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Numerical Verification Methods for the Solutions of Nonlinear Elliptic and Evolution Problems

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Abstract

In this paper, we consider a numerical technique to enclose the solutions with guaranteed error bounds for nonlinear elliptic bundary value problems as well as its extension to the evolution equations. Using a finite element solution and explicit error estimate for certain simple linear problem, we construct, in computer, a set of functions which satisfies the condition of Schauder's or other fixed point theorem under some appropriately function spaces. In order to obtain such a numerical set, we use a kind of multivalued iterative procedure with efficient use of an initial approximate solution. Some numerical examples are illustrated.

1. Introduction

In recent years, several techniques have been developed to use computers in proving existence and /or uniquness of exact solutions for functional equations[3],[4],[6]etc.. Some of them have partially worked upon the integral equations, ordinary differential equations and the special functional equations. In particular, there are not a few approaches for ordinary differntial equations and some of them have already attained sufficiently practical level. It seems, however, few such attempts resulted in success for partial differential equations up to now.

In this report, we will describe some of our results on the numerical approaches for the proof of the existence of weak solutions for elliptic boundary value problems by the computer as well as its extension to the evolution equations.
Recently, Plum[16],[17] proposed another verification techniques for elliptic equations which are different from our method. His method is essentially based upon the numerical enclosure of eigenvalues using the homotopy method for linearlized elliptic operator.

In the following section, we first reformulate the elliptic boundary value problem as the fixed point equation of the compact operator. Next, we introduce two concepts the rounding and the rounding error which correspond to the projection to some finite dimensional subspace and the error estimates, respectively. Then, the verification condition based upon Schauder's fixed point theorem is clarified and we describe the verification procedure using a successively iterative technique. Furthermore, we mention about the Newton-like technique which enable us to apply our method for more general elliptic problems. In the section 3, we extend our method to the second order parabolic initial boundary value problem. Finally, in the section 4, we show that the similar verification principle to those in the previous sections can be also applied to the hyperbolic case of second order. In each case, some numerical examples are presented.

2. Elliptic problems

2.1 Problem and fixed point formulation

Consider the following nonlinear elliptic boundary value problem:

$$\begin{cases} -\Delta u &= f(x,u,\nabla u), & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ $(1 \le n \le 3)$ with piecewise smooth boundary $\partial\Omega$.

Here, we assume that on f:
A1. $f : H^1(\Omega) \longrightarrow L^2(\Omega)$ is continuous.
A2. If $U \subset H^1(\Omega)$ is bounded, then $f(\cdot, U, \nabla U) \subset L^2(\Omega)$ is also bounded.

Here, for an intger m, let $H^m(\Omega) \equiv H^m$ means L^2-Sobolev space of order m on Ω. And set $H_0^1 \equiv \{\phi \in H^1 | tr(\phi) = 0 \text{ on } \partial\Omega\}$ with inner product $< \phi, \psi > \equiv (\nabla\phi, \nabla\psi)$, where $(\cdot,\cdot)$ means inner product on $L^2(\Omega)$.

For example, when $n = 2$, $f(\cdot, u, \nabla u) = g_1 \cdot \nabla u + g_2 u^p$ satisfies above assumptions, where $g_1 = (g_1^1, g_1^2)$, g_2 are $L^\infty(\Omega)$ functions and p is an arbitrary nonnegative integer.

It is well known that for arbitrary $\psi \in L^2(\Omega)$ there exists a unique solution $\phi \in H^2 \cap H_0^1$ of the following Poisson's equation:

$$\begin{cases} -\Delta\phi &= \psi, & x \in \Omega, \\ \phi &= 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

When we denote the solution of (2) by $\phi \equiv A\psi$, from the compactness of the imbedding: $H^2 \hookrightarrow H^1$, the map $A : L^2 \longrightarrow H_0^1$ is compact. Therefore, if we define a nonlinear map by $F \equiv Af$, then from A1 and A2, $F : H_0^1 \longrightarrow H_0^1$ also becomes compact. Based upon the following weak formulation of (2);

$$(\nabla\phi, \nabla v) = (\psi, v), \qquad v \in H_0^1,$$

we define a weak solution for (1) by

$$(\nabla u, \nabla v) = (f(\cdot, u, \nabla u), v), \quad v \in H_0^1. \tag{3}$$

Then (3) can be rewritten as $u = Fu$ which is a fixed point formulation of the original problem (1).

Now let U be a bounded, closed, convex and nonempty subset in H_0^1 such that

$$Af(\cdot, U, \nabla U) \subset U. \tag{4}$$

Then, by Schauder's fixed point theorem, there exists $u \in U$ such that $u = Fu$. Hence, the problem is reduced to finding a set U, in computer, satisfying (4).

2.2 Rounding and verification condition

Since the operator F is infinite dimensional, it is impossible to compute FU for a given $U \subset H_0^1$ directly in computer. In order to treat such an infinite dimensional problem by finite procedures, we introduce the rounding $R(FU)$ into an appropriate finite dimensional subspace of H_0^1 and the rounding error $RE(FU)$ by the following manner:
Let S_h be a finite dimensional subspace of H_0^1 dependent on h $(0 < h < 1)$. Usually, S_h is taken to be a finite element subspace with mesh size h.
Now let $P_h : H_0^1 \longrightarrow S_h$ be an H_0^1 projection defined by

$$(\nabla\phi - \nabla(P_h\phi), \nabla v) = 0, \ {}^\forall v \in S_h.$$

We suppose the following approximation property of P_h:

A3. $\quad {}^\forall\phi \in H^2 \cap H_0^1 \implies ||\phi - P_h\phi||_{H_0^1} \leq C_1 h|\phi|_{H^2},$

where $|\phi|_{H^2}^2 \equiv \sum_{i,j=1}^n ||\frac{\partial^2\phi}{\partial x_i \partial x_j}||_{L^2}^2$. Here, C_1 is a positive constant numerically determined and independent of h. This assumption holds for many finite element subspace of piecewise linear polynomials with quasi-uniform partition([1]). But it will be easily seen, by arguments in the below, that the dependency of C_1 on h is not essential problem for the present case. Also let C_2 be another constant determined by

$$|\phi|_{H^2} \leq C_2||\psi||,$$

where ϕ and ψ satisfy the same relation as (2). For example, we can take as $C_2 = 1$ for the convex polygonal domain in R^2([2]). Then, for $U \subset H_0^1$, we define

Rounding : $R(FU) \equiv \{u_h \in S_h | u_h = P_h(FU), u \in U\}$

and

Rounding error:

$$RE(FU) \equiv \{\phi \in S_h^\perp | \ ||\phi||_{H_0^1} \leq C_1 C_2 h||f(U)||_{L2}\},$$

where $||f(U)||_{L2} \equiv \sup_{u\in U}||f(\cdot, u, \nabla u)||_{L2}$. Note that these quantities can be numerically obtained for a given set U using above constants C_1, C_2. Then, we have

$${}^\forall U \subset H_0^1 \ \textit{bounded} \Longrightarrow FU \subset R(FU) \oplus RE(FU).$$

Therefore, by Schauder's fixed point theorem, if

$$R(FU) \oplus RE(FU) \subset U, \tag{5}$$

then there exists an element $u \in U$ such that $u = Fu$.

2.3 Verification procedures by computer

In order to construct the set U satisfying the verification condition (5) in computer, we use an iterative technique described below.

Let $\{\phi_j\}_{j=1,\cdots M}$ be a basis of S_h. And let $\mathcal{S}_{h,I}$ be the set of all linear combinations with interval coefficients of $\{\phi_j\}$. For $\alpha \in R^+$, the set of all nonnegative real numbers, we set

$$[\alpha] \equiv \{\phi \in S_h^\perp | \ ||\phi||_{H_0^1} \leq \alpha\}.$$

We now generate the following iterative sequence $\{(u_h^{(i)}, \alpha_i)\}_{i=0,1\cdots}$, where $(u_h^{(i)}, \alpha_i) \in \mathcal{S}_{h,I} \times R^+$.

For $i = 0$, $u_h^{(0)} \in \mathcal{S}_{h,I}$, and $\alpha_0 \in R^+$ are appropriately chosen, normally as $u_h^{(0)} = \{u_h\}$, where $u_h \in S_h$ is an approximate solution of the problem, and as $\alpha_0 = 0$.

For $i \geq 1$, first, for given $0 < \delta << 1$, define

$$\begin{cases} \bar{u}_h^{(i-1)} & \equiv & u_h^{(i-1)} + \sum_{j=1}^{M}[-1,1]\delta\phi_j, \\ \bar{\alpha}_{i-1} & \equiv & \alpha_{i-1} + \delta, \end{cases} \tag{6}$$

which is so-called $\delta - inflation$([18]) of $(u_h^{(i-1)}, \alpha_{i-1})$.
Next, set

$$\begin{cases} u_h^{(i)} & \equiv & R(\ F(\bar{u}_h^{(i-1)} + [\bar{\alpha}_{i-1}])), \\ \alpha_i & \equiv & C_1 C_2 h\ ||f(\bar{u}_h^{(i-1)} + [\bar{\alpha}_{i-1}])||_{L^2}. \end{cases} \tag{7}$$

Note that the former of (7) means that $u_h^{(i)}$ is determined by the interval vector solution for the M dimensional linear system of equations with interval right hand side.
Then we have the following verification condition in computer.

Theorem 1. (verification condition)

If, for an integer N,

$$u_h^{(N)} \subset \bar{u}_h^{(N-1)} \text{ and } \alpha_N < \bar{\alpha}_{N-1},$$

then there exists $u \in u_h^{(N)} \oplus [\alpha_N]$ such that $u = Fu$. Here, the first relation implies that each interval in $u_h^{(N)}$ is included to the corresponding interval in $\bar{u}_h^{(N-1)}$.

Verification examples:

Let Ω be a rectangular domain $(0,1) \times (0,1) \subset \mathbf{R}^2$ and let $S_h(x)$ denote the set of continuous linear polynomials on $(0,1)$ with homogeneous boundary conditions. And set $S_h \equiv S_h(x) \otimes S_h(y)$. Then, $M = dimS_h = (N-1)^2$ and previously appeared constants can be taken as $C_1 = \dfrac{1}{\pi}$, $C_2 = 1$, respectively(e.g.[8]).
We could verify the following problems by the above method :

i) $-\Delta u + [-2,2]u^2 = [0,7]$,

where, the intervals mean the sets of L^∞ functions whose ranges are included in the same interval.

ii) $-\Delta u + \lambda e^u = 0$,
where λ is a real parameter, e.g. $\lambda = 1$.
Because of the strong nonlinearity, in this case we need some additional techniques which are not described here(see [13]).

2.4 In case that F is not retractive

In the above, we implicitly assumed that the map F is retractive near the fixed point. When such an assumption does not hold, we use a Newton-like method as follows.
Let S_h and u_h be the same finite dimensional subspace and approximate solution as in the previous section, respectively. Also let $F'(u_h) : H_0^1 \longrightarrow H_0^1$ be the Fréchet derivative of F at u_h.
Suppose that

A4. restriction of the operator $P_h[I - F'(u_h)] : H_0^1 \longrightarrow S_h$ to S_h has an inverse $[I - F'(u_h)]_h^{-1} : S_h \longrightarrow S_h$, where I means the identity map on H_0^1.

This assumption corresponds to the unique existence of the finite element solution in S_h to the linearlized equation of the original problem (1) or (3).

Next, let ε be a fixed positive parameter such that $0 < \varepsilon < 1$.
We now define a nonlinear operator $T : H_0^1 \longrightarrow H_0^1$ as follows:

$$Tu \equiv \{I - ([I - F'(u_h)]_h^{-1} P_h + \varepsilon I)(I - F)\}(u).$$

Then T becomes a condensing map([19]), i.e. T can be decomposed as the sum of contraction and compact operators. It is seen that if ε^{-1} does not coincide with any eigenvalue of the operator $P_h[I - F'(u_h)]$ on S_h then $u = Fu$ becomes equivalent to $u = Tu$. Thus, we can introduce the rounding and the rounding error by the similar technique as in the previous subsection and present the verification procedure using Sadovskii's fixed point theorem[19].

A verification example[15]:

iii) $-\Delta u = \lambda u(u - a)(1 - u)$,
which appears in mathematical biology (e.g., $\lambda = 150,\ a = 0.01$).

3. Parabolic problems

Consider the following nonlnear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,t,u), & (x,t) \in \Omega \times J, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times J, \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{8}$$

where $J=(0,T)$.
Now for ${}^{\forall}\psi \in L^2(\Omega \times J)$ define $\phi \equiv A\psi \in H^1(J;L^2)\cap L^2(J;H^2\cap H_0^1)$ by

$$\left\{\begin{array}{lcll} \dfrac{\partial \phi}{\partial t} - \Delta\phi & = & \psi & (x,t) \in \Omega \times J, \\ \phi(x,t) & = & 0, & (x,t) \in \partial\Omega \times J, \\ \phi(x,0) & = & 0, & x \in \Omega, \end{array}\right. \tag{9}$$

where $H^1(J;L^2)$ and $L^2(J;H^2\cap H_0^1)$ mean the Sobolev spaces of time-dependent type. Then by the compactness of the imbedding : $H^1(J;L^2)\cap L^2(J;H^2\cap H_0^1) \hookrightarrow L^2(J;H_0^1)$ the map A is compact. Thus (8) is equivalent to the fixed point equation: $u = Af(\cdot,\cdot,u)$ for the compact map Af under certain assumptions on f. Therefore, similar techniques, as in the elliptic case, can be applied for verification of the solution of (8). That is, we can define the rounding and the rounding error, which are analogous concepts to that in subsection 2.2, by the use of some appropriate finite dimensional subspace of $L^2(J;H_0^1)$ and the projection into it. And the verification procedures are the same as line of the previous subsection.
In [13], we presented a verification procedure based upon the error estimates for the projection $P_h\ :\ L^2(J;H^2\cap H_0^1)\cap H^1(J;L^2) \longrightarrow S_h$ defined by the following simultaneous discretization scheme for space and time for one space dimensional case:

$$\int_0^T \{(\phi_t^h, v)_\Omega + (\nabla\phi^h, \nabla v)_\Omega\}dt = \int_0^T (\psi, v)_\Omega dt, \qquad v \in S_h,$$

where $\phi^h \equiv P_h\phi$ and S_h is the space of piecewise linear polynomials for space and time. Also $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product on Ω.
Then the error estimates are provided as follows: for $e \equiv \phi - \phi^h$

$$||\nabla e||^2 \le \inf_{v\in S_h} 2(||e_t||\ ||\phi - v|| + ||\nabla(\phi - v)||^2/2),$$

where $||\cdot||$ implies the L^2 norm on $\Omega \times J$. Using above estimates, approximation property of S_h and an *a priori* estimate for the solution of (9)(e.g.[7]), the rounding can be defined.
A verification example[12]:
iv) $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = pu^2 + [q_1, q_2],$

$$\text{e.g., } p = 1,\ [q_1, q_2] = [0, \frac{1}{\pi}].$$

4. Hyperbolic problems

Consider the following nonlinear hyperbolic homogeneous initial boundary value problem.

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u &= f(x,t,u), & (x,t) \in \Omega \times J, \\
u(x,t) &= 0, & (x,t) \in \partial\Omega \times J, \\
u(x,0) &= 0, & x \in \Omega, \\
\dfrac{\partial u}{\partial t}(x,0) &= 0, & x \in \Omega,
\end{cases} \tag{10}
$$

where Ω and J are the same as in the previous section. For ${}^\forall\psi \in H^1(J; L^2)$, define $\phi \in H^{2,2}(\Omega \times J) \equiv L^2(J; H^2 \cap H_0^1) \cap H^2(J; L^2)$ as the solution of the following linear problem:

$$
\begin{cases}
\dfrac{\partial^2 \phi}{\partial t^2} - \Delta \phi &= \psi & (x,t) \in \Omega \times J, \\
\phi(x,t) &= 0, & (x,t) \in \partial\Omega \times J, \\
\phi(x,0) &= 0, & x \in \Omega, \\
\dfrac{\partial \phi}{\partial t}(x,0) &= 0, & x \in \Omega.
\end{cases} \tag{11}
$$

When we denote the relation (11) by $\phi = A\psi$, from an *a priori* estimate for the solution of (11)[5] and the compactness of imbedding $H^{2,2}(\Omega \times J) \hookrightarrow H^{1,1}(\Omega \times J)$ A also becomes compact on $H^{1,1}(\Omega \times J)$. Therefore, the solution u for (10) can be defined, under certain conditions on f, as the fixed point of compact map $Fu \equiv Af(\cdot, \cdot, u)$.

Next, in order to define the rounding: $R(Fu)$ and the rounding error: $RE(Fu)$, we use a simultaneous discretization for space and time of (11) and the error estimates as in the previous section. For example, in [14], we defined a projection $P_h : H^{2,2}(\Omega \times J) \longrightarrow S_h \subset H^{2,2}(\Omega \times J)$ for the solution of (11) by the following somewhat peculier scheme in one space dimensional case.

$$
\int_0^T \int_0^t \{(\phi_{ss}^h, v_s)_\Omega + (\nabla\phi^h, \nabla v_s)_\Omega\} ds dt = \int_0^T \int_0^t (\psi, v_s)_\Omega ds dt, \qquad v \in S_h,
$$

where $\phi^h \equiv P_h\phi$ and S_h is the space of C^2 class piecewise cubic polynomials, i.e. cubic spline functions, both directions for space and time.
Then the folowing error estimates are derived: for $e \equiv \phi - \phi^h$

$$
||e_t||^2 + ||\nabla e||^2 \le 2(||\psi - \phi_{tt}^h + \Delta\phi^h|| + T||\psi_t - \phi_{ttt}^h + \Delta\phi_t^h||) \inf_{v \in S_h} ||\phi - v||.
$$

Furthermore, from the *a priori* estimate for the solution of (11) and the approximation property of S_h, we can define the rounding error and the verification procedures.

A numerical example

viii) $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = Ku^2 + P\sin^2\pi x(2 + t^2\pi^2 - KPt^4 \sin\pi x)$,

where K and P means real parameters. The exact solution of this equation is

$u = Pt^2 \sin \pi x$. We could verify several cases, e.g. $K = 0.5, P = 0.1$ ($\Omega \times J = (0,1) \times (0,1)$). Owing to the limitation of our computing facility, in the present stage, we could not verify for large K, P.

5. Concluding remarks

As described above, it was proved that our verification methods based on the rounding and the rounding error can be applied in principle to all types of partial differential equations , i.e. elliptic, parabolic and hyperbolic cases. Particularly, the various results of numerical experiments show that the present method can verify with sufficient accuracy and effectiveness for the elliptic problems. On the other hand, for parabolic and hyperbolic cases, the explicit error estimates with high accuracy are not yet obtained for the simple linear problems (9) and (11), respectively. Therefore, in order to attain the practical level for such cases, it is neccessary, as the future work, to find some approximation schemes for these linear problems which enable us efficient constructive error estimates.

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