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TRANSFER IMAGE FOR STUNTED PROJECTIVE SPACES

和歌山大学教育学部 今岡光範 MITSUNORI IMAOKA

Let $N_i \xrightarrow{r} M_i = L_i N_i \xrightarrow{s} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i$ be the cofiber sequence such that $S^0 = N_0 \xrightarrow{r} M_0 \xrightarrow{rs} M_1 \to \ldots$ is the geometric realization by Ravenel [Ra] of the chromatic resolution, where L_i is the Bousfield localization [Bo] with respect to the $v_i^{-1}BP_*$-homology. Throughout this note, we assume that spectra are always localized at an odd prime p. The authors of [BC] revealed that the double S^1-transfer

$$t_2 : \Sigma^2 CP_0^\infty \wedge CP_0^\infty \to S^0$$

factors through $\delta_1\delta_2 : \Sigma^{-2}N_2 \to S^0$. The purpose of the present note is to remark that their result is also valid for the double S^1-transfers

$$t_2 : \Sigma^{-2(m+n-1)} CP_m^\infty \wedge CP_n^\infty \to S^0 \text{ for all integers } m \text{ and } n.$$

Here CP_k^∞ denotes the suspension spectrum of the Thom space for the k-Whitney sum $k\xi$ of the canonical complex line bundle ξ over CP^∞.

Theorem. *For the second stage N_2 of the chromatic filtration at an odd prime p, there is a map $u : \Sigma^{-2(m+n-1)} CP_m^\infty \wedge CP_n^\infty \to \Sigma^{-2}N_2$ satisfying $\delta_1\delta_2 \circ u = t_2$.*

We will prove the theorem using an analogous method as in [BC], that is, we will see that the strategy in [BC;Th.5.2] is applicable to our situation. The crucial point is to prove the following lemma, in which a map $\bar{U}_1$ is defined later in (2).

Lemma 1. *There is a map U_2 satisfying the following homotopy commutative diagram:*

$$\begin{array}{ccc} \Sigma^{2m} CP^{\infty}_{n+1} & \xrightarrow{i\wedge 1} & CP^{\infty}_{m} \wedge CP^{\infty}_{n+1} \\ \bar{U}_1 \downarrow & & U_2 \downarrow \\ \Sigma^{2(m+n)} N_1 & \xrightarrow{r} & \Sigma^{2(m+n)} M_1 \end{array}$$

where i is the bottom inclusion.

t_2 is the composition $t(t \wedge 1)$ of the transfer maps $t : \Sigma^{-2k+1} CP^{\infty}_{k} \to S^0$ for $k = m$ and n, and t is homotopic to the attaching map to the bottom cell of $\Sigma^{-2k+2} CP^{\infty}_{k-1}$ ([Kn]). Thus the lemma induces a required map $u : CP^{\infty}_{m+1} \wedge CP^{\infty}_{n+1} \to \Sigma^{2(m+n)} N_2$ in our theorem, and the rest of this note is devoted to the proof of this lemma.

Let $U_1 \in \pi^{2k}(CP^{\infty}_{k}; Q)$ be the Thom class of $k\xi$ for the rational theory $SQ = HQ$, $q : SQ \to SQ/Z_{(p)}$ the mod $Z_{(p)}$ quotient and $c : CP^{\infty}_{k} \to CP^{\infty}_{k+1}$ the collapsing map. Then, we have an element $\bar{U}_1 \in \pi^{2k}(CP^{\infty}_{k+1}; Q/Z_{(p)})$ satisfying $q_*(U_1) = c^*(\bar{U}_1)$. Since $N_1 = SQ/Z_{(p)}$, $\bar{U}_1$ represents a map

$$\bar{U}_1 : CP^{\infty}_{n+1} \to \Sigma^{2n} N_1 \tag{2}$$

which is the map in Lemma 1.

Lemma 1 will be established by using Lemma 8 below, and before it we need to prepare some generalization of a result due to Miller [Mi]. Let E be a ring spectrum such that E_*E is flat over $E_* = \pi_*E$ and $H^0(E; Q) \cong Q$. Furthermore, we assume that E is oriented by $x \in E^2(CP^{\infty})$. Then, $E^*(CP^{\infty}_{k}) \cong E^*[[x]]\{U\}$ for a Thom class U of $k\xi$, and $E_*(CP^{\infty}_{k}) \cong E_*\{\beta_k, \beta_{k+1}, \dots\}$, where β_i is the dual element of Ux^{i-k}. We put $\hat{\beta}_k(T) = \sum_{i \geq k} \beta_i T^{i-k} \in E_*(CP^{\infty}_{k})[[T]]$.

Let $\log^E T$ be the power series which gives a strict isomorphism from the formal group law defined by the orientation class x of E to the additive formal group law, over $E_* \otimes Q$. Then, by the method designed in [Mi], we have the following:

Lemma 3. $(\bar{U}_1)_*(T\hat{\beta}_{k+1}(T)) = \left(\dfrac{\log^E T}{T}\right)^k - 1$ *in* $\pi_*(E;Q/Z)[[T]]$.

Now, we consider the spectrum $E(1)$ which represents a wedge summand of the complex K-theory $K_{(p)}$ localized at p and whose coefficient group is $E(1)_* = Z_{(p)}[v_1, v_1^{-1}]$ for $v_1 \in E(1)_{2(p-1)}$. Then Lemma 3 holds for the case of $E = E(1)$, and in this case the formula of $\log^{E(1)} T$ is given by

Theorem 4 (S.Araki [Ar]).

$$\log^{E(1)} T = \sum_{i \geq 0} \frac{v_1^{\frac{p^i - 1}{p-1}}}{p^i} T^{p^i}$$

Let $\exp^{E(1)} T$ be the formal power inverse of $\log^{E(1)} T$, and put

$$\left(\frac{T}{\exp^{E(1)} T}\right)^k = \sum_{i \geq 0} B(k,i) T^i \in (E(1)_* \otimes Q)[[T]] \tag{5}$$

for $B(k,i) \in E(1)_{2i} \otimes Q$. Clearly $B(m,0) = 1$, and $B(1,i)$ is the $E(1)$-theory Bernoulli number in the sense of [Mi].

Let $\psi^\gamma : E(1) \to E(1)$ be the stable Adams operation for a positive integer γ which generates the unit group of Z/p^2. Then ψ^γ is a ring homomorphism on $E(1)^*(\ )$, and it holds that $\psi^\gamma(v_1) = \gamma^{p-1} v_1$ and $\psi^\gamma(x) = (1/\gamma)[\gamma](x)$, where $x \in E(1)^2(CP^\infty)$ is the orientation class and $[\gamma](x)$ means the formal group sum of γ numbers of x. The following is easy to see by these properties, (5) and Theorem 4.

Lemma 6. *Let* $U_k^{E(1)} \in E(1)^2(CP_k^\infty)$ *be a Thom class of* $k\xi$. *Then we have the following :*

(1) $\psi^\gamma(U_k^{E(1)}) = U_k^{E(1)}(1 + \sum_{i>0}(\gamma^i - 1)B(-k,i)x^{-k}(\log^{E(1)} x)^{k+i})$;
(2) $\psi^\gamma(\log^{E(1)} x) = \log^{E(1)} x$.

Let Ad be the fiber spectrum of $\psi^\gamma - 1 : E(1) \to E(1)$, and $j : Ad \to E(1)$ the inclusion. A unit $\iota \in E(1)_0$ induces a map $\iota : N_1 = SQ/Z_{(p)} \to E(1) \wedge SQ/Z_{(p)}$, and we have $\iota = j_*(\iota')$ for a unique $\iota' \in \pi_0(Ad; Q/Z)$, since $(\psi^\gamma - 1)_*(\iota) = 0$ and j_* is monomorphic. By [Ra], there is an equivalence

$$(7) \quad \zeta : M_1 \simeq AdQ/Z_{(p)} \quad \text{with} \quad \iota' \simeq \zeta \circ r : SQ/Z_{(p)} = N_1 \to Ad \wedge SQ/Z_{(p)}.$$

Lemma 8. *There is an element* $u_2 \in E(1)^{2(m+n)}(CP_m^\infty \wedge CP_{n+1}^\infty) \otimes Q$ *satisfying*

(1) $(i \wedge 1)^* q_*(u_2) = \iota_*(\bar{U}_1)$ *in* $E(1)^{2(m+n)}(\Sigma^{2m} CP_{n+1}^\infty; Q/Z_{(p)})$ *and*

(2) $(\psi^\gamma - 1)_*(u_2) \in E(1)^{2(m+n)}(CP_m^\infty \wedge CP_{n+1}^\infty)$,

where $i : S^{2m} \to CP_m^\infty$ *and* $q : E(1) \wedge SQ \to E(1) \wedge SQ/Z_{(p)}$ *are the bottom inclusion and the mod* $Z_{(p)}$ *quotient respectively.*

By (2) in this lemma, we have $q_*(u_2) = j_*(u_2')$ for some $u_2' \in Ad^{2(m+n)}(CP_m^\infty \wedge CP_{n+1}^\infty; Q/Z)$. Since $j_* : Ad^{2(m+n)}(CP_m^\infty \wedge CP_{n+1}^\infty; Q/Z) \to E(1)^{2(m+n)}(CP_m^\infty \wedge CP_{n+1}^\infty; Q/Z)$ is monomorphic, we have $(u_2') \circ (i \wedge 1) = (\iota' \circ \bar{U}_1)$ in $Ad^{2(m+n)}(\Sigma^{2m} CP_{n+1}; Q/Z)$ by (1) in Lemma 8. Thus, by (7), we can take U_2 in Lemma 1 to be $(\zeta)^{-1} \circ u_2'$, and thus Lemma 8 yields Lemma 1.

We put

$$g_n(T) = T^{-1}\left(\left(\frac{\log^{E(1)} T}{T}\right)^n - 1\right) \in (E(1)_* \otimes Q)[[T]],$$

and consider the following element of $(E(1)_* \otimes Q)[[S,T]]$ by using $B(k,i)$ in (5):

$$h_{m,n}(S,T) = \sum_{k,l>0} a_{k,l} B(-m,k) B(-n,l) S^{-m} (\log^{E(1)} S)^{m+k} T^{-n-1} (\log^{E(1)} T)^{n+l},$$

where $a_{k,l} = (\gamma^k - 1)/(\gamma^{k+l} - 1)$. Then $g_n(T) = (\bar{U}_1)_*(\hat{\beta}_{n+1}(T))$ in $\pi_*(E(1); Q/Z)[[T]]$ by Lemma 3. By putting $E(1)^*(CP_0^\infty \wedge CP_0^\infty) = E(1)^*[[x,y]]$, we regard $E(1)^*(CP_m^\infty \wedge CP_{n+1}^\infty)$ as a free $E(1)^*[[x,y]]$-module with $U_m^{E(1)} U_{n+1}^{E(1)}$ as a base. Then Lemma 8 follows from the next Lemma.

Lemma 9. *The element* $u_2 = U_m^{E(1)} U_{n+1}^{E(1)}(g_n(y) + h_{m,n}(x,y))$ *of* $E(1)^{2(m+n)}$ $(CP_m^\infty \wedge CP_{n+1}^\infty) \otimes Q$ *satisfies (1) and (2) of Lemma 8.*

The proof of Lemma 9 is straightforward using Lemmas 3 and 6, and we can complete the proof. ·

References

[Ar] S. Araki, *Typical Formal Groups in Complex Cobordism and K–theory*, Lecture Notes Math., Kyoto Univ. 6, Kinokuniya Book Store, 1973.

[BC] A. Baker, D. Carlisle, B. Gray, S. Hilditch, N. Ray and R. Wood, *On the iterated complex transfer*, Math.Z.199 (1988), 191–207.

[Bo] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[Kn] K. Knapp, *Some applications of K-theory to framed bordism : e-invariant and transfer*, Habilitationsshrift (1979).

[Mi] H. R. Miller, *Universal Bernoulli numbers and the* S^1*–transfer*, Can.Math.Soc.Conf.Proc. **2** (1982), 437–439.

[Ra] D. C. Ravenel, *A geometric realization of the chromatic resolution*, Proc.J.C.Moore Conf., Princeton (1983), 168–179.

[R2] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.