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SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I

- $\mathcal{P}$-congruence pairs -

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Yamada and Sen introduced the new concept of $\mathcal{P}$-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$-regular semigroups by using "$\mathcal{P}$-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular *-semigroups. Also, for a given congruence ρ on a $\mathcal{P}$-regular semigroup S, the maximum and the minimum congruences on S whose traces coincide with the trace of ρ (= $\rho \cap E(S)\times E(S)$) are determined.

1. Introduction. Let S be a regular semigroup and E the set of idempotents of S. Let $P \subset E$. If S satisfies the following, it is called a $\mathcal{P}$-<u>regular</u> <u>semigroup</u>:

(1) $P^2 \subset E$,

(2) $qPq \subset P$ for any $q \in P$,

(3) for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of a) such that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$.

In such a case, S is denoted by S(P) and P is called a C-<u>set</u> in S. Throughout this paper, let S(P) be a $\mathcal{P}$-regular semigroup with a C-set P such that the set of idempotents of S is E. Let $a \in S(P)$ and $a^+ \in V(a)$. If a^+ satisfies that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$, it is called a $\mathcal{P}$-inverse of a, and the set of $\mathcal{P}$-inverses of a is denoted by $V_P(a)$. An element of a C-set P in S is called a projection. The class of $\mathcal{P}$-regular semigroups contains both the classes of orthodox semigroups and regular $*$-semigroups. A good account of the concept of $\mathcal{P}$-regularity can be seen in [8] and [9].

A congruence on S is sometimes called a $\mathcal{P}$-congruence on S(P). Let ρ be a $\mathcal{P}$-congruence on S(P), and put $\overline{x} = x\rho$ for any $x \in S$, $\overline{S} = \{\overline{x}: x \in S\}$ and $\overline{P} = \{\overline{q}: q \in P\}$. Then $\overline{S}(\overline{P})$ is also a $\mathcal{P}$-regular semigroup with a C-set $\overline{P}$. So $\overline{S}(\overline{P})$ is called the factor $\mathcal{P}$-regular semigroup of S(P) mod. ρ, and it is denoted by $S(P)/(\rho)_{\mathcal{P}}$.

Let ρ be a $\mathcal{P}$-congruence on S(P). Then it is called an orthodox $\mathcal{P}$-congruence on S(P) if $S(P)/(\rho)_{\mathcal{P}}$ is an orthodox semigroup, and it is called a strong $\mathcal{P}$-congruence on S(P) if it satisfies that for $a \in S(P)$ and $e \in P$,

$$a \;\rho\; e \text{ implies } a^+ \;\rho\; e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if ρ is a strong $\mathcal{P}$-congruence on S(P), then $S(P)/(\rho)_{\mathcal{P}}$ becomes a regular $*$-semigroup with the set $\{e\rho: e \in P\}$ of projections if the $*$-operation # on $S(P)/(\rho)_{\mathcal{P}}$ is defined by $(a\rho)^{\#} = a^+\rho$ $(a \in S(P),\ a^+ \in V_P(a))$.

The set $\{a \in S(P): a \;\rho\; e \text{ for some } e \in E\ [e \in P]\}$ is called the $[\mathcal{P}$-$]$kernel of ρ, and it denoted by $[\mathcal{P}$-$]\mathrm{ker}\rho$. The restriction $\rho \cap (E\times E)$ $[\rho \cap (P\times P)]$ of ρ is called the $[\mathcal{P}$-$]$trace of ρ, and it is denoted by $[\mathcal{P}$-$]\mathrm{tr}\rho$.

For any subset A of S(P), define the terminology as follows:

A is [$\mathcal{P}$-]<u>full</u> if $E \subset A$ [$P \subset A$],

A is a $\mathcal{P}$-<u>subset</u> if $V_P(a) \subset A$ for any $a \in A$,

A is a $\mathcal{P}$-<u>self-conjugate</u>

if $x^+Ax \subset A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$,

A is <u>weakly closed</u> if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

Result 1.1 (due to [8] and [9]). *Let* a, b $\in$ S(P), e $\in$ E *and* q $\in$ P. *Then*

(i) $V_P(b)V_P(a) \subset V_P(ab)$,

(ii) *if* $a^+ \in V_P(a)$, *then* $a \in V_P(a^+)$,

(iii) $V_P(e) \subset E$, (iv) $q \in V_P(q)$.

Result 1.2 (due to [2]). *Let* ρ *be a* $\mathcal{P}$-*congruence on* S(P) *and* a, b $\in$ S(P). *Then* a ρ b *if and only if*

$$ba' \in \ker\rho,\ aa' \ \rho\ bb'aa',\ b'b \ \rho\ b'ba'a$$

for some $a' \in V(a)$ *and* $b' \in V(b)$.

In section 2, for a given $\mathcal{P}$-congruence ρ on a $\mathcal{P}$-regular semigroup S(P), the maximum and the minimum $\mathcal{P}$-congruences on S(P) whose traces coincide with $\mathrm{tr}\rho$ are determined, and the properties for those $\mathcal{P}$-congruences are given.

The concept introduced in section 3 is "$\mathcal{P}$-congruence pairs". This concept is a characterization of the pair ($\mathrm{tr}\rho$, $\ker\rho$) associated with a given $\mathcal{P}$-congruence ρ on S(P), and the pair uniquely determines the $\mathcal{P}$-congruence κ such that $tr\kappa = tr\rho$ and $ker\kappa = ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence ρ on S(P), define a relation ρ_{max} on S(P) as follows:

$$\rho_{max} = \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ \ \rho\ beb^+aea^+,\ beb^+ \ \rho\ aea^+beb^+,\ a^+ea \ \rho\ a^+eab^+eb \text{ and } b^+eb \ \rho\ b^+eba^+ea \text{ for all } e \in P\}.$$

Then we can easily see that

$$\rho_{max} = \{(a,b): aea^+ \ \rho\ beb^+aea^+,\ beb^+ \ \rho\ aea^+beb^+,\ a^+ea \ \rho\ a^+eab^+eb \text{ and } b^+eb \ \rho\ b^+eba^+ea \text{ for all } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}$$

Lemma 2.1. <u>Let</u> ρ <u>be</u> <u>a</u> $\mathcal{P}$-<u>congruence</u> <u>on</u> S(P) <u>and</u> a, b $\in$ S(P). <u>If</u> a ρ_{max} b, <u>then</u>

$$aa^+ \ \rho\ bb^+aa^+,\ bb^+ \ \rho\ aa^+bb^+,\ a^+a \ \rho\ a^+ab^+b,\ b^+b \ \rho\ b^+ba^+a$$

<u>for</u> <u>any</u> $a^+ \in V_P(a)$ <u>and</u> $b^+ \in V_P(b)$.

Theorem 2.2. <u>For</u> <u>any</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on</u> <u>a</u> $\mathcal{P}$-<u>regular</u> <u>semigroup</u> S(P), ρ_{max} <u>is</u> <u>the</u> <u>greatest</u> $\mathcal{P}$-<u>congruence</u> <u>on</u> S(P) <u>whose</u> <u>trace</u> <u>coincides</u> <u>with</u> $tr\rho$.

Theorem 2.3. <u>For</u> <u>any</u> <u>orthodox</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on</u> S(P),

ρ_{max} is the greatest orthodox $\mathcal{P}$-congruence on S(P) whose trace coincides with $tr\rho$.

From now on, denote the maximum idempotent-separating congruence on a semigroup T by μ_T.

Corollary 2.4 (compare with [8, Proposition 4.1]). *The maximum idempotent-separating $\mathcal{P}$-congruence $\mu_{S(P)}$ on S(P) is given as follows*:

$$\mu_{S(P)} = \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ = beb^+aea^+,\ beb^+ = aea^+beb^+,\ a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}.$$

$$= \{(a,b): aea^+ = beb^+aea^+,\ beb^+ = aea^+beb^+,\ a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}$$

Let S be an orthodox semigroup and E the band of idempotents of S. Then it is easy to check that S(E) is a $\mathcal{P}$-regular semigroup with a C-set E in S. So we have immediately

Corollary 2.5 ([1, Theorem 4.2]). *Let ρ be a congruence on an othodox semigroup S with the band E of idempotents of S. Then*

$$\rho_{max} = \{(a,b): \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \ \rho\ beb'aea',\ beb' \ \rho\ aea'beb',\ a'ea \ \rho\ a'eab'eb,\ b'eb \ \rho\ b'eba'ea \text{ for any } e \in E\}$$

$$= \{(a,b): aea' \ \rho\ beb'aea',\ beb' \ \rho\ aea'beb',\ a'ea \ \rho$$

a'eab'eb, b'eb ρ b'eba'ea <u>for any</u> a' $\in$ V(a), b' $\in$ V(b) <u>and</u> e $\in$ E}

<u>is the greatest congruence on</u> S <u>whose trace coincides with</u> trρ.

On the other hand, the minimum $\mathcal{P}$-congruence on S(P) with the same trace is given as follows:

Theorem 2.6. <u>For any</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on a</u> $\mathcal{P}$-<u>regular semigroup</u> S(P), <u>define a relation</u> ρ_0 <u>on</u> S(P) <u>by</u>

$$\rho_0 = \{(a,b): \text{there exist } x, y \in S(P)^1 \text{ and } e, f \in E \text{ such that } a = xey,\ b = xfy \text{ and } e\ \rho\ f\}$$

<u>Then</u> $\rho_{min} = \rho_0^{\ t}$, <u>the transitive closure of</u> ρ_0, <u>is the least</u> $\mathcal{P}$-<u>congruence on</u> S(P) <u>whose trace coincides with</u> trρ. <u>In other words, the least</u> $\mathcal{P}$-<u>congruence on</u> S(P) <u>with</u> trρ <u>as its trace is the</u> $\mathcal{P}$-<u>congruence on</u> S(P) <u>generated by</u> trρ.

The following corollary gives us the characterization which is defferent from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

Corollary 2.7. <u>For any congruence</u> ρ <u>on an orthodox semigroup</u> S, <u>the congruence generated by</u> trρ <u>is the least congruence on</u> S <u>whose trace coincides with</u> trρ.

Proposition 2.9. <u>For any</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on</u> S(P), $\rho = \rho_{max}$ <u>if and only if</u> $S(P)/(\rho)_{\mathcal{P}}$ <u>is a fundamental</u> $\mathcal{P}$-<u>regular</u>

semigroup.

For any $\mathcal{P}$-congruences ρ and σ on S(P) such that $\rho \subset \sigma$, define a relation σ/ρ on $S(P)/(\rho)_{\mathcal{P}}$ by

$$\sigma/\rho = \{(a\rho, b\rho) : (a,b) \in \sigma\}$$

Proposition 2.10. For any $\mathcal{P}$-congruence ρ on S(P), ρ_{max}/ρ is the maximum idempotent-separating $\mathcal{P}$-congruence on $S(P)/(\rho)_{\mathcal{P}}$.

Let Λ be the lattice of all $\mathcal{P}$-congruences on S(P). Define a relaton Θ on Λ as follows: for any ρ, $\sigma \in \Lambda$,

$$\rho \ \Theta \ \sigma \quad \text{if and only if} \quad \mathrm{tr}\rho = \mathrm{tr}\sigma .$$

It immediately follows from Theorems 2.2 and 2.6 that $\rho\Theta$, the Θ-class containing $\rho \in \Lambda$, is the interval $[\rho_{min}, \rho_{max}]$ of Λ.

Proposition 2.11 ([6, Theorem 5.1]). If $\mathcal{P}$-congruences ρ and σ on S(P) are Θ-equivalent, then $\rho\sigma = \sigma\rho$. Therefore, for any $\rho \in \Lambda$, $\rho\Theta$ is a complete modular subsemilattice of Λ.

Proposition 2.12. Let $\xi \in \Lambda$, and let Γ be the lattice of all idempotent-separating $\mathcal{P}$-congruences on $S(P)/(\xi_{min})_{\mathcal{P}}$. Then the mapping $\rho \rightarrow \rho/\xi_{min}$ is a complete isomorphism of $\xi\Theta$ onto Γ.

3. $\mathcal{P}$-congruence pairs. Let ξ be an equivalence on E. Then ξ is called a normal equivalence on E if it satisfies the following conditions: for any $a \in S(P)$ and e, f, g, h, i, j, k $\in$

E,

(a) if $e \xi f$ and $aea^+, afa^+ \in E$ for some $a^+ \in V_P(a)$, then $aea^+ \xi afa^+$,

(b) if $e \xi f$, $g \xi h$ and $eg, fh \in E$, then $eg \xi fh$,

(c) if $\square \neq (e\xi)(f\xi) \cap E \subset h\xi$, $\square \neq (f\xi)(g\xi) \cap E \subset i\xi$ and $\square \neq (e\xi)(i\xi) \cap E \subset j\xi$ $[\square \neq (h\xi)(g\xi) \cap E \subset k\xi]$, then $\square \neq (h\xi)(g\xi) \cap E$ $[\square \neq (e\xi)(i\xi) \cap E]$ and $j \xi k$.

Let ξ be a normal equivalence on E. Define a partial binary operation $\bullet$ on E/ξ as follows: for any $e, f, g \in E$,

$$e\xi \bullet f\xi = g\xi, \text{ where } \square \neq (e\xi)(f\xi) \cap E \subset g\xi.$$

It is easy to verify that the partial binary operation $\bullet$ is well-defined. The partial groupoid E/ξ satisfies the following:

(w) if $e\xi \bullet f\xi$, $f\xi \bullet g\xi$ and $e\xi \bullet (f\xi \bullet g\xi)$ $[(e\xi \bullet f\xi) \bullet g\xi]$ are defined in E/ξ, then $(e\xi \bullet f\xi) \bullet g\xi$ $[e\xi \bullet (f\xi \bullet g\xi)]$ is defined in E/ξ and $(e\xi \bullet f\xi) \bullet g\xi = e\xi \bullet (f\xi \bullet g\xi)$.

Let K be a weakly closed full $\mathcal{P}$-subset of S(P) and ξ a normal equivalence on E. Then the pair (ξ, K) is called a $\mathcal{P}$-<u>congruence</u> <u>pair</u> for S(P) if itsatisfies the following conditions: for any $a, b, c \in S(P)$, $c^+ \in V_P(c)$, $e, f, g \in E$ and $q \in P$,

(C1) $a \in K$ implies $a^+a \ \xi \ a^+a^+aa$ for any $a^+ \in V_P(a)$,

(C2) $aefb \in K$ and $e\xi \bullet f\xi = (a^+a)\xi$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

(C3) $ab^+ \in K$ and $aa^+ \ \xi \ bb^+aa^+$, $b^+b \ \xi \ b^+ba^+a$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $aqb^+ \in K$ and $aqa^+ \ \xi \ bqb^+aqa^+$, $b^+qb \ \xi \ b^+qba^+qa$, (C4) $a, b \in K$, $aa^+ \ \xi \ ee^+aa^+$, $ee^+ \ \xi \ aa^+ee^+$, $a^+a \ \xi \ a^+ae^+e$, $e^+e \ \xi \ e^+ea^+a$, $bb^+ \ \xi \ ff^+bb^+$, $ff^+ \ \xi \ bb^+ff^+$, $b^+b \ \xi \ b^+bf^+f$, $f^+f \ \xi \ f^+fb^+b$ and $e\xi \circ f\xi = g\xi$ for some $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$ imply $ab \in K$,

(C5) $aq \in K$ and $aa^+ \ \xi \ qaa^+$, $q \ \xi \ qa^+a$ for some $a^+ \in V_P(a)$ imply $cac^+ \in K$.

For any $\mathcal{P}$-congruence pair (ξ, K) for S(P), define a relation $\kappa_{(\xi,K)}$ on S(P) as follows:

$$\kappa_{(\xi,K)} = \{(a,b)\colon ab^+ \in K \text{ and } aa^+ \ \xi \ bb^+aa^+,\ bb^+ \ \xi \ aa^+bb^+,\ a^+a \ \xi \ a^+ab^+b,\ b^+b \ \xi \ b^+ba^+a \text{ for some [any] } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b)\}.$$

Now we can determine $\mathcal{P}$-congruences on S(P) by $\mathcal{P}$-congruence pairs.

Theorem 3.1. <u>For any $\mathcal{P}$-congruence pair (ξ, K) for a $\mathcal{P}$-regular semigroup S(P), $\kappa_{(\xi,K)}$ is a $\mathcal{P}$-congruence on S(P) such that</u> $\mathrm{tr}\,\kappa_{(\xi,K)} = \xi$ <u>and</u> $\ker\kappa_{(\xi,K)} = K$. <u>Conversely, for any $\mathcal{P}$-congruence ρ on S(P), $(\mathrm{tr}\,\rho, \ker\rho)$ is a $\mathcal{P}$-congruence pair for S(P) and</u> $\rho = \kappa_{(\mathrm{tr}\,\rho,\ker\rho)}$.

Let $\mathscr{A}$ be the set of $\mathcal{P}$-congruence pairs for S(P). Define an order $\leqslant$ on $\mathscr{A}$ by

$$(\xi_1, K_1) \leqslant (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2,\ K_1 \subset K_2.$$

Corollary 3.2. <u>The mappings</u>

$$(\xi, K) \to \kappa_{(\xi, K)}, \quad \rho \to (\mathrm{tr}\rho, \ker\rho)$$

are mutually inverse order-preserving mappings of $\mathscr{A}$ onto Λ and of Λ onto $\mathscr{A}$, respectively. Therefore, $\mathscr{A}$ forms a complete lattice.

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