\le(f(Q_{4R,R^\theta+(4^p-1)R^p}(t_0,x_0)))^{p+\varepsilon}. \tag{4.16}$$

Here note that by taking $R_0>0$ sufficiently small, $R^\theta+(4^p-1)R^p\le(4R)^\theta$ holds for any $0<R<R_0$. Combining (4.15) and (4.16) with (4.12) we have

$$\frac{1}{|Q_R^\theta|}\iint_{Q_R^\theta}|Du|^{p+\varepsilon}dtdx$$
$$\le\gamma\frac{4^\theta([4^p]+1)}{4^p}\delta^{\varepsilon/p}R^{-\varepsilon}\frac{1}{|Q_{4R}^\theta|}\iint_{Q_{4R,R^\theta+(4^p-1)R^p}(t_0,x_0)}|Du|^pdtdx+\gamma(f(Q_{4R}^\theta(t_0,x_0)))^{p+\varepsilon} \tag{4.17}$$

Substituting (4.11) and (4.17) into (4.10) and noting that $0<R<1$ and $\theta\le p$, we have

$$\iint_{Q_R^\theta}|Du|^p|v-u|dtdx\le\gamma\delta^{\frac{\varepsilon}{p+\varepsilon}}\iint_{Q_{4R}^\theta}|Du|^pdtdx$$
$$+\gamma|Q_R^\theta|\Big(\frac{1}{|B_{4R}|}\iint_{Q_{4R}^\theta}|Du|^pdtdx\Big)^{\varepsilon/(p+\varepsilon)}\Big\{\Big(\iint_{Q_{4R}^\theta}|Du|^{2(1-\alpha_2)\beta_2}dtdx\Big)^{\frac{p\alpha_1}{\beta_1(p-2+2\alpha_1(1-\alpha_2))}}$$
$$+\Big(\iint_{Q_{4R}^\theta}|Du|^{2(1-\tilde\alpha_2)\tilde\beta_2}dtdx\Big)^{\frac{p}{\tilde\beta_2(p-2\tilde\alpha_2)}}\Big\}^{mp/(m+2)} \tag{4.18}$$

Combining (4.18) and (4.5) with (4.9) gives that

$$\iint_{Q_r^\theta}|Du|^pdtdx\le\gamma\Big\{\Big(\frac{r}{R}\Big)^{m+\theta-\alpha p}+\delta^{\frac{\varepsilon}{p+\varepsilon}}\Big\}\Big(\iint_{Q_{4R}^\theta}|Du|^pdtdx+1\Big)$$
$$+\gamma|Q_R^\theta|\Big(\frac{1}{|B_{4R}|}\iint_{Q_{4R}^\theta}|Du|^pdtdx\Big)^{\varepsilon/(p+\varepsilon)}\Big\{\Big(\iint_{Q_{4R}^\theta}|Du|^{2(1-\alpha_2)\beta_2}dtdx\Big)^{\frac{p\alpha_1}{\beta_1(p-2+2\alpha_1(1-\alpha_2))}}$$
$$+\Big(\iint_{Q_{4R}^\theta}|Du|^{2(1-\tilde\alpha_2)\tilde\beta_2}dtdx\Big)^{\frac{p}{\tilde\beta_2(p-2\tilde\alpha_2)}}\Big\}^{mp/(m+2)}. \tag{4.19}$$

Again noting (4.13) and iterating (4.19) similarly as Lemma 2.1 in [8],p86 (also see [9],p446) we have that for all $0<\alpha<1$, there exists a positive constant γ depending only on m,p, α and $\iint_Q|Du|^pdtdx$ such that

$$\iint_{Q_r^\theta(t_0,x_0)}(1+|Du|^p)dtdx\le\gamma r^{m+\theta-\alpha p} \tag{4.20}$$

holds for any $0 < r < R_0/4$ and $(t_0, x_0) \in Q^2_{R_0/4}$.

From a partition argument(see (4.12)) and (4.20), we obtain (4.1).

Proof of theorem. Let $(\bar{t}, \bar{x})$ satisfy (4.1). Exploiting Lemma 4.1 and estimating similarly as in the proof of Prop.3.3 in [13], pp118-120, we deduce that, for any $0 < \alpha < 1$ there exists a positive constant γ depending only on m, p, α and $\iint_Q |Du|^p dtdx$ such that

$$\frac{1}{|Q^2_r|} \iint_{Q^2_r(t_0,x_0)} |u - \bar{u}_{Q^2_r(t_0,x_0)}|^p dtdx \le \gamma r^{p(1-\alpha)} \tag{4.21}$$

holds for all $(t_0, x_0) \in Q^2_{R_0/4}(\bar{t}, \bar{x})$ and any $0 < r < R_0/4$.

From (4.21) and Prop.2.1 with setting $Q = Q^2_{R_0/4}(\bar{t}, \bar{x})$, $\theta = 2$ and $\mu = 2(1 - \alpha)$ we conclude that $u \in C^{0,\beta}(Q^2_{R_0/4})$ for any $0 < \beta < 1$.

To obtain the assertion of Theorem, we have only to recall Prop.3.2 in [9],p447(also see [8]) and to note the L^q−estimate for $|Du|$ (Lemma 3.1).

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