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Numerical Verification of Solutions of Parametrized Nonlinear Boundary Value Problems with Turning Points

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Abstract. Nonlinear boundary value problems (NBVPs in abbreviation) with parameters are called parametrized nonlinear boundary value problems. This paper studies numerical verification of solutions of parametrized NBVPs defined on one-dimensional bounded intervals. Around turning points the original problem is extended so that the extented problem has an invertible Fréchet derivative. Then, the usual procedure of numerical verification of solutions can be applied to the extended problem. A numerical examples is given.

Key words. parametrized nonlinear boundary value problems, numerical verification of solutions, regular branches, turning points

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1. Introduction.

For the past several years a theory for numerical verification of solutions of differential equations has been developed [N1-5]. By the theory the existence of *exact* solutions of differential equations are verified on computers by certain procedures in finite steps.

Let $\Lambda \subset \mathbb{R}$ be a bounded interval for parameter. Here we deal with the following nonlinear two-point boundary value problem with a parameter $\lambda \in \Lambda$ on the bounded interval $J := (a, b)$:

$$(1.1) \qquad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where $f : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is a given smooth function. Since (1.1) has the parameter λ, the set of the solutions of (1.1) would form one dimensional curves. There, however, may exist singular points on the curves. For example, a solution curve might fold (the folding point is call a **turning point**), or several solution curves might intersect at one point (the intersecting point is called a **bifurcation point**). In this paper we consider the case of turning points.

Let (λ, u) be a solution of (1.1). The above singularities occur when the following eigenvalue problem has the eigenvalue $\mu = 0$:

$$(1.2) \qquad L\psi = \mu\psi,$$

where the differential operator L is defined by

$$L\psi := -\psi'' - f_y(\lambda, x, u)\psi,$$

and $f_y(\lambda, x, y)$ denotes the derivative of f with respect to y. More precisely, if $\mu = 0$ is *not* an eigenvalue of (1.2), by the implicit function theorem, there exists a unique solution curve around (λ, u), and it is parametrized by λ. Such a solution curve is called a **regular branch**. On regular branches the usual procedure of numerical verification of solutions of (1.1) can be applied.

However, during the solution branch following, the usual procedure may become divergent when we get closer to a turning point: the number of iteration becomes bigger or smaller mesh size may be needed. Moreover, at a turning point, our theory cannot be applied, and we have to find a new theory of numerical verification.

Our goal is to overcome this difficulty and establish a new procedure for numerical verification around turning points. The main idea is as follows: In [TB1] the original equation is extended around turning points so that the extended equation has an invertible Fréchet derivative. Then a straightforward modification of the usual numerical verification procedure works well around tuning points.

In the last section a numerical examples is given.

2. Parametrized NBVP.

As is stated in Section 1, we consider the two-point boundary value problem

$$(2.1)\qquad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where $J := (a, b) \subset \mathbb{R}$ is a bounded interval, and $\lambda \in \Lambda \subset \mathbb{R}$ is a parameter.

Let $H_0^1(J)$, $H^{-1}(J)$, etc. are the usual Sobolev spaces. In notation we omit '(J)' whenever there is no danger of confusion. The weak form of (2.1) is written as

$$(2.2)\qquad \textit{Find}\quad u \in H_0^1 \quad \textit{such that}\quad (u', v') = (f(\lambda, x, u), v), \quad \text{for } \forall v \in H_0^1,$$

where $(\cdot, \cdot)$ is the inner product of L^2 defined by $(g, h) := \int_J ghdx$ for $g, h \in L^2$. Now, define the operators $L : \Lambda \times H_0^1 \to H^{-1}$ and $F : \Lambda \times H_0^1 \to L^2 \subset H^{-1}$ by, for $(\lambda, u) \in \Lambda \times H_0^1$,

$$(2.3)\qquad < L(\lambda, u), v > := \int_J u'v'dx, \qquad \forall v \in H_0^1,$$

$$(2.4)\qquad < F(\lambda, u), v > := \int_J f(\lambda, x, u)vdx, \qquad \forall v \in H_0^1,$$

where $< \cdot, \cdot >$ is the duality pair of H^{-1} and H_0^1. Since the inclusion $\iota : L^2 \hookrightarrow H^{-1}$ is compact, the operator $L - F : \Lambda \times H_0^1 \to H^{-1}$ is a Fredholm operator of index 1.

For F to be smooth, we suppose the following assumption:

A function $\psi : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is called **Carathéodory continuous** if ψ satisfies the following conditions: for $(\lambda, x, y) \in \Lambda \times J \times \mathbb{R}$,

$$\begin{cases} \psi(\lambda, x, y) \text{ is continuous with respect to } \lambda \text{ and } y \text{ for almost all } x, \\ \psi(\lambda, x, y) \text{ is Lebesgue measurable with respect to } x \text{ for all } \lambda \text{ and } y. \end{cases}$$

If $\psi(\lambda, x, y)$ is Carathéodory continuous, $\psi(\lambda, x, u(x))$ is Lebesgue measurable with respect to x for any Lebesgue measurable function u.

Let $\alpha = (\alpha_1, \alpha_2)$ be usual multiple index with respect to λ and y. That is, for $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha f(\lambda, x, y)$ means $\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha_1} \partial y^{\alpha_2}} f(\lambda, x, y)$.

Let $d \geq 1$ be an integer. For α, $|\alpha| \leq d$, we define the map $\mathsf{F}^\alpha(\lambda, u)$ for $(\lambda, u) \in \Lambda \times H_0^1$ by

$$\mathsf{F}^\alpha(\lambda, u)(x) := D^\alpha f(\lambda, x, u(x)). \tag{2.5}$$

We then assume that

Assumption 2.1. *Let $d \geq 2$. For all α, $|\alpha| \leq d$, we suppose that*

(1) *For almost all $x \in J$, $D^\alpha f(\lambda, x, y)$ exists at any $(\lambda, y) \in \Lambda \times \mathbf{R}$, and that is Carathéodory continuous.*

(2) *The mapping F^α defined by (2.5) is a continuous operator from $\Lambda \times H_0^1$ to L^2, and the image $\mathsf{F}^\alpha(U)$ of any bounded subset $U \subset \Lambda \times H_0^1$ is bounded.* $\lhd$

Assumption 2.1 is satisfied if $f : \Lambda \times J \times \mathbf{R} \to \mathbf{R}$ is, for instance, C^d function.

Lemma 2.2. *Suppose that Assumption 2.1 holds. Then, the operator $F : \Lambda \times H_0^1 \to H^{-1}$ is of C^d class, and its partial derivatives are written as*

$$\begin{aligned} < D_u F(\lambda, u)\psi, v > &= \int_J f_y(\lambda, x, u(x))\psi v dx, \\ < D_\lambda F(\lambda, u)\eta, v > &= \eta \int_J f_\lambda(\lambda, x, u(x)) v dx, \end{aligned}$$

for $\psi, v \in H_0^1$, and $\eta \in \mathbf{R}$. $\lhd$

By the theory due to Fink and Rheinboldt [R], we have the following fact (also see [BRR2]). Let $\mathcal{R}(L - F) \subset \Lambda \times H_0^1$ be defined by

$$\mathcal{R}(L - F) := \left\{ (\lambda, u) \in \Lambda \times H_0^1 \;\middle|\; D(L - F)(\lambda, u) \text{ is onto} \right\}.$$

Theorem 2.3. *Suppose that f satisfies Assumption 2.1. Also, suppose that $0 \in (L - F)(\mathcal{R}(L - F))$. Then, the set of solutions of (2.2)*

$$\mathcal{M} = \mathcal{M}_0 := \left\{ (\lambda, u) \in \mathcal{R}(L - F) \;\middle|\; (L - F)(\lambda, u) = 0 \right\}$$

is a one-dimensional C^d-manifold without boundary. $\lhd$

Now, let $L_0 := L|_{H_0^1}$. Then, $L_0 : H_0^1 \to H^{-1}$ is an isomorphism. Hence, if we define $\Phi \in \mathcal{L}(H^{-1}, H_0^1)$ by $\Phi := L_0^{-1}$, there exists a constant C_1 such that

$$\|\Phi f\|_{H^2} \leq C_1 \|f\|_{L^2} \tag{2.6}$$

for any $f \in L^2$. Note that in this case the constant C_1 is easily determined. That is, C_1 is available in numerical verification procedures.

Let $(\lambda, u) \in \mathcal{M}_0$ be such that

$$D_\lambda(L-F)(\lambda,u) = -D_\lambda F(\lambda,u) \neq 0.$$

By assumptions, we have $\dim \mathrm{Ker} D(L-F)(\lambda,u) = 1$. Let $(\mu,\psi) \in \mathbb{R} \times H_0^1$ be the basis of $\mathrm{Ker} D(L-F)(\lambda,u)$. By [TB1,Lemma8.1], we have $\psi \neq 0$. Let $x_0 \in J$ be such that $\psi(x_0) \neq 0$. Define the map $G : \Lambda \times H_0^1 \to \mathbb{R} \times H_0^1$ by

$$G(\lambda,u) := (\lambda - u(x_0) + \gamma, \Phi \circ F(\lambda,u)), \tag{2.7}$$

where $\gamma \in \mathbb{R}$ is given. Note that, since $F(\lambda,u) \in L^2$ for any $(\lambda,u) \in \Lambda \times H_0^1$, $\Phi \circ F$ is a compact operator.

As in [TB1,2], the equation (2.1) is rewritten as

$$\begin{cases} -u'' = f(\lambda,x,u), \\ u(x_0) = \gamma, \qquad u(a) = u(b) = 0, \end{cases} \tag{2.8}$$

provided $D_\lambda(L-F)(\lambda,u) \neq 0$. Using G defined by (2.7), the equation (2.8) can be written as a fixed point problem:

$$(\lambda,u) = G(\lambda,u), \qquad (\lambda,u) \in \Lambda \times H_0^1. \tag{2.9}$$

That is, a solution $(\lambda,u) \in \Lambda \times H_0^1$ of (2.1) is a fixed point of G provided $D_\lambda(L-F)(\lambda,u) \neq 0$. Note that by [TB1,Lemma8.1] the Fréchet derivative $I - DG(\lambda,u)$ is an isomorphism for any $(\lambda,u) \in \mathcal{R}(L-F)$. Here and in the sequel, I is the identity of $\mathbb{R} \times H_0^1$.

Remark 2.4. One may wonder how $x_0 \in J$ can be taken. In this paper, to compute finite element solutions of (2.8), we use the continuation program package PITCON developed by Rheinboldt and his colleagues. During path following, PITCON picks up a certain nodal point of the finite element space in use. From the design of PITCON, we may expect that the nodal point satisfies what x_0 has to satisfy (see [TB1,Remark8.3]).

In Section 6, we present a verification procedure which verifies that the selection of the nodal point x_0 is correct: for the basis $(\mu,\psi) \in \mathbb{R} \times H_0^1$ of $\mathrm{Ker} D(L-F)(\lambda_h,u_h)$, we have $\psi(x_0) \neq 0$, where $(\lambda_h,u_h) \in \Lambda \times S_h$ is the obtained finite element solution. $\triangleleft$

3. Formulation of Numerical Verification.

Let $S_h \subset H_0^1$ be a finite element space. The projection $P_{h0} : H_0^1 \longrightarrow S_h$ is defined by

$$((u - P_{h0}u)', v_h') = 0, \qquad \forall v_h \in S_h.$$

For S_h, we suppose the following assumption:

Assumption 3.1. *There exists a computable constant C_2 which is independent of h and u, and satisfies the following estimate:*

$$\|u - P_{h0}u\|_{H_0^1} \leq C_2 h |u|_{H^2}, \qquad \forall u \in H_0^1 \cap H^2. \quad \lhd \tag{3.1}$$

It is well known that the finite element space of piecewise linear functions satisfies Assumption 3.1.

The projection $P_h : \mathbb{R} \times H_0^1 \longrightarrow \mathbb{R} \times S_h$ is defined by

$$P_h(\mu, u) := (\mu, P_{h0}u), \qquad \text{for} \quad (\mu, u) \in \mathbb{R} \times H_0^1. \tag{3.2}$$

As stated in Remark 2.4, we suppose that a nodal point $x_0 \in J$ of S_h is taken in a certain way so that $I - DG(\lambda, u)$ is an isomorphism for any $(\lambda, u) \in \mathcal{R}(L - F)$. The finite element solution $(\lambda_h, u_h) \in \mathbb{R} \times S_h$ of (2.8) is defined naturally by

$$(u_h', v_h') = (f(\lambda_h, x, u_h), v_h), \quad \forall v_h \in S_h, \quad \text{and } u_h(x_0) = \gamma. \tag{3.3}$$

Assumption 3.2. *At the computed finite element solution $(\lambda_h, u_h) \in \mathbb{R} \times S_h$ of (3.3), the restricted operator $P_h(I - DG(\lambda_h, u_h))|_{\mathbb{R}\times S_h}$ has the inverse*

$$[I - DG^h]_h^{-1} : \mathbb{R} \times S_h \to \mathbb{R} \times S_h. \quad \lhd$$

In the sequel, we denote $DG(\lambda_h, u_h)$ and $DF(\lambda_h, u_h)$ by DG^h and DF^h, respectively.

Assumption 3.2 means that, for all $(\mu, w_h) \in \mathbb{R} \times S_h$, there exists the unique solution $(\delta, y_h) \in \mathbb{R} \times S_h$ of the equation $P_h(I - DG^h)(\delta, y_h) = (\mu, w_h)$. Since $DG^h(\delta, y_h) = (\delta - y_h(x_0), DF^h(\delta, y_h))$, we see that $(I - DG^h)(\delta, y_h) = (y_h(x_0), y_h - DF^h(\delta, y_h))$, and

$$\begin{cases} \mu = y_h(x_0) \\ ((y_h - DF^h(\delta, y_h) - w_h)', v_h') = 0, \quad \forall v_h \in S_h. \end{cases} \tag{3.4}$$

Let $M := \dim S_h$. Let $\{\phi_j\}_{j=1}^{M}$ be the basis of S_h and $y_h = \sum_{j=1}^{M} a_j\phi_j$, $w_h = \sum_{j=1}^{M} b_j\phi_j$. Then, Assumption 3.2 implies that the equation

$$(3.5)\quad \begin{cases} \mu = a_p \qquad (p \text{ is the index such that } \phi_p(x_0) = 1) \\ \sum_{j=1}^{M} a_j((I - D_uF^h)\phi_j)' - \delta(D_\lambda F^h)', \phi_k') = \sum_{j=1}^{M} b_j(\phi_j', \phi_k'), \quad k = 1, ..., M \end{cases}$$

is uniquely solvable for any $(b_1, ..., b_M, \mu)$. Therefore, we can verify on computer whether or not Assumption 3.2 holds.

4. Rounding and Rounding Error.

Let ϵ, $(0 < \epsilon < 1)$ be a parameter. We first define the operator $T_\epsilon : \Lambda \times H_0^1 \to \mathbf{R} \times H_0^1$ by

$$(4.1)\qquad T_\epsilon := I - ([I - DG^h]_h^{-1} P_h + \epsilon I)(I - G).$$

Note that if $[I - DG^h]_h^{-1} P_h + \epsilon I$ has an inverse operator, the two fixed point equations $(\lambda, u) = G(\lambda, u)$ and $(\lambda, u) = T_\epsilon(\lambda, u)$ are equivalent. Our main tool of numerical verification has been the following fixed point theorem (for instance, see [Z]):

Theorem 4.1 (Sadovskii's Fixed Point Theorem). *Let X be a Banach space and $U \subset X$ a nonempty, bounded, convex, closed subset. Suppose that the nonlinear operator $T : U \to U$ is a condensing map. Then, there exists a fixed point $u \in U$ of T:*

$$\exists u \in U \quad \text{such that} \quad u = Tu. \quad \triangleleft$$

Since T_ϵ can be rewritten as

$$T_\epsilon = (1 - \epsilon)I + [I - DG^h]_h^{-1} P_h(I - G) + \epsilon G,$$

T_ϵ is a condensing map from $\Lambda \times H_0^1$ to $\mathbf{R} \times H_0^1$. Hence, if we have a nonempty, bounded, convex, closed subset $U \subset \Lambda \times H_0^1$ such that $T_\epsilon U \subseteq U$, we can conclude that there exists a fixed point of T_ϵ. Moreover, if $[I - DG^h]_h^{-1} + \epsilon I$ is invertible, the fixed point of T_ϵ is a solutuon of (2.2). Hence, our verification is reduced to the construction of such U on the memory of computer.

The approximations of an element $u \in H_0^1$, a sebset $U \subset H_0^1$, and operators defined on H_0^1 in a certain finite element space S_h are called their **rounding**. The error of the rounding is called **rounding error**. These notions are defined by projection.

The rounding $\tilde{T}_\epsilon$ of T_ϵ is defined by $\tilde{T}_\epsilon := P_h \circ T_\epsilon$, where P_h is the projection defined by (3.2). Then, we see that

$$\tilde{T}_\epsilon = \tilde{I} - ([I - DG^h]_h^{-1} + \epsilon\tilde{I})(\tilde{I} - \tilde{G}), \tag{4.2}$$

where $\tilde{I} := P_h \circ I_{\mathbb{R}\times H_0^1}$ and $\tilde{G} := P_h \circ G$. Let $U \subset H_0^1$. The rounding $R(T_\epsilon U)$ is defined as the image of $\tilde{T}_\epsilon$:

$$R(T_\epsilon U) := \left\{(\mu, v) \in \mathbb{R} \times S_h \;\middle|\; (\mu, v) = \tilde{T}_\epsilon(\lambda, u),\ (\lambda, u) \in U\right\}. \tag{4.3}$$

We define the rounding error $RE(T_\epsilon U)$ of T_ϵ by

$$\alpha := \sup_{(\mu,u)\in U} \|T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u)\|_{\mathbb{R}\times H_0^1}, \tag{4.4}$$

$$C := C_1C_2, \qquad (C_1, C_2 \text{ are defined by (2.6), (3.1), respectively.}), \tag{4.5}$$

$$RE(T_\epsilon U) := \{0\} \times \left\{\psi \in S_h^\perp \;\middle|\; \|\psi\|_{H_0^1} \le \alpha,\ \|\psi\|_{L^2} \le Ch\alpha\right\} \subset \{0\} \times H_0^1. \tag{4.6}$$

Then, we have

Theorem 4.2. *Let $U \subset \Lambda \times H_0^1$ be a nonempty, bounded, convex, closed subset. If*

$$R(T_\epsilon U) \oplus RE(T_\epsilon U) \overset{\circ}{\subset} U, \tag{4.7}$$

for some ϵ, $0 < \epsilon < 1$, then, there exists a solution $(\lambda, u) \in U$ of the fixed point problem $(\lambda, u) = G(\lambda, u)$. Here, $A \overset{\circ}{\subset} B$ means closure$(A) \subset$ interior(B).

Proof. First, we claim that $T_\epsilon U \subseteq R(T_\epsilon U) \oplus RE(T_\epsilon U)$. For any $(\mu, u) \in U$, we have $T_\epsilon(\mu, u) = \tilde{T}_\epsilon(\mu, u) + (T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u))$. Thus, we just need to show that $T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u) \in RE(T_\epsilon U)$ to prove our claim.

Define the projection $\pi : \mathbb{R} \times H_0^1 \to \mathbb{R}$ by $\pi(\mu, u) = u$ for $(\mu, u) \in \mathbb{R} \times H_0^1$. Let arbitrary $\psi \in L^2$ be taken. Let $\phi := \Phi\psi$, where $\Phi := (L|_{H_0^1})^{-1}$. Then, from (4.4), (4.5), we find that

$$\begin{aligned}(\pi(T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u)), \psi) &= (\pi(T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u)), -\phi'') \\ &= ((\pi(T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u)))', (\phi - P_{0h}\phi)') \\ &\le \|T_\epsilon(\mu, u) - \tilde{T}_\epsilon(\mu, u)\|_{\mathbb{R}\times H_0^1} \|\phi - P_{0h}\phi\|_{H_0^1} \\ &\le C\alpha h\|\psi\|_{L^2}.\end{aligned} \tag{4.8}$$

In (4.8), we use the fact that

$$\|\pi(T_\epsilon(\mu,u)-\tilde{T}_\epsilon(\mu,u))\|_{H_0^1}=\|T_\epsilon(\mu,u)-\tilde{T}_\epsilon(\mu,u)\|_{\mathbf{R}\times H_0^1},$$

since the restricted operator $P_h|_{\mathbf{R}}$ is the identity of $\mathbf{R}$, and there is no "error" of $\tilde{T}_\epsilon$ with respect to the entry of $\mathbf{R}$. By (4.8), we obtain

$$\|\pi(T_\epsilon(\mu,u)-\tilde{T}_\epsilon(\mu,u))\|_{L^2}=\sup_{\psi\in L^2}\frac{|\pi(T_\epsilon(\mu,u)-\tilde{T}_\epsilon(\mu,u))|}{\|\psi\|_{L^2}}\leq Ch\alpha,$$

and conclude that $T_\epsilon U\subseteq R(T_\epsilon U)\oplus RE(T_\epsilon U)$. Therefore, by Theorem 4.1, there exists $(\lambda,u)\in U$ such that $(\lambda,u)=T_\epsilon(\lambda,u)$.

The equation $(\lambda,u)=T_\epsilon(\lambda,u)$ is written as

$$([I-DG^h]_h^{-1}P_h+\epsilon I)(I-G)(\lambda,u)=0. \tag{4.9}$$

The operator $[I-DG^h]_h^{-1}P_h+\epsilon I$ is invertible if and only if $-\epsilon$ is not an eigenvalue of the operator $[I-DG^h]_h^{-1}P_h$. Since $[I-DG^h]_h^{-1}P_h$ is compact, all its eigenvalues are isolated. If (4.7) holds for some ϵ, it also holds for ϵ_0 such that $|\epsilon-\epsilon_0|$ is sufficiently small. Hence, we may assume without loss of generality that $-\epsilon$ is not an eigenvalue of $[I-DG^h]_h^{-1}P_h$. Therefore, from (4.9), we conclude that there esists $(\lambda,u)\in U$ such that $(\lambda,u)=G(\lambda,u)$. $\lhd$

5. Numerical Verification.

By Theorem 4.2, in the set $U\subseteq\Lambda\times H_0^1$ which satisfies (4.7), there exists at least one solution of the fixed point problem $(\lambda,u)=G(\lambda,u)$. Therefore, if we construct such U on the memory of computer, the solution of the fixed point problem is said to verified numerically. This is what we shall do in this section.

Let $\{\phi_j\}_{j=1}^M$ be the basis of S_h. Let Θ_h be the set of linear combinations of intervals and ϕ_j:

$$\Theta_h:=\left\{(A_0,\sum_{j=1}^M A_j\phi_h)\;\middle|\;A_j\subset\mathbf{R}\text{ are interval}\right\}. \tag{5.1}$$

That is, an element $\omega\in\Theta_h$ is the set

$$\omega=(A_0,\sum_{j=1}^M A_j\phi_h):=\left\{(a_0,\sum_{j=1}^M a_j\phi_h)\;\middle|\;a_j\in A_j\right\}.$$

Let $\mathbb{R}^+$ be the set of nonnegative reals. For $\alpha \in \mathbb{R}^+$, we define the set $[\alpha] \subset \{0\} \times S_h^\perp \subset \{0\} \times H_0^1$ by

$$[\alpha] := \{0\} \times \left\{ \phi \in S_h^\perp \;\middle|\; \|\phi\|_{H_0^1} \le \alpha,\ \|\phi\|_{L^2} \le Ch\alpha \right\}. \tag{5.2}$$

We define the following iteration:

Definition 5.1. *Let $(\lambda_h, u_h) \in \Lambda \times S_h$ be the finite element solution defined by* (3.3).
(1) *We set $\Delta(\lambda_h^0, u_h^0) := \{(\lambda_h, u_h)\}$ and $\alpha_0 := 0$ as the initial values.*
(2) *For $n \ge 1$, we define $U^{n-1} \subset \mathbb{R} \times H_0^1$, $\Delta(\lambda_h^n, u_h^n) \subset \mathbb{R} \times S_h$, and $\alpha_n \in \mathbb{R}^+$ inductively by*

$$\begin{cases} U^{n-1} := \Delta(\lambda_h^{n-1}, u_h^{n-1}) + [\alpha_{n-1}], \\ \Delta(\lambda_h^n, u_h^n) := \tilde{T}_\epsilon U^{n-1}, \\ \alpha_n := Ch \sup\limits_{(\mu,v)\in U^{n-1}} \|f(\mu, x, v)\|_{L^2}. \qquad \lhd \end{cases} \tag{5.3}$$

Note that it is very difficult or impossible to estimate $\Delta(\lambda_h^n, u_h^n)$ and α_n in (5.3) exactly. It is, however, possible and easy to enclose each coefficient interval by a slightly bigger interval, that is, overestimate them (cf. [WN]).

Now, let $\delta > 0$ be a small real. We define

$$\begin{cases} \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n) := \Delta(\lambda_h^n, u_h^n) + \left([-1,1]\delta, \sum\limits_{j=1}^{M} [-1,1]\delta\phi_h \right), \\ \tilde{\alpha}_n := \alpha_n + \delta. \end{cases} \tag{5.4}$$

The definition of (5.4) is called **δ-extension**. Let $\tilde{U} := \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n) + [\tilde{\alpha}_n]$. Let $\Delta(\bar{\lambda}_h, \bar{u}_h) \subset \mathbb{R} \times S_h$ and $\bar{\alpha}_n \in \mathbb{R}^+$ be obtained by the iteration (5.3) from $\tilde{U}$:

$$\begin{cases} \Delta(\bar{\lambda}_h, \bar{u}_h) := T_\epsilon \tilde{U}, \\ \bar{\alpha}_n := Ch \sup\limits_{(\mu,v)\in \tilde{U}} \|f(\mu, x, v)\|_{L^2}. \end{cases} \tag{5.5}$$

For these sets, the inclusion $\Delta(\bar{\lambda}_h, \bar{u}_h) \overset{\circ}{\subset} \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n)$ is defined by $B_j \overset{\circ}{\subset} A_j$ $(j = 0, 1, \dots, M)$, where $\Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n) = (A_0, \sum\limits_{j=1}^{m} A_j\phi_j)$ and $\Delta(\bar{\lambda}_h, \bar{u}_h) = (B_0, \sum\limits_{j=1}^{m} B_j\phi_j)$.

To judge whether or not $\tilde{U}$ is what we want, we have the following theorem:

Theorem 5.2. *If we find*

$$\begin{cases} \Delta(\bar{\lambda}_h, \bar{u}_h) \overset{\circ}{\subset} \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n), \\ \bar{\alpha}_n < \tilde{\alpha}_n, \end{cases} \tag{5.6}$$

we conclude that there exists a solution $(\lambda, u) \in \tilde{U}$ of the fixed point problem $(\lambda, u) = G(\lambda, u)$.

Proof. By Theorem 4.2, we only have to show that $R(T_\epsilon\widetilde{U}) \oplus RE(T_\epsilon\widetilde{U}) \overset{\circ}{\subset} \widetilde{U}$.

For any $(\mu, v) \in R(T_\epsilon\widetilde{U})$, there exists $(\lambda, u) \in \widetilde{U}$ such that $(\mu, v) = \widetilde{T}_\epsilon(\lambda, u)$ because of the definition (4.3). Since $T_\epsilon\widetilde{U} = \Delta(\bar{\lambda}_h, \bar{u}_h)$ and (5.6), we have

$$R(T_\epsilon\widetilde{U}) \subset \Delta(\bar{\lambda}_h, \bar{u}_h) \overset{\circ}{\subset} \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n) \subseteq \widetilde{U}. \tag{5.7}$$

By (4.1) and (4.2), we have

$$T_\epsilon(\lambda, u) - \widetilde{T}_\epsilon(\lambda, u) = (1-\epsilon)(I - \widetilde{I})(\lambda, u) + \epsilon(G - \widetilde{G})(\lambda, u).$$

Since $\widetilde{U} = \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n) + [\tilde{\alpha}_n]$, there exist $(\lambda_h, u_h) + (\mu, \omega) \in \Delta(\tilde{\lambda}_h^n, \tilde{u}_h^n)$ and $\beta \in [\tilde{\alpha}_n]$ so that $(\lambda, u) = (\lambda_h + \mu, u_h + \omega + \beta)$. Thus, we obtain $(I - \widetilde{I})(\lambda, u) = (0, \beta) \in [\tilde{\alpha}_n]$.

By Assumption3.1 and (5.6), we have

$$\begin{aligned} \|G(\lambda, u) - \widetilde{G}(\lambda, u)\|_{\mathbb{R}\times H_0^1} &\le C_2 h|\Phi \circ F(\lambda, u)|_{H^2} \le Ch\|f(\lambda, x, u)\|_{L^2} \\ &\le \sup_{(\mu,v)\in\widetilde{U}} \|f(\mu, x, v)\|_{L^2} = \bar{\alpha}_n < \tilde{\alpha}_n. \end{aligned}$$

Therefore, we conclude that $\|T_\epsilon(\lambda, u) - \widetilde{T}_\epsilon(\lambda, u)\|_{\mathbb{R}\times H_0^1} \le (1-\epsilon)\tilde{\alpha}_n + \epsilon\bar{\alpha}_n < \tilde{\alpha}_n$, and

$$RE(T_\epsilon\widetilde{U}) \overset{\circ}{\subset} [\tilde{\alpha}_n] \subseteq \widetilde{U}. \tag{5.8}$$

By (5.7) and (5.8), the proof is completed. ◁

6. The Linearized Equation and Uniqueness.

We iterate the procedure (5.3) until (5.6) is satisfied. Once we obtain $\widetilde{U}$ which satisfies (5.6), we are now sure that there exists at least one solution of the equation $(\lambda, u) = G(\lambda, u)$. We, however, cannot say anything about uniqueness of the solution. Moreover, as mentioned in Remark 2.4, we still have some uncertainty about the choice of the nodal point $x_0 \in J$. This is the motivation of this section.

We suppose that the set $\widetilde{U} \subset \Lambda \times H_0^1$ which satisfies (5.6) has been constructed by computer. Then, we consider the following linearized equation of $I - G$:

$$(I - DG(\widetilde{U}))(\mu, \psi) = (1, 0) \in \mathbb{R} \times H_0^1. \tag{6.1}$$

The equation (6.1) is equivalent to

$$\begin{cases} \psi(x_0) = 1, \\ (L - DF(\widetilde{U}))(\mu, \psi) = 0. \end{cases} \tag{6.2}$$

Note that the equation (6.1) and (6.2) have *interval coefficients*, and thus their solutions are sets.

We try to verify the solution of (6.1) and (6.2) in the exactly same way as before:
(1) Define the operators $T_\epsilon, \widetilde{T}_\epsilon : \Lambda \times H_0^1 \to \mathbb{R} \times H_0^1$ by

$$T_\epsilon := I - ([I - DG^h]_h^{-1} P_h + \epsilon I)(I - DG(\widetilde{U})),$$

and $\widetilde{T}_\epsilon := P_h T_\epsilon$.
(2) Let $(\mu_h, \psi_h) \in \mathbb{R} \times S_h$ be the finite element solution defined by

$$(\psi_h', v_h') = (f_y(\lambda_h, x, u_h)\psi_h + \mu_h f_\lambda(\lambda_h, x, u_h), v_h), \quad \forall v_h \in S_h, \quad \text{and } \psi_h(x_0) = 1.$$

(3) Set $\Delta(\mu_h^0, \psi_h^0) := \{(\mu_h, \psi_h)\}$, $\alpha := 0$, and $n := 1$.
(4) Compute $V^{n-1} \subset \mathbb{R} \times H_0^1$, $\Delta(\mu_h^n, \psi_h^n) \subset \mathbb{R} \times H_0^1$, and $\alpha_n \in \mathbb{R}^+$ by (5.3). Set $n := n+1$.
(5) Compute the δ-extension $\Delta(\tilde{\mu}_h^n, \tilde{\psi}_h^n)$ and $\tilde{\alpha}_n$ by (5.4) from $\Delta(\mu_h^n, \psi_h^n)$ and α_n. Also, compute $\Delta(\bar{\mu}_h^n, \bar{\psi}_h^n)$ and $\bar{\alpha}_n$ by (5.5). Check whether or not they satisfy the condition (5.6). If so, the solution of (6.1) (or (6.2)) is verified. If not, go to (4). $\lhd$

Now, suppose that we have constructed $\widetilde{V} := \Delta(\tilde{\mu}_h^n, \tilde{\psi}_h^n) + [\tilde{\alpha}_n]$ which satisfies (5.6). Then, we conclude that there exists at least one solution of (6.1) in $\widetilde{V}$. Moreover, since the inclusion of (5.6) is strict, the union of solutions is bounded in $\mathbb{R} \times H_0^1$. This means that the kernal of $I - DG(\widetilde{U})$ is trivial: For each $(\eta, w) \in \widetilde{U}$, the kernal of $I - DG(\eta, w)$ is trivial. Therefore, the solution $(\lambda, u) \in \widetilde{U}$ of $(\lambda, u) = G(\lambda, u)$ is unique, at least, locally.

Also, in the set $\widetilde{V}$, there should be some (μ, ψ) which satisfies $(I - DG(\lambda, u)) = (\mu, \psi)$, that is, $\psi(x_0) = 1$ and $(L - DF(\lambda, u))(\mu, \psi) = 0$. This means that for the basis (μ, ψ) of the kernel of $L - DF(\lambda, u)$, we have $\psi(x_0) \neq 0$, and the choice of $x_0 \in J$ is correct.

7. A Numerical Example.

In this section we present an example of numerical verification for the following equation: $J := (0, 1)$ and

$$\text{(7.1)} \qquad \begin{cases} -u'' = \lambda u(u - a)(1 - u), & \text{in } J, \\ u(0) = u(1) = 0, \end{cases}$$

where $a = 0.25$.

Let $N := 100$. We divide J equally into N small intervals. Let $x_i := i/N$ and S_h the finite element space of piecewise linear funtions. As mentioned in Remark 2.4, we use PITCON to follow the solution branch. It is known that this equation has a turning point. According to output of PITCON, the turning point occurs at $\lambda_h = 79.860...$, and PITCON picks up $x_0 = x_{46}$ as the continuation point. We tried to verify the solution (λ_h, u_h) at the point. In the verification we use the values ϵ:=1.0D-6 and δ:=1.0D-4.

The following are the result of verification. We show $\tilde{\alpha}_n$ and the constructed set $\tilde{U} = (A_0, \sum_{j=1}^{99} A_j\phi_j)$, where $A_j := [a_j, b_j]$.

The iteration number = 6,

$\tilde{\alpha}_n = 1.64497D - 2,$

$\lambda_h = 79.8606 \in A_0 = (79.7810, 79.9398)$ and *the width of* $|A_0| = 0.15878$.

Table 7.1: The result of verification.

x_i	a_i	b_i	$\lvert b_j - a_j\rvert$
0.1	0.196815	0.197189	3.7412D–4
0.2	0.396851	0.397447	5.9590D–4
0.3	0.569046	0.569569	5.2240D–4
0.4	0.680851	0.681073	2.2164D–4
0.46	0.712718	0.712718	3.7348D–9
0.5	0.718697	0.718847	1.5025D–4
0.6	0.680702	0.681221	5.1960D–4
0.7	0.568907	0.569708	8.0060D–4
0.8	0.396742	0.397556	8.1412D–4
0.9	0.196757	0.197247	4.9044D–4

After the verification of the solution $(\lambda, u) \in \tilde{U}$, we verified the local uniqueness of the solution and the correctness of the choice of $x_0 = 0.46$. It was done using the same parameters. After only one iteration, the verification was completed.

References

[BRR1] F. **Brezzi**, J. **Rappaz, and** P.A. **Raviart**, Finite Dimensional Approximation of Nonlinear Problems, Part I: Branches of Nonsingular Solutions, Numer. Math., 36 (1980), pp.1-25.

[BRR2] F. **Brezzi**, J. **Rappaz, and** P.A. **Raviart**, Finite Dimensional Approximation of Nonlinear Problems, Part II: Limit Points, Numer. Math., 37 (1981), pp.1-28.

[C] P.G. **Ciarlet**, The Finite Element Methods for Elliptic Problems, North-Holland, 1978.

[N1] M.T. **Nakao**, A numerical approach to the proof of existence of solutions for elliptic problems, Japan J. Appl. Math., 5, (1988), 313-332.

[N2] M.T. **Nakao**, A computational verification method of existence of solutions for nonlinear elliptic equations, Lecture Notes in Num. Appl. Anal., 10, (1989), 101-120.

[N3] M.T. **Nakao**, A numerical approach to the proof of existence of solutions for elliptic problems II, Japan J. Appl. Math., 7, (1990), 477-488.

[N4] M.T. **Nakao**, Solving nonlinear parabolic problems with result verificaion, *to appear in J. Comp. Appl. Math., 38 (1991).*

[N5] M.T. **Nakao**, A numrical verification method for the existence of weak solutions for nonlinear BVP, *to appear in J. Math. Anal. Appl.*

[R] W.C. **Rheinboldt**, Numerical Analysis of Parametrized Nonlinear Equations, Wiley, 1986.

[TB1] T. **Tsuchiya and** I. **Babuška**, A prioir error estimates of finite element solutions of parametrized nonlinear equations, *preprint.*

[TB2] T. **Tsuchiya and** I. **Babuška**, A posteriori error estimates of finite element solutions of parametrized nonlinear equations, *preprint.*

[WN] Y. **Watanabe and** M.T. **Nakao**, Numerical verifications of solutions for nonlinear elliptic equations, Research Report of Mathematics of Computation, Kyushu University, RMC 66-09, (1991), 15 pages.

[Z] E. **Zeidler**, Nonlinear Functional Analysis and Its Application I, Springer, (1986).