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CONTINUOUS FUNCTIONALS ON FUNCTION SPACES

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In this note, we assume that all spaces are Tychonoff. Let $C(X)$ be the set of all real-valued continuous functions on X. We call a real-valued function on $C(X)$ a *functional.* $C_p(X)$, $C_k(X)$ and $C_n(X)$ denote function spaces over X with the pointwise convergent topology, the compact-open topology and the sup-norm topology respectively. For a family $\mathcal{A}$ of sets, we write $\bigcap\mathcal{A} = \cap\{A : A \in \mathcal{A}\}$. For a function f on X and a subset M of X, the restriction of f to M is denoted by $f|_M$. The symbol $\pi_M : C_k(X) \to C_k(M)$ denotes the restriction map from X to a subspace M. $\mathbf{R}$, ω and ω_1 denote the real line, the first infinite ordinal and the first uncountable ordinal respectively.

First, we consider linear continuous functionals on $C_p(X)$. For any point x in X, we can suppose that x is a functional, which carries f into $f(x)$ for any f in $C(X)$, on $C(X)$. Obviously x is a linear continuous functional on $C_p(X)$. The following fact is well-known.

Fact 1. *Let λ be a non-constant linear continuous functional on $C_p(X)$. There exist a finite subset $\{x_1, \ldots, x_n\}$ and non-zero numbers $\{\alpha_1, \ldots, \alpha_n\}$ such that $\lambda = \sum_{i=1}^{n} \alpha_i x_i$.*

By Fact 1, we have;

- (1) *For any pair (f, g) of functions in $C_p(X)$, if $f|_{\{x_1,\ldots,x_n\}} = g|_{\{x_1,\ldots,x_n\}}$ holds, then $\lambda(f) = \lambda(g)$ holds,*

• (2) *There exists a real-valued continuous function* $\tilde{\lambda}$ *on* $\mathbf{R}^{\{x_1,\dots,x_n\}}$ *such that* $\lambda = \tilde{\lambda} \circ \pi_{\{x_1,\dots,x_n\}}$.

In (2), the continuity of $\tilde{\lambda}$ is deduced by the following fact.

Fact 2. *Let* F *be a closed subset of* X *and* π_F *the restriction map from* $C_p(X)$ *into* $C_p(F)$. *Then* π_F *is an open map onto* $\pi_F(C_p(X))$.

Below, we shall deal with non-linear functionals in general. In view of (1), (2) and Fact 2, we define a notion.

Definition. Let ξ be a functional on $C(X)$. A subset S of X is said to be a *support* for ξ if S is closed in X and $\xi(f) = \xi(g)$ holds for any pair (f, g) of functions in $C(X)$ such that $f|_S = g|_S$. $Supp\,\xi$ denotes the set of all supports for a functional ξ on $C(X)$.

By Fact 2, if ξ is a continuous functional on $C_p(X)$ and S is a support for ξ, then there exists a real-valued continuous function $\tilde{\xi}$ on $\pi_S(C_p(X))$ such that $\xi = \tilde{\xi} \circ \pi_S$.

Moreover, we have a condition on the set $\{x_1, \dots, x_n\}$ in Fact 1.

• (3) *If* S *is a support for* λ, *then* $\{x_1, \dots, x_n\} \subset S$ *holds.*

(3) says that the set $\{x_1, \dots, x_n\}$ is minimal in supports for λ in Fact 1. In general, we define a concept;

Definition. Let ξ be a functional on $C(X)$ and S a support for ξ. S is said to be *minimal* if every support for ξ contains S.

By (1) and (3), we have that every linear continuous functional on $C_p(X)$ has the finite minimal support. Generally, we have;

Theorem 3. ([1]) *The minimal support S for any continuous functional on $C_p(X)$ exists and S is a separable subspace of X.*

In the proof of Theorem 3, we show that, for any continuous functional ξ on $C_p(X)$, $\bigcap Supp\,\xi$ is a support for ξ.

By Theorem 3, we have an operation from the set of all continuous functionals on $C_p(X)$ to the set of all closed separable subspaces of X. The following is remarkable.

Remark 4. *For any countable subset A of X, there exists a continuous functional ξ_A on $C_p(X)$ such that $\bigcap\ Supp\,\xi_A = \overline{A}$.*

Using the same idea in the proof of Theorem 3, we can prove the following theorem.

Theorem 5. *Let $\mathcal{F}$ be a non-empty proper closed subset of $C_p(X)$. We put*

$$Supp\,\mathcal{F} = \{S \subset X :\ S \text{ is closed in } X,\ \pi_S^{-1}(\pi_S(\mathcal{F})) = \mathcal{F}\}.$$

Then the set $\bigcap Supp\,\mathcal{F}$ belongs to $Supp\,\mathcal{F}$.

This theorem gives a result on the minimal support.

Theorem 6. ([1]) *Let ξ be a non-constant continuous functional on $C_p(X)$. For an $r \in \xi(C_p(X))$, we put $S_r = \bigcap Supp\,\xi^{-1}(r)$. Then we have*

$$\bigcap Supp\,\xi = \overline{\bigcup\{S_r : r \in \xi(C_p(X))\}}.$$

For function spaces with the compact-open topology, we have a similar result.

Theorem 7. ([2]) *The minimal support for any continuous functional on $C_k(X)$ exists.*

Making a comparison between Theorem 3 and Theorem 7, we have the following question naturally.

Question. Let S be the minimal support in Theorem 7. Does S have a dense σ-compact subset ?

Below, we consider this question. For the proofs of the following results, see [2]. Let τ be a cardinal. A space X is said to be *almost τ-compact* if for any $\alpha < \tau$, there exists a non-empty compact subset K_α of X such that $X = \overline{\bigcup\{K_\alpha : \alpha < \tau\}}$. Almost ω-compact spaces are said to be *almost σ-compact.* The smallest cardinal τ such that X is almost τ-compact, is denoted by $cd(X)$.

Definition. A space X has the *property (σ)* if, for any continuous functional ξ on $C_k(X)$, the closed subset $\bigcap Supp\,\xi$ of X is almost σ-compact.

First, we give a sufficient condition of the property (σ).

Theorem 8. *If the space $C_k(X)$ satisfies the countable chain condition, then X has the property (σ).*

Vidossich [4] and Nakhmanson [3] proved that $C_k(X)$ satisfies the countable chain condition if X is submetrizable. We have the following corollary.

Corollary 9. *If X is submetrizable (in particular, metrizable), then X has the property (σ).*

Proposition 10. *The space ω_1 has the property (σ).*

Remark 11. *Nakhmanson [3] noted that $C_k(\omega_1)$ does not satisfy the countable chain condition.*

In special cases, we have a condition that the property (σ) necessarily satisfies.

Theorem 12. *Let X be a space which has a closed-and-open subset Y such that $cd(Y) = \omega_1$. If X has the property (σ), then every compact subset of X is metrizable.*

Using Theorem 12, we have a space which does not have the property (σ).

Example. Let $D(\omega_1)$ be the discrete space whose cardinarity is ω_1. The space $D(\omega_1) \bigoplus (\omega_1 + 1)$ does not have the property (σ).

Remark 13. *The above example shows that the property (σ) is not preserved by topological sums in general. In fact, since $C_k(D(\omega_1)) = C_p(D(\omega_1))$ holds, every continuous functional on $C_k(D(\omega_1))$ has the countable minimal support. Obviously the space $\omega_1 + 1$ has the property (σ).*

Final Remarks. Theorem 5 and Theorem 6 are valid for $C_k(X)$ (See [2]). Theorem 3 and Theorem 5 are not valid for $C_n(X)$. For any f in $C_n(\omega_1)$, $\overline{f}$ denotes the unique extention of f to $\omega_1 + 1$. We define a functional ξ on $C_n(\omega_1)$ by the rule $\xi(f) = \overline{f}(\omega_1)$ for any f in $C_n(\omega_1)$. Then ξ is continuous ovbiously. Since $[\alpha, \omega_1) \in Supp\, \xi$ holds for any $\alpha < \omega_1$, we have $\bigcap Supp\, \xi = \emptyset$. Put $\mathcal{F} = \{f \in C_n(\omega_1) : \overline{f}(\omega_1) = 0\}$. Then $\mathcal{F}$ is a non-empty proper closed subset of $C_n(\omega_1)$. Similarly, we have $\bigcap Supp\, \mathcal{F} = \emptyset$ also.

References

[1] K. Morishita, *The minimal support for a continuous functional on a function space*, to appear in Proc. Amer. Math. Soc..

[2] K. Morishita, *The minimal support for a continuous functional on a function space II*, preprint.

[3] L. B. Nakhmanson, *The Souslin number and caliber of rings of continuous functions*, Izvestiya VUZ. Mathematika, Vol. 28, 1984, pp. 49 - 55.

[4] G. Vidossich, *Function spaces which are pseudo-$\aleph$-compact spaces*, preprint 1972.