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On real James numbers

大阪市大理　大嶋秀明　(Hideaki Ōshima)

1. Introduction

The purpose of this note is to determine the real James numbers. Throughout the note n, l, k denote integers with $n \geq l \geq k \geq 1$ and $n \geq 2$. Let P_k denote the real projective space of dimension $k-1$, $P_{l,k} = P_l/P_{l-k}$ the stunted projective space, and $V_{l,k} = O(l)/O(l-k)$ the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^l$. Note that $P_{k,k}$ is the union of P_k and a disjoint base point. Write $q : V_{l,k} \to V_{l,1} = S^{l-1}$ for the projection on to the last component, and $q : P_{l,k} \to P_{l,1} = S^{l-1}$ for the quotient map. There is a commutative square [6]:

$$\begin{array}{ccc} V_{n,k} & \xrightarrow{\ q\ } & V_{n,1} \\ \cup\uparrow & & \| \\ P_{n,k} & \xrightarrow[\ q\ ]{} & P_{n,1} \end{array}$$

The unstable real James numbers $V\{n,k\}$ and $P\{n,k\}$ are non-negative integers which generate respectively the images of

$$q_* : \pi_{n-1}(V_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z},$$
$$q_* : \pi_{n-1}(P_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z}.$$

In the same way, replacing homotopy group $\pi_{n-1}(-)$ by stable homotopy group ${}^s\pi_{n-1}(-)$ we have the stable real James numbers $V^s\{n,k\}$ and $P^s\{n,k\}$. Let us denote the exponent of 2 in a positive integer k by $\nu_2(k)$; define $\varphi(k)$ to be the number of integers s such that $0 < s < k$ and $s \equiv 0,1,2,4 \pmod 8$. Our results are

THEOREM (1.1). *We have* $P^s\{n,k\} = V^s\{n,k\} = V\{n,k\}$ *which is equal to* 0, 1, *or* 2 *according as* $n \equiv 1 \pmod 2$ *and* $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, *or* $1 \leq \nu_2(n) < \varphi(k)$.

THEOREM (1.2). *We have* $V\{n,k\} = P\{n,k\}$ *except for the following cases:* (1) *if* $(n,k) = (4,3), (8,5), (8,6), (8,7), (16,9)$, *then* $V\{n,k\} = 1$ *and* $P\{n,k\} = 2$; (2) *if* $n = k = 2m$ *with* $m = 1,2,4$, *then* $V\{n,k\} = 1$ *and* $P\{n,k\} = 0$; (3) *if* $n = k = 2m$ *with* $m \neq 1,2,4$, *then* $V\{n,k\} = 2$ *and* $P\{n,k\} = 0$.

Let $p_n : S^{n-1} \to P_n$ be the canonical double covering map and $p_{n,k} : S^{n-1} \to P_{n,k}$ $(n > k)$ the composition of p_n with the quotient map.

COROLLARY (1.3). *The rank of* $\pi_{n-1}(P_{n,k})$, *for* $n > k$, *is* 0, 2, *or* 1 *according as* $n \equiv 1 \pmod 2$ *and* $2 \leq k \leq n-2$, $n = 2k$ *and* $k \equiv 0 \pmod 2$, *or otherwise. The map* $p_{n,k}$ *generates a free direct summand of* $\pi_{n-1}(P_{n,k})$ *if and only if* $n = k+1 \geq 3$, $P\{n,k\} = 2$ *or* $(n,k) = (4,2), (8,4), (16,8)$.

Note that a part of (1.1) is not new. Indeed $V\{n,k\}$ was already known [**1**, **4**, **5**]. We shall calculate it again by using codegree [**3**, **8**, **9**]. We shall prove (1.1) in §2, and (1.2), (1.3) in §3.

2. $V\{n,k\}$

The symbol $a \mid b$ means that $b = ma$ for some integer m.

LEMMA (2.1). (1) $V\{n,n\} = V\{n,n-1\}$; $P^s\{n,1\} = V^s\{n,1\} = V\{n,1\} = P\{n,1\} = 1$; $P\{n,n\} = 0$.

(2) $V^s\{n,k\} \mid V\{n,k\} \mid P\{n,k\}$; $V\{n,k\} \mid V\{n,l\}$ *and* $P\{n,k\} \mid P\{n,l\}$ *if* $n \geq l \geq k \geq 1$.

(3) $V\{2,2\} = V\{4,4\} = V\{8,8\} = V\{16,9\} = 1$.

(4) ([7; 4.2]) $P^s\{n,k\} = V^s\{n,k\}$.

(5) *If* $n \geq 2k$, *then* $P^s\{n,k\} = V^s\{n,k\} = V\{n,k\} = P\{n,k\}$.

(6) ([10; 23.4, 25.6], [5; 2.3]) *If* n *is even or* $k = 1$, *then* $V\{n,k\} = 1$ *or* 2. *If* n *is odd and* $k \geq 2$, *then* $V\{n,k\} = 0$.

PROOF. By definition, (1) and (2) are obvious. As is well-known, if $n = 2,4,8$, then $V\{n,n\} = 1$ (cf., [11; p. 200]). By [6; p. 4], we have $V\{16,9\} = 1$. This proves (3). Since $P_{n,k}$ is $(n-k-1)$-connected, it follows from suspension theorem that $P^s\{n,k\} = P\{n,k\}$ if $n \geq 2k$. Hence (5) follows from (2) and (4).

PROPOSITION (2.2). *The number* $V^s\{n,k\}$ *is* 0, 1, *or* 2 *according as* $n \equiv 1 \pmod 2$ *and* $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, *or* $1 \leq \nu_2(n) < \varphi(k)$.

PROOF. Let $L_k \to P_k$ be the canonical line bundle. Then L_k is of order $2^{\varphi(k)}$ in the J-group of P_k [2]. If a positive integer m satisfies $m + n \equiv 0 \pmod{2^{\varphi(k)}}$, then $P^s\{n,k\} = {}^s\mathrm{cdg}(P_k^{mL}, m)$ by stable duality [6; (7.9)], where ${}^s\mathrm{cdg}(-)$ is the stable codegree [3, 8, 9] which was denoted by $\mathrm{cd}(mL_k)$ in [8], and P_k^{mL} is the Thom space of mL_k. Then the assertion follows from (2.1)(4) and [8; 3.5] (cf., [3]).

PROPOSITION (2.3). $V^s\{n,k\} = V\{n,k\}$.

To prove (2.3), we need

LEMMA (2.4). (1) *If* $k \geq 10$, *then* $2^{\varphi(k)} > 2k$. *If* $1 \leq k \leq 9$, *then* $2^{\varphi(k)} < 2k$.

(2) *Conditions* $2k > n \geq k \geq 2$ *and* $n \equiv 0 \pmod{2^{\varphi(k)}}$ *are satisfied if and only if* (n,k) *is* (2,2), (4,3), (4,4), (8,5), (8,6), (8,7), (8,8) *or* (16,9).

(3) $V\{n,k\} = 1$ *for every* (n,k) *in* (2).

PROOF. Write $k - 1 = 8x + y$ with $0 \leq y \leq 7$. Then $\varphi(k) = 4x + z$ such that z is 0 (if $y = 0$), 1 (if $y = 1$), 2 (if $y = 2,3$), and 3 (if $4 \leq y \leq 7$).

If $x \geq 2$, that is, if $k \geq 17$, then $2^{\varphi(k)} \geq 2^{4x} > 16(x+1) \geq 2k$. Hence the following table completes the proof of (1).

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$2k$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$2^{\varphi(k)}$	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128

If $2k > n \geq k \geq 2$ and $n \equiv 0 \pmod{2^{\varphi(k)}}$, then $k \leq 9$ by (1), hence (2) follows from the table. We have (3) by (2.1)(2)(3).

Proof of Proposition (2.3). By (2.1)(5)(6) and (2.2), it suffices to consider the case: $2k > n \equiv 0 \pmod 2$. If $1 \leq \nu_2(n) < \varphi(k)$ and $n < 2k$, then $V^s\{n,k\} = V\{n,k\} = 2$ by (2.1)(2)(6) and (2.2). If $2k > n \equiv 0 \pmod{2^{\varphi(k)}}$, then $V^s\{n,k\} = V\{n,k\} = 1$ by (2.2) and (2.4)(3).

Proof of Theorem (1.1). This follows from (2.1)(4), (2.2) and (2.3).

Let us write $n = (2a+1)2^{b+4c}$, where a, b, c are integers and $0 \leq b \leq 3$; let us define $\rho(n) = 2^b + 8c$. As is easily shown, $\nu_2(n) \geq \varphi(k)$ if and only if $\rho(n) \geq k$. Hence we have the following by Theorem (1.1).

THEOREM (2.5) (Eckmann, Adams). *The fibration $q : V_{n,k} \to V_{n,1}$ has a cross section if and only if $\rho(n) \geq k$.*

3. $P\{n,k\}$

Let $\iota_k \in \pi_k(S^k)$ be the class of the identity map of S^k. Then the following is well-known.

LEMMA (3.1). *The homotopy class of $p_{n,1}$ is $2\iota_{n-1}$ or 0 according as n is even or odd; $\pi_{n-1}(P_n) = \mathbb{Z}\{\varepsilon{\cdot}p_n\}$, where ε is 1 or $1/2$ according as $n \geq 3$ or $n = 2$.*

LEMMA (3.2). *If n is even, then $P\{n,n-1\} = 2$ for $n \geq 4$ and $P\{n,k\} = 1$ or 2 for $n > k$. If n is odd and $k \geq 2$, then $P\{n,k\} = 0$.*

PROOF. If n is even, then $P\{n, n-1\}$ is 1 or 2 according as $n = 2$ or $n \geq 4$ by (2.1)(1) and (3.1), hence $P\{n, k\} \mid 2$ provided $n > k$ by (2.1)(2). The second assertion follows from (2.1)(2)(6).

LEMMA (3.3). *If* $n = 2, 4, 8$, *then* $\pi_{2n-1}(P_{2n,n+1}) = \mathbb{Z}\{p_{2n,n+1}\} \oplus \mathrm{Tor}$.

PROOF. Let $n = 2$, 4, 8. The assertion is obvious by (3.1) when $n = 2$. Let $\omega_n : S^{2n-1} \to S^n$ be the Hopf map. We denote by Tor the torsion subgroup of any group. Then $\pi_{2n-1}(S^n) = \mathbb{Z}\{\omega_n\} \oplus \mathrm{Tor}$. Let $\mathcal{TOR}$ be the class of torsion groups. By mod $\mathcal{TOR}$ Hurewicz theorem, $\pi_*(P_{2n-1,n})$ is a torsion group for $n = 4$, 8. It then follows from the homotopy exact sequence of the pair $(P_{2n,n+1}, P_{2n-1,n})$ that the rank of $\pi_{2n-1}(P_{2n,n+1})$ is 1 and $p_{2n,n+1}$ is of infinite order for $n = 4$, 8. To complete the proof, it suffices to prove

(3.4) $$\pi_{2n-1}(P_{2n,n}) = \mathbb{Z}\{p_{2n,n}\} \oplus \mathbb{Z} \oplus \mathrm{Tor} \quad \text{for } n = 2,\ 4,\ 8.$$

We shall prove (3.4). Since the manifold P_n is parallelizable and the Whitney sum of the tangent bundle of P_n with a trivial line bundle is nL_n, we have $P_{2n,n} = P_n^{nL} = S^n \wedge P_{n,n} = S^n \vee (S^n \wedge P_n) = S^n \vee (S^n \wedge P_{n-1}) \vee S^{2n-1}$ up to homotopy. Hence $\pi_{2n-1}(P_{2n,n}) \cong \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n \wedge P_{n-1}) \oplus \pi_{2n-1}(S^{2n-1})$ by [11; (1.5) in p.492, (7.12) in p.368], where the isomorphism is induced by inclusion maps, and the rank of $\pi_{2n-1}(P_{2n,n})$ is 2, since $\pi_*(S^n \wedge P_{n-1})$ is a torsion group by mod $\mathcal{TOR}$ Hurewicz theorem. We can write $p_{2n,n} \equiv i_{1*}(a_n\omega_n) + i_{3*}(2\iota_{2n-1})$ (mod Tor) by (3.1), where $a_n \in \mathbb{Z}$ and i_k is a respective inclusion map. As is well-known, $\omega_n = f \circ p_{2n,n}$ for some map $f : P_{2n,n} \to S^n$. Write $f|_{S^n} = x\iota_n$ and $f|_{S^{2n-1}} \equiv z\omega_n$ (mod Tor) with $x, z \in \mathbb{Z}$. Then $\omega_n = f \circ p_{2n,n} \equiv (a_n x^2 + 2z)\omega_n$ (mod Tor), hence a_n is odd, therefore (3.4) follows. This completes the proof of (3.3).

Proof of Theorem (1.2). As is easily shown, $\nu_2(n) \geq \varphi(n)$ if and only if $n = 2, 4, 8$. Then the assertion for $n = k$ follows from (1.1) and (2.1)(1).

If $V\{n,k\}$ is 0 or 2, then $P\{n,k\} = V\{n,k\}$ by (1.1), (2.1)(2) and (3.2). Suppose that $V\{n,k\} = 1$ and $n > k$. Then $n \equiv 0 \pmod{2^{\varphi(k)}}$ by (1.1). If $k \geq 10$ or $k \leq 9$ and $n \geq 2k$, then $V\{n,k\} = P\{n,k\}$ by (2.1)(5) and (2.4)(1). If $k \leq 9$ and $n < 2k$, then (n,k) is (4,3), (8,5), (8,6), (8,7), or (16,9) by (2.4)(2), and $P\{n,k\} = 2$ except for $(n,k) = (8,6)$ by (3.1), (3.2) and (3.3). We then have $P\{8,6\} = 2$ by (2.1)(2).

Proof of Corollary (1.3). The assertions are obvious when $k = 1$ or $k = n-1$, by (3.1). Suppose $2 \leq k \leq n-2$. If n is odd, then $\pi_*(P_{n,k}) \in \mathcal{TOR}$ for k even by mod $\mathcal{TOR}$ Hurewicz theorem, and $i_* : \pi_*(S^{n-k}) \to \pi_*(P_{n,k})$ is a $\mathcal{TOR}$-isomorphism for k odd by mod $\mathcal{TOR}$ Whitehead theorem. Let $j : P_{n-1,k-1} \to P_{n,k}$ be the inclusion map and $f : S^{n-1} \to P_{n,k}$ a map with $q_*(f) = P\{n,k\}\iota_{n-1}$. If n is even, then, by mod $\mathcal{TOR}$ Whitehead theorem, $q_* : \pi_*(P_{n,k}) \to \pi_*(S^{n-1})$ and $(f \vee j)_* : \pi_*(S^{n-1} \vee P_{n-1,k-1}) \to \pi_*(P_{n,k})$ are $\mathcal{TOR}$-isomorphisms when k is odd and even respectively. Then the assertions can be proved easily by using (1.1), (1.2) and (3.4).

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