数理解析研究所講究録
第785巻 1992年 152-165

Asymptotic behavior as $t \to \infty$ of solutions of quasi-linear heat equations

TADASHI KAWANAGO (川中子 正)

Osaka University

0. Introduction

We consider the large time behavior of weak solutions of the following initial-boundary value problem:

$$
\text{(I)} \quad \begin{cases} u_t = \Delta\phi(u) & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}
$$

Here, $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We assume that

(A1) $\quad \phi \in C^1(\mathbf{R})$ and $\phi'(r) > 0$ if $r \neq 0$,

(A2) $\quad u_0 \in L^2(\Omega)$, $u_0 \not\equiv 0$ in Ω.

If we set $k = \phi'$, then we can rewrite the equation as

$$
u_t = \nabla \cdot (k(u)\nabla u).
$$

We shall study the following two cases:

1° (The degenerate case) When $k(0) = 0$, the eqaution arises in gas flow through porous media.

2° (The nondegenerate case) When $k(0) > 0$, the eqaution arises in heat flow through solids.

In this article we intend to extend and make more precise some results of Kawanago [5]. We shall recall a definition of weak solutions of (I) by the nonlinear semigroup theory. We define operator $A : L^1(\Omega) \to L^1(\Omega)$ by

$$Au = -\Delta\phi(u) \quad \text{for} \quad u \in D(A)$$

with $D(A) = \{u \in L^1(\Omega);\ \phi(u) \in W_0^{1,1}(\Omega),\ \Delta\phi(u) \in L^1(\Omega)\}$. The operator A is m-accretive in $L^1(\Omega)$ under the condition (0.1). Therefore A generates the contraction semigroup $S_A(t)$. Hence we can define a unique weak solution of (I) by $S_A(t)u_0$ for any $u_0 \in \overline{D(A)} = L^1(\Omega)$.

In the proofs of our result we often compute formally. We remark that it is not difficult to make our formal proofs rigorous.

Notation.

1. $\{\lambda_\nu\}_{\nu=1}^{\infty}$ $(0 < \lambda_1 < \lambda_2 < \cdots)$ are all distinct eigenvalues of $-\Delta$ with zero-Dirichlet condition.
2. P_j is the orthogonal projection of the eigenspace of λ_j and $R(P_j)$ is the eigenspace of λ_j for $i \in \mathbf{N}$.
3. $(\cdot\,,\,\cdot)$ denotes the inner product in $L^2(\Omega)$.
4. $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.
5. $f(t) = o(g(t))$ means that $\limsup_{t\to\infty} |f(t)/g(t)| = 0$, $f(t) = O(g(t))$ that $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$, and $f(t) \sim g(t)$ that $a < \liminf_{t\to\infty} |f(t)/g(t)| \le \limsup_{t\to\infty} |f(t)/g(t)| < b$ for some a, $b > 0$.

1. The degenerate case

We state the large time behavior for solutions of (I) under the condition: $k(0) = 0$ (and also $k(0) \ge 0$). When $\phi(r) = |r|^{m-1}r$ $(m > 1)$ and $u_0 \ge 0$ in Ω, Aronson and

Peletier [2] showed that

$$\|(1+t)^{1/(m-1)}u(t)-h\|_\infty \leq C(1+t)^{-1} \quad \text{for} \quad t \geq 0,$$

where $h(x)$ is the unique positive solution of the problem:

$$\begin{cases} \Delta(h^m)+1/(m-1)h=0 & \text{in} \quad \Omega, \\ h(x)=0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The corresponding results were obtained under milder assumption on ϕ:

(1.1) $\quad 0<\alpha<\phi(r)\phi''(r)/[\phi'(r)]^2 \leq 1$

in some nonempty neighborhood of $r=0$ for some $\alpha \in (0,1)$,

by Bertsch and Peletier [3]. We remark that the solutions of equations they considered decay at most algebraically and the decay order does not depend on Ω. We shall give some decay results under a different assumption. And we shall show that there are some degenerate equations of which solutions decay exponentially and the decay order depends on Ω (see Example following Theorem 1).

Theorem 1. *Let* (A1-2) *be satisfied and* $k(0) \geq 0$. *We assume that* $k(r)$ *is non-decreasing (non-increasing) on* $(0,\varepsilon)$ *and that*

$$(1.2) \qquad \frac{\phi(r)}{r} \geq k(\alpha r) \quad (\text{resp. } \frac{\phi(r)}{r} \leq k(\alpha r)\ ) \quad \textit{for} \quad 0<r\leq\varepsilon$$

for some $\alpha \in (0,1)$ *and* $\varepsilon>0$. *Also assume that there exists* $\omega \in R(P_1)$ *such that* $u_0 \geq \omega > 0$ *(resp.* $u_0 \leq \omega$*) in* Ω. *Then,* the weak solution u of (I) satisfies

$$(1.3) \qquad (u(t),e_1) \sim y(t),$$

where e_1 is an element of $R(P_1)$ such that $e_1 > 0$ in Ω and $\|e_1\|_2 = 1$. And $y(t)$ is any fixed positive solution of the ordinary equation: $y'(t) = -\lambda_1 \phi(y)$.

Remark. All examples of ϕ given in [3] satisfy the condition (1.2). Hence it seems that the condition (1.2) is substantially weaker than (1.1).

Sketch of proof of Theorem 1. We state only the case when $k(r)$ is non-decreasing on $(0,\varepsilon)$ because the proof for the other case when $k(r)$ is non-increasing is similar. We denote by $y(t;y_0)$ the solution of the ordinary equation: $y'(t) = -\lambda_1\phi(y)$ with the initial value $y_0 > 0$. Then we can easily verify that $y(t;a) \sim y(t;b)$ for any $a, b \in (0,\infty)$. Therefore we have only to show that $(u(t), e_1) \sim y(t;a)$ for any fixed $a \in (0,\infty)$. In what follows, we choose $\omega_1 \in R(P_1)$ such that $\omega_1 > 0$ in Ω and $0 < \|\omega_1\|_\infty \le \alpha\,(< 1)$. We divide the proof into two steps.

Step 1. We shall prove that

$$(u(t), e_1) \le C y(t) \quad \text{for} \quad t \ge 0 \tag{1.4}$$

for some $C \in (0,\infty)$. We already see in [5] that

$$\|u(t)\|_\infty \to 0 \quad \text{as} \quad t \to \infty \tag{1.5}$$

(see [5, Proposition 4.1]). Therefore we may assume without loss of generality that $\|u(t)\|_\infty \le \varepsilon$ for $t \ge 0$. By the convexity of ϕ and Jensen's inequality, we have

$$\frac{d}{dt}(u(t),\omega_1) = -\lambda_1 \int \phi(u)\,\omega_1 dx \le -\lambda_1 \int \phi(u\,\omega_1)dx \le -\lambda_1|\Omega|\phi(\frac{1}{|\Omega|}(u,\omega_1)),$$

where we denote by $|\Omega|$ the Lebesque measure of $|\Omega|$. It follows that

$$\frac{1}{|\Omega|}(u(t),\omega_1) \le y(\,t;\frac{1}{|\Omega|}(u_0,\omega_1)\,).$$

Hence we have proved (1.4).

Step 2. We shall prove that

$$(1.6) \qquad (u, e_1) \geq Cy(t) \quad \text{for} \quad t \geq 0$$

for some $C \in (0, \infty)$. We shall show that $z(x,t) = \omega_1(x)y(t;a)$ ($a > 0$) is a subsolution of (I). Indeed we obtain from (1.2) and the convexity of ϕ that

$$(1.7) \qquad z_t - \Delta\phi(z) \leq -\lambda_1\omega_1 y\{\frac{\phi(y)}{y} - k(\omega_1 y)\} \leq -\lambda_1\omega_1 y\{k(\alpha y) - k(\omega_1 y)\} \leq 0.$$

Hence $z(x,t)$ is a subsolution of (I). Moreover, we may choose $a \in (0, \infty)$ so small that

$$(1.8) \qquad z(x,0) = a\,\omega_1(x) \leq \omega(x) \leq u_0(x) \quad \text{in} \quad \Omega.$$

It follows from (1.7), (1.8) and the comparison principle that

$$u(x,t) \geq z(x,t) = \omega_1(x)y(t),$$

which implies (1.6). ∎

Example. *When $k(r) = \theta/(-\log r)^\rho$ for $0 < r \leq \varepsilon$ (θ, $\rho \in (0,\infty)$ are constants), (1.2) holds. And $y(t)$ satisfies that*

$$y(t) \sim \exp\{-((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)}\}$$

Moreover, in this example the weak solution u of (I) satisfies that

$$(*) \qquad \frac{u(t)}{(u(t), e_1)} \underset{t\to\infty}{\longrightarrow} e_1 \quad in \quad L^2(\Omega)$$

when $N = 1$.

Sketch of proof of (*) in the above example. We set $v = u(t) - (u, e_1)e_1$. It is sufficient to derive that $\|v(t)\|_2 = o(y(t))$. In view of (1.5), we may assume that $\sup_{t\geq 0}\|u(t)\|_\infty$ is sufficiently small. We set $h(r) = \int_0^r \sqrt{k(s)}\,ds$. Then,

$$\begin{aligned}\frac{d}{dt}\int v^2 &= -2\int \{h(u)_x\}^2 + 2\lambda_1(u,e_1)(\phi(u),e_1)\\ &= -2\sum_{j=1}^{\infty}\lambda_j(h(u),e_j)^2 + 2\lambda_1(u,e_1)(\phi(u),e_1)\\ &\leq 2(\lambda_2-\lambda_1)(h(u),e_1)^2 - 2\lambda_2\int h(u)^2 + 2\lambda_1(u,e_1)(\phi(u),e_1).\end{aligned}$$

We set $g(r) = h(\sqrt{r}\,)^2$. By Jensen's inequality,

$$\frac{d}{dt}\int v^2 \leq -2\lambda_2|\Omega|\,g(\frac{1}{|\Omega|}\int u^2) + 2(\lambda_2-\lambda_1)(h(u),e_1)^2 + 2\lambda_1(u,e_1)(\phi(u),e_1).$$

We may assume by rescaling that $|\Omega| = 1$. We can easily verify that

$$g(r) \geq 2^\rho\phi(r)\{1 - \frac{C}{(-\log r)^2}\} \quad \text{for some} \quad C \in (0,\infty).$$

It follows that

(1.9)
$$\begin{aligned}\frac{d}{dt}\int v^2 \leq -\frac{2^{\rho+1}\theta\lambda_2}{(-\log\int u^2)^\rho}\{\int v^2 + (u,e_1)^2\} + \frac{C}{(-\log\int u^2)^{\rho+2}}\int u^2\\ +2(\lambda_2-\lambda_1)(h(u),e_1)^2 + 2\lambda_1(u,e_1)(\phi(u),e_1).\end{aligned}$$

We can easily verify that

$$h(r) \leq \frac{\sqrt{\theta}\,r}{(-\log r)^{\rho/2}}. \tag{1.10}$$

We obtain from (1.10) that

(1.11)

$$
\begin{aligned}
I &= -\frac{2^{\rho+1}\theta\lambda_2}{(-\log\int u^2)^\rho}(u,e_1)^2 + 2(\lambda_2-\lambda_1)(h(u),e_1)^2 + 2\lambda_1(u,e_1)(\phi(u),e_1)\\
&\le 2\theta(u,e_1)^2\{\frac{-\lambda_2}{(-\log\|u\|_2)^\rho} + \frac{\lambda_2-\lambda_1}{(-\log\|u\|_\infty)^\rho} + \frac{\lambda_1}{(-\log\|u\|_\infty)^\rho}\}\\
&= 2\theta\lambda_2(u,e_1)^2\{\frac{1}{(-\log\|u\|_\infty)^\rho} - \frac{1}{(-\log\|u\|_2)^\rho}\}.
\end{aligned}
$$

On the other hand, it follows from [6, Lemma 3.3] that

$$
(1.12)\quad \|\phi(u(t))\|_\infty \le C\|\nabla\phi(u(t))\|_2 \le \sqrt{\frac{k(\|u(t-t_0)\|_\infty)}{2t_0}}\|u(t-t_0)\|_2 \quad \text{for} \quad t>t_0
$$

for any $t_0 \in (0,\infty)$. By (1.12) and [5, (4.12)],

$$
(1.13)\qquad \|u(t)\|_\infty \le Ct^\sigma \exp\{-((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)}\} \quad \text{for} \quad t\ge 0
$$

for some $\sigma \in (0,\infty)$. It follows from (1.11), (1.13) and the mean value theorem that

$$
(1.14)\qquad I \le \frac{C\log t}{t}y(t)^2.
$$

In view of (1.9) and (1.14), we have

$$
\frac{d}{dt}\int v^2 \le -\frac{2\theta\lambda_2}{((\rho+1)\theta\lambda_1 t)^{\rho/(\rho+1)}+C}\int v^2 + \frac{C\log t}{t}\exp\{-2((\rho+1)\theta\lambda_1 t)^{1/(\rho+1)}\},
$$

which implies that

$$
\|v\|_2 \le Ct^{\varepsilon-1/(2\rho+2)}y(t) \quad \text{for} \quad t\ge 0
$$

for any small $\varepsilon \in (0,1)$. ∎

2. The nondegenerate case

We consider the nondegenerate case. By rescaling, we can assume without loss of generality that $k(0)=1$.

We shall recall a result of [5].

Theorem 2. ([5]) *Let* (A1-2) *be satisfied. We assume that*

$$|k(r)-1| \le \theta/(-\log|r|)^{\rho} \quad \textit{for} \quad |r| \le \varepsilon \tag{2.1}$$

for some $\theta \in (0,\infty)$, $\rho \in (1,\infty)$ *and* $\varepsilon \in (0,1)$. Let u be the weak solution of (I). *Then, there exists an eigenvector* $\omega_1 \in R(P_1)$ *satisfying*

$$e^{\lambda_1 t}u(t) \underset{t\to\infty}{\longrightarrow} \omega_1 \quad \textit{in} \quad H_0^1(\Omega). \tag{2.2}$$

Moreover, $\omega_1 > 0$ *in* Ω *if* $u_0 \ge 0$ *in* Ω.

The condition (2.1) is a sufficient and an almost necessary condition for (2.2). Indeed if the graph of the heat conduction $k(r)$ is not 'mild' near $r = 0$, then, by Theorem 2, the decay order of solutions of (I) is different from that of the linear case, i.e. $\phi(r) \equiv r$ (see Theorem 3 and Example following it). On the other hand, when the graph of $k(r)$ is 'mild' near $r = 0$, we are interested in some delicate difference of large time behavior between the solutions of quasilinear equation and those of linear equation. In this case we shall study the behavior for solutions of quasilinear equations in detail by observing some first terms of the asymptotic expansion of the solutions (see Theorem 4 and Example 2 following it).

Theorem 3. *Let* (A1-2) *be satisfied and* $k(0) = 1$. *We assume that* $k(r)$ *is non-decreasing* (resp. *non-increasing*) *on* $(0,\varepsilon)$ *and that*

$$\frac{\phi(r)}{r} \ge k(\alpha r) \quad (\text{resp. } \frac{\phi(r)}{r} \le k(\alpha r)\,) \quad \textit{for} \quad 0 < r \le \varepsilon \tag{2.3}$$

for some $\varepsilon > 0$ *and* $\alpha \in (0,1)$. *Also assume that there exists* $\omega \in R(P_1)$ *such that* $u_0 \ge \omega > 0$ (resp. $u_0 \le \omega$) *in* Ω. *Then,* the weak solution u of (I) satisfies that

$$\frac{u(t)}{(u(t),e_1)} \underset{t\to\infty}{\longrightarrow} e_1 \quad \textit{in} \quad L^2(\Omega), \tag{2.4}$$

$$(u(t), e_1) \sim y(t), \tag{2.5}$$

where e_1 is an element of $R(P_1)$ such that $e_1 > 0$ in Ω and $\|e_1\|_{L^2} = 1$. And $y(t)$ is any fixed positive solution of the ordinary equation: $y'(t) = -\lambda_1 \phi(y)$.

Example. When $k(r) = 1 + \theta/(-\log r)^\rho$ for $0 < r \le \varepsilon$ ($\theta \in \mathbf{R}$ and $\rho \in (0,1]$ are constants), (2.3) holds. In this case, we can verify that the decay order $y(t)$ satisfies that

$$\log y(t) = -\lambda_1 t - \frac{\theta(\lambda_1 t)^{1-\rho}}{1-\rho} + o(t^{1-\rho}) \quad \text{if} \quad \rho \in (0,1),$$
$$y(t) \sim (t+1)^{-\theta} e^{-\lambda_1 t} \quad \text{if} \quad \rho = 1.$$

Sketch of proof of Theorem 3. In view of Theorem 1, we have only to prove that

$$\|u(t) - (u, e_1)e_1\|_2 = o(y(t)). \tag{2.6}$$

We set $v = u(t) - (u, e_1)e_1$. We remark that $(v, e_1) = (\nabla v, \nabla e_1) = 0$.

$$\begin{aligned} \frac{d}{dt}\int v^2 &= 2\int \nabla v \cdot k(u)\nabla u \\ &= -2\int k(u)|\nabla v|^2 - 2(u, e_1)\int \{k(u) - 1\}\nabla v \cdot \nabla e_1. \end{aligned} \tag{2.7}$$

In view of (1.5), we may assume without loss of generality that $k(u(x,t)) \ge 1 - \varepsilon$ for any $(x,t) \in \Omega \times [0, \infty)$, where $\varepsilon \in (0,1)$ is any small constant. Therefore, we have

$$\begin{aligned} \frac{d}{dt}\int v^2 &\le -2(1-\varepsilon)\|\nabla v\|_2^2 + 2(u, e_1)\|k(u) - 1\|_\infty \|\nabla v\|_2 \|\nabla e_1\|_2 \\ &\le -2(1-2\varepsilon)\|\nabla v\|_2^2 + C\|k(u) - 1\|_\infty^2 (u, e_1)^2. \end{aligned}$$

It follows that

$$\|v(t)\|_2^2 \le e^{-2(1-2\varepsilon)\lambda_2 t}\|v(0)\|_2^2 + Ce^{-2(1-2\varepsilon)\lambda_2 t}\int_0^t e^{2(1-2\varepsilon)\lambda_2 s}\|k(u(s))-1\|_\infty^2 (u(s),e_1)^2 ds.$$

On the other hand, we can easily verify that $y(t) \ge Ce^{-(\lambda_1-\varepsilon)t}$ for $t \ge 0$. We can choose ε so small that $\lambda_1 - \varepsilon - 2(1-2\varepsilon)\lambda_2 < 0$. It follows from l'Hospital's theorem that

$$\begin{aligned}\limsup_{t\to\infty}\frac{\|v(t)\|_2^2}{y(t)^2} &\le C\lim_{t\to\infty}\frac{\int_0^t e^{2(1-2\varepsilon)\lambda_2 s}\|k(u(s))-1\|_\infty^2 y(s)^2 ds}{e^{2(1-2\varepsilon)\lambda_2 t}y(t)^2}\\ &= C\lim_{t\to\infty}\frac{e^{2(1-2\varepsilon)\lambda_2 t}\|k(u(t))-1\|_\infty^2 y(t)^2}{\{e^{2(1-2\varepsilon)\lambda_2 t}y(t)^2\}'}\\ &= C\lim_{t\to\infty}\frac{\|k(u(t))-1\|_\infty^2}{2(1-\varepsilon)\lambda_2 - 2\lambda_1\phi(y)/y}\\ &= 0.\end{aligned}$$

Hence, we have $\|v(t)\|_2 = o(y(t))$. ∎

Next, we shall consider the large time behavior under the following condition on ϕ:

(A3) $$|k(r)-1| \le a|r|^\alpha \quad \text{for} \quad |r| \le \varepsilon$$

for some a, α, $\varepsilon \in (0,\infty)$.

And we shall study the following more general form of equation:

$$\text{(II)}\quad \begin{cases} u_t = \Delta\phi(u) - f(u) & \text{in} \quad \Omega\times(0,\infty),\\ u(x,t) = 0 & \text{on} \quad \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in} \quad \Omega.\end{cases}$$

Here, we assume that

(A4) $f:\mathbf{R}\to\mathbf{R}$ is a locally Lipschitz continuous function satisfying that $rf(r)\ge 0$

for $r \in \mathbf{R}$ and that there exist constants $b \geq 0$ and $p > 1$ such that $|f(r)| \leq b|r|^p$ in some nonempty neighborhood of $r = 0$,

We can not define weak solutions of (II) by the nonlinear semigroup theory since we does not assume that f is monotone. Instead, we define weak solutions by the following:

Definition. A weak solution u of (II) on $t \in (0, \infty)$ is a locally Hölder continuous function in $\Omega \times \mathbf{R}^+$ with the properties:

(i) $u(x,t) \in L^\infty(\Omega \times \mathbf{R}^+)$,

(ii)
$$\int_\Omega \{u_0(x)\eta(x,0) - u(x,T)\eta(x,T)\}dx + \int_0^T dt \int_\Omega \{u\eta_t + \phi(u)\Delta\eta - f(u)\eta\}dx = 0$$

for any $T > 0$ and for any $\eta \in C^2(\overline{\Omega} \times [0,T])$ such that $\eta(x,t) = 0$ on $\partial\Omega \times [0,T]$.

Proposition 1. *We assume* (A1-4). *Then* (II) *has a unique weak solution* u.

Proof. For the uniqueness, see [1, Theorem 12 (i)]. For the existence, the proof is similar to that of [6, Proposition 1.1]. ∎

Theorem 4. ([7]) *Let the conditions* (A1-4) *be satisfied.* Let u be the weak solution of (II). *Then we have the following:*

(i) *There exist* $m \in \mathbf{N}$ *and a non-zero eigenvector* $\omega_m \in R(P_m)$ *satisfying*

$$e^{\lambda_m t}u(t) \underset{t\to\infty}{\longrightarrow} \omega_m \quad \text{in} \quad H_0^1(\Omega).$$

(ii) *More precisely, let* $\kappa = \min\{\alpha + 1, p\}$ *and* $n = \max\{j \in \mathbf{N}\,;\, \lambda_j < \kappa\lambda_m\}$ $(\geq m)$. *Then, also for each* $m < j \leq n$, *there exists eigenvector* $\omega_j \in R(P_j)$ *satisfying*

$$u(t) - \sum_{j=m}^{n} e^{-\lambda_j t}\omega_j = \begin{cases} O(e^{-\kappa\lambda_m t}) & \text{if} \quad \kappa\lambda_m < \lambda_{n+1} \\ O(te^{-\kappa\lambda_m t}) & \text{if} \quad \kappa\lambda_m = \lambda_{n+1} \end{cases} \quad \text{in} \quad H_0^1(\Omega). \tag{2.8}$$

We can derive Theorem 4 by combining the computations in [5] and those in [4].

Example 1. When $k(r) = 1 + e^{-1/|r|}$ and $f \equiv 0$, we can easily obtain from Theorem 4 the following asymptotic expansion in $H_0^1(\Omega)$:

$$u(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \omega_j,$$

where $\omega_j \in R(P_j)$ for $j \in \mathbf{N}$.

This expansion is just the same form as the linear case: $k \equiv 1$. Example 1 is, however, an extreme and exceptional case. We shall study a typical example: $k(r) = 1 + a|r|^\alpha + o(|r|^\alpha)$ and $f(r) = b|r|^{p-1}r + o(|r|^p)$, from which we see that the estimate (2.8) is optimal.

Example 2. We shall state the case: $k(r) = 1 + a|r|^\alpha + o(|r|^\alpha)$ and $f(r) = b|r|^{p-1}r + o(|r|^p)$ (a, $b > 0$ are constants.) We use the same notations as in the statement of Theorem 4 and define $\chi_j(\alpha, p)$ ($j \in \mathbf{N}$) by

$$\chi_j(\alpha,p) = \begin{cases} a\lambda_j/(\alpha+1) & \text{if} \quad \alpha+1 < p, \\ a\lambda_j/(\alpha+1) + b & \text{if} \quad \alpha+1 = p, \\ b \quad \text{if} \quad \alpha+1 > p. \end{cases}$$

Then we have the following:

(i) Let $\kappa\lambda_m < \lambda_{n+1}$. Then,

$$u(t) = \sum_{j=m}^{n} e^{-\lambda_j t} \omega_j + e^{-\kappa\lambda_m t} \nu_1 + o(e^{-\kappa\lambda_m t})$$

in $L^2(\Omega)$, where

$$\nu_1 = \sum_{j=1}^{\infty} \frac{\chi_j(\alpha,p)}{\kappa\lambda_m - \lambda_j} P_j(|\omega_m|^{\kappa-1}\omega_m) \not\equiv 0 \quad \text{in} \quad \Omega.$$

(ii) Let $\kappa\lambda_m = \lambda_{n+1}$. Then,

$$u(t) = \sum_{j=m}^{n} e^{-\lambda_j t}\omega_j + te^{-\lambda_{n+1}t}\nu_2 + o(te^{-\lambda_{n+1}t})$$

in $L^2(\Omega)$, where

$$\nu_2 = -\chi_{n+1}(\alpha,p)P_{n+1}(|\omega_m|^{\kappa-1}\omega_m).$$

Moreover, if $\nu_2 \equiv 0$ in Ω, then there also exists $\omega_{n+1} \in R(P_{n+1})$ such that

$$u(t) = \sum_{j=m}^{n+1} e^{-\lambda_j t}\omega_j + e^{-\lambda_{n+1}t}\nu_3 + o(e^{-\lambda_{n+1}t})$$

in $L^2(\Omega)$, where

$$\nu_3 = (\sum_{j=1}^{n} + \sum_{j=n+2}^{\infty})\frac{\chi_j(\alpha,p)}{\kappa\lambda_m - \lambda_j}P_j(|\omega_m|^{\kappa-1}\omega_m) \not\equiv 0 \quad \text{in} \quad \Omega.$$

References

[1] D. G. Aronson, M. G. Crandall and L. A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Analysis **6** (1982), 1001–1022.

[2] D. G. Aronson and L. A. Peletier, *Large time behavior of solutions of the porous medium equation in bounded domains*, J. Diff. Equations **39** (1981), 378–412.

[3] M. Bertsch and L. A. Peletier, *The Asymptotic Profile of Solutions of Degenerate Diffusion Equations*, Arch. Rat. Mech. Anal. **91** (1986), 207–229.

[4] C. Foias and J. C. Saut, *Asymptotic behavior, as $t \to \infty$ of solutions of Navier-Stokes equations and Nonlinear spectral manifolds*, Indiana Univ. Math. J. **33** (1984), 459–477.

[5] T. Kawanago, *On the behavior of solutions of quasilinear heat conduction equations*, in Evolution Equations and Applications to Nonlinear Problems, RIMS Kokyuroku **730** (1990).

[6] T. Kawanago, *The behavior of solutions of quasi-linear heat equations*, Osaka J. Math. **27** (1990), 769–796.

[7] T. Kawanago, *Large time behavior of solutions of quasi-linear heat conduction equations*, Preprint.