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DIMENSION AND SUPERPOSITION OF BOUNDED CONTINUOUS FUCTIONS ON LOCALLY COMPACT, SEPARABLE METRIC SPACES

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Let $\mathbf{R}$ and $\mathbf{I}=[0,1]$ be the space of the real line and the closed unit interval respectively. For a space X let $C(X)$ denote the space of all continuous, real valued functions of X equipped with the compact-open topology, and $C^*(X)$ be the set of all bounded, continuous, real-valued functions of X.

In 1957, Kolmogorov [2] proved a superposition theorem for continious functions in $\mathbf{I}^n$ giving a solution to Hilbert's Problem 13 (Kolmogorov's superposition theorem) : For each integer $n \geq 2$ there are $2n+1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(\mathbf{I}^n)$ of the form

$$\varphi_i(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \varphi_{i,j}(x_j), \ \ (x_1, x_2, \dots, x_n) \in \mathbf{I}^n,$$

$$\varphi_{i,j} \in C(\mathbf{I}), \ \ 1 \leq i \leq 2n+1, \ \ 1 \leq j \leq n,$$

such that each $f \in C(\mathbf{I}^n)$ is representable as

$$f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x)), \ \ x = (x_1, x_2, ..., x_n) \in \mathbf{I}^n,$$

where $g_i \in C(\mathbf{R})$, $i = 1, \ldots, 2n + 1$.

Definition ([8]). Let X be a space and $\varphi_i \in C(X), i = 1, ..., k$. Then, $\{\varphi_i\}_{i=1}^k$ is said to be a *basic family* on X if each $f \in C^*(X)$ is representable in the form

$$f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)), x \in X,$$

where $g_i \in C(\mathbf{R}), i = 1, ..., k$.

In compact metric spaces, it is known that the existence of such φ_i's essentially depends on dimension of a space. In fact, Ostrand [6] proved that for every compact metric space X with $\dim X \leq n$ ($n \geq 1$) there are $2n+1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(X)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on X. On the other hand, Sternfeld [7] proved the converse of the Ostrand's theorem: For a compact metric space X and $n \geq 1$ if there are $2n + 1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(X)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on X, then $\dim X \leq n$. (We notice that a simpler proof of the theorem is recently presented by Levin [3].) Hence, a superposition of continuous functions characterizes dimension of a compact metric space.

In non-compact spaces, a few results on superposition of continuous functions are known. Demko [1] proved a superposition theorem for bounded continuous functions on $\mathbf{R}^n$: For each integer $n \geq 2$ there are $2n + 1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(\mathbf{R}^n)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on $\mathbf{R}^n$. This generalized the Kolmogorov's superposition theorem. In connection with the Demko's theorem and the Ostrand's one, Sternfeld posed the following problem ([8, Problem 6.12]): Does the Demko's theorem extend to every n-dimensional separable metric space?

In particular, does it extend to every n-dimensional, locally compact, separable metric space?

We shall prove that the Demko's theorem extends to n-dimensional, locally compact, separable metric spaces, which gives a solution to the second part of the Sternfeld's problem. However, the general problem of Sternfeld still remains open.

For a subset A of a space X we denote by Int A and Bd A the interior and the boundary of A in X respectively. For a mapping f of a space X to a space Y and a subspace A of X we denote by $f|A$ the restriction of f to A. By dimension we mean covering dimension of a space. (However, since we shall consider only separable metric spaces, three fundamental dimensions ind, Ind and dim coincide.) We refer the reader to [5] for dimension theory. We also refer the reader to [8] for the relations between dimension and superposition of continuous functions in compact metric spaces.

1. Results

Our main result is the following.

Theorem. *Let n be an integer with* $n \geq 1$ *and* X *be a locally compact, separable metric space with* $\dim X \leq$ n. *Then, there are* $2n+1$ *functions* $\varphi_1, \ldots, \varphi_{2n+1} \in C(X)$ *such that* $\{\varphi_i\}_{i=1}^{2n+1}$ *is a basic family on* X.

As suggested in [8], the theorem is proved by combining an argument due to Demko [1] with the Ostrand's covering theorem.

Ostrand's covering theorem ([6] or see [5]). *A metric space X is of dimension $\leq n$ if and only if for each open cover $\mathcal{U}$ of X and each integer $k \geq n+1$ there are k many discrete families $\mathcal{V}_1, \dots, \mathcal{V}_k$ of open sets of X such that the union of any $n+1$ of $\mathcal{V}_i$' s is a cover of X and refines $\mathcal{U}$.*

Now, let X be a space, $\{\varphi_i\}_{i=1}^{k} \subset C(X)$ be a basic family on X and A a closed subspace of X. It is clear that $\{\varphi_i | A\}_{i=1}^{k}$ is a basic family on A. Hence, by our theorem and the Sternfeld's theorem above, we have the following characterization theorem.

Corollary. *Let n be an integer with $n \geq 1$ and X be a locally compact, separablet metric space. Then $\dim X \leq n$ if and only if there are $2n+1$ many functions $\varphi_1, \dots, \varphi_{2n+1} \in C(X)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on X.*

2. Proof of the theorem.

We shall sketch an outline of the proof of Theorem. As mentioned above, a framework of our proof is due to Demko [1]. Our main task is to extend Lemmas 2 and 3 in [1] to an n-dimensional, locally compact, separable metric space.

Let X be a locally compact, separable, metric space with $\dim X \leq$ n. Let $\{K_m : m \in \omega\}$ be a countable cover of X by compact sets such that $K_0 = \emptyset$ and $K_m \subset \text{Int } K_{m+1}$ for each m. For each $m \in \omega$ we put

$$L_m = K_m - \text{Int} K_{m-1},$$

and

$$U_m = \begin{cases} \text{Int } K_1, & \text{if } m = 0, \\ \text{Int } K_{m+1} - K_{m-1}, & \text{if } m \geq 1. \end{cases}$$

We notice that $\ell = m$ or $m+1$ if $U_m \cap U_\ell \neq \emptyset$. By the Ostrand's covering theorem, for each integer $k \geq 1$ there are $2n+1$ many families $\mathcal{C}_k^1, ..., \mathcal{C}_k^{2n+1}$ of compact subsets of X satisfying the following conditions.

(1) Each $\mathcal{C}_k^i$ is discrete in X.

(2) For each $k \geq 1$ and each $x \in X$ $\left| \left\{ C \in \bigcup_{i=1}^{2n+1} \mathcal{C}_k^i : x \in C \right\} \right| \geq n+1$.

(3) mesh $\mathcal{C}_k^i (= \sup\{\text{diam} C : C \in \mathcal{C}_k^i\}) < 1/k$ for each i and k.

(4) $\bigcup_{i=1}^{2n+1} \mathcal{C}_k^i$ refines $\{U_m : m = 1, 2, ...\}$.

Lemma 1. *There are $2n+1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(X)$ such that*

(5) *for each i and each m $\varphi_i(L_m) \subset [m, m+2]$, where $[a, b]$ is a closed interval $\{t : a \leq t \leq b\}$,*

(6) *for each pair N, $m \geq 1$ of integers there is $k \geq N$ such that $\{\varphi_i(C) : C \in \mathcal{C}_k^i$ and $C \subset K_m\}$ is mutually disjoint for each i.*

Lemma 2. *Let $f \in C(X)$ with* supp $f \subset \bigcup_{j=0}^{\ell} L_{m+j}$ *and θ be a real number with $n(n+1)^{-1} < \theta < 1$. Then, there are $2n+1$ many functions $g_1, ..., g_{2n+1} \in C(\mathbf{R})$ satisfying the following conditions.*

(7) $\|g_i\| \leq \dfrac{1}{n+1}\|f\|$ *for each i.*

(8) $\left| f(x) - \sum_{i=1}^{2n+1} g_i(\varphi_i(x)) \right| < \theta \|f\|$ *for each $x \in X$.*

(9) supp $g_i \subset [m-1, m+\ell+3]$ *for each i.*

The proof of the following lemma is parallel to that of [1, Lemma 4].

Lemma 3. *Let $f \in C(X)$ with* supp $f \subset L_m \cup L_{m+1}$ *and θ be a real number such that $n(n+1)^{-1} < \theta < 1$. Then there are $2n+1$ many functions $g_1, ..., g_{2n+1} \in C(\mathbf{R})$ such that*

(10) $f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x))$ *for each $x \in X$, and*

(11) $\|g_i|[k, k+1]\| \leq \frac{\|f\|}{\theta(1-\theta)} \theta^{\frac{|m-k|}{3}}$ *for each $k \geq 1$.*

Proof of the theorem. We shall show that the family $\{\varphi_i\}_{i=1}^{2n+1}$ constructed in Lemma 1 is a basic family on X. To do this, let $f \in C^*(X)$. Let U_m and L_m, $m = 1, 2, ...$, be subsets of X described in the top of this section. Let $\{h_m : m = 1, 2, ...\}$ be a locally finite partition of unity subordinated to $\{U_m : m = 1, 2, ...\}$. For each m we put $f_m(x) = f(x)h_m(x)$, $x \in X$. Then the function f_m is continuous, supp $f_m \subset U_m \subset L_m \cup L_{m+1}$, $\|f_m\| \leq \|f\|$ and $f(x) = \sum_{m=1}^{\infty} f_m(x)$. By Lemma 3, for each m there are $2n+1$ functions $g_1^m, ..., g_{2n+1}^m \in C(\mathbf{R})$ such that

(12) $f_m(x) = \sum_{i=1}^{2n+1} g_i^m(\varphi_i(x))$, for each $x \in X$, and

(13) $\left\|g_i^m|[k, k+1]\right\| \leq \frac{1}{\theta^2(1-\theta)} \|f_m\| \theta^{\frac{|m-k|}{3}}$ for each $k \geq 1$.

By (13) and the Weierstrass M-test, $\sum_{m=1}^{\infty} g_i^m|[k, k+1]$ is continuous. Hence, we put $g_i(t) = \sum_{i=1}^{\infty} g_i^m(t)$, $t \in \mathbf{R}$. Then, g_i is continuous and

$$f(x) = \sum_{m=1}^{\infty} \sum_{i=1}^{2n+1} g_i^m(\varphi_i(x)) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x)),$$

for each $x \in X$ by (12). This completes the proof of the theorem.

3. A remark.

Let X be a σ-compact, metric space. If there are $2n+1$ many functions $\varphi_1, ..., \varphi_{2n+1} \in C(X)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on X, then it follows from the Sternfeld's theorem [7] that $\dim X \leq n$. Thus, in connection with our corollary in section 1,we ask the following question, which is a special case of the problem of Sternfeld [8, Problem 6.12].

Question. Let X be an n-dimensional, σ-compact, metric space. Are there $2n+1$ functions $\varphi_1, ..., \varphi_{2n+1} \in C(X)$ such that $\{\varphi_i\}_{i=1}^{2n+1}$ is a basic family on X?

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